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BOUNDARY TERMS FOR MASSLESS FERMIONIC FIELDS

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Abstract. Local supersymmetry leads to boundary conditions for fermionic fields in one-loop quantum cosmology involving the Euclidean normal ${}_e n_A^{\ A'}$ to the boundary and a pair of independent spinor fields ψ^A and $\widetilde{\psi}^{A'}$. This paper studies the corresponding classical properties, i.e. the classical boundary-value problem and boundary terms in the variational problem. If $\sqrt{2}\ {}_e n_A^{\ A'} \ \psi^A \mp \widetilde{\psi}^{A'} \equiv \Phi^{A'}$ is set to zero on a 3-sphere bounding flat Euclidean 4-space, the modes of the massless spin-$\frac{1}{2}$ field multiplying harmonics having positive eigenvalues for the intrinsic 3-dimensional Dirac operator on S^3 should vanish on S^3. Remarkably, this coincides with the property of the classical boundary-value problem when spectral boundary conditions are imposed on S^3 in the massless case. Moreover, the

boundary term in the action functional is proportional to the integral on the boundary of $\Phi^{A'} \; {}_{e}n_{AA'} \; \psi^{A}$.

Locally supersymmetric boundary conditions have been recently studied in quantum cosmology to understand its one-loop properties. They involve the normal to the boundary and the field for spin $\frac{1}{2}$, the normal to the boundary and the spin-$\frac{3}{2}$ potential for gravitinos, Dirichlet conditions for real scalar fields, magnetic or electric field for electromagnetism, mixed boundary conditions for the 4-metric of the gravitational field (and in particular Dirichlet conditions on the perturbed 3-metric). The aim of this letter is to describe the corresponding classical properties in the case of massless spin-$\frac{1}{2}$ fields.

For this purpose, we consider flat Euclidean 4-space bounded by a 3-sphere of radius a. The spin-$\frac{1}{2}$ field, represented by a pair of independent spinor fields ψ^A and $\widetilde{\psi}^{A'}$, is expanded on a family of 3-spheres centred on the origin as [1-3]

$$\psi^A = \frac{\tau^{-\frac{3}{2}}}{2\pi} \sum_{n=0}^{\infty} \sum_{p=1}^{(n+1)(n+2)} \sum_{q=1}^{(n+1)(n+2)} \alpha_n^{pq} \left[m_{np}(\tau) \rho^{nqA} + \widetilde{r}_{np}(\tau) \overline{\sigma}^{nqA} \right] \tag{1}$$

$$\widetilde{\psi}^{A'} = \frac{\tau^{-\frac{3}{2}}}{2\pi} \sum_{n=0}^{\infty} \sum_{p=1}^{(n+1)(n+2)} \sum_{q=1}^{(n+1)(n+2)} \alpha_n^{pq} \left[\widetilde{m}_{np}(\tau) \overline{\rho}^{nqA'} + r_{np}(\tau) \sigma^{nqA'} \right] \quad . \tag{2}$$

With our notation, τ is the Euclidean-time coordinate, the α_n^{pq} are block-diagonal matrices with blocks $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, the $\rho-$ and σ-harmonics obey the identities described in [1,3]. Last but not least, the modes m_{np} and r_{np} are regular at $\tau = 0$, whereas the modes $\widetilde{m}_{np}$ and $\widetilde{r}_{np}$ are singular at $\tau = 0$ if the spin-$\frac{1}{2}$ field is massless. Bearing in mind that the harmonics ρ^{nqA} and $\sigma^{nqA'}$ have positive eigenvalues $\frac{1}{2}\left(n + \frac{3}{2}\right)$ for the 3-dimensional Dirac operator on the bounding S^3 [3], the decomposition (1-2) can be re-expressed as

$$\psi^A = \psi^A_{(+)} + \psi^A_{(-)} \tag{3}$$

$$\widetilde{\psi}^{A'} = \widetilde{\psi}^{A'}_{(+)} + \widetilde{\psi}^{A'}_{(-)} \quad . \tag{4}$$

In (3-4), the (+) parts correspond to the modes m_{np} and r_{np}, whereas the (−) parts correspond to the singular modes $\widetilde{m}_{np}$ and $\widetilde{r}_{np}$, which multiply harmonics having negative eigenvalues $-\frac{1}{2}\left(n+\frac{3}{2}\right)$ for the 3-dimensional Dirac operator on S^3. If one wants to find a classical solution of the Weyl equation which is regular $\forall \tau \in [0,a]$, one is thus forced to set to zero the modes $\widetilde{m}_{np}$ and $\widetilde{r}_{np}$ $\forall \tau \in [0,a]$ [1]. This is why, if one requires the local boundary conditions [3]

$$\sqrt{2}\ {}_{e}n_{A}^{\ \ A'}\ \psi^{A} \mp \widetilde{\psi}^{A'} = \Phi^{A'} \text{ on } S^3 \tag{5}$$

such a condition can be expressed as [3]

$$\sqrt{2}\ {}_{e}n_{A}^{\ \ A'}\ \psi^{A}_{(+)} = \Phi^{A'}_{1} \text{ on } S^3 \tag{6}$$

$$\mp \widetilde{\psi}^{A'}_{(+)} = \Phi^{A'}_{2} \text{ on } S^3 \tag{7}$$

where $\Phi^{A'}_1$ and $\Phi^{A'}_2$ are the parts of the spinor field $\Phi^{A'}$ related to the $\overline{\rho}$- and σ-harmonics respectively. In particular, if $\Phi^{A'}_1 = \Phi^{A'}_2 = 0$ on S^3 as in [2,3], one finds

$$\sum_{n=0}^{\infty} \sum_{p=1}^{(n+1)(n+2)} \sum_{q=1}^{(n+1)(n+2)} \alpha_n^{pq}\ m_{np}(a)\ {}_{e}n_{A}^{\ \ A'}\ \rho^{A}_{nq} = 0 \tag{8}$$

$$\sum_{n=0}^{\infty} \sum_{p=1}^{(n+1)(n+2)} \sum_{q=1}^{(n+1)(n+2)} \alpha_n^{pq}\ r_{np}(a)\ \sigma^{A'}_{nq} = 0 \tag{9}$$

where a is the 3-sphere radius. Since the harmonics appearing in (8-9) are linearly independent, these relations lead to $m_{np}(a) = r_{np}(a) = 0\ \forall n,p$. Remarkably, this simple calculation shows that the classical boundary-value problems for regular solutions of the Weyl equation subject to local or spectral conditions on S^3 share the same property provided $\Phi^{A'}$ is set to zero in (5): the regular modes m_{np} and r_{np} should vanish on the bounding S^3.

To study the corresponding variational problem for a massless fermionic field, we should now bear in mind that the spin-$\frac{1}{2}$ action functional in a Riemannian 4-geometry takes the form [2,3]

$$I_E = \frac{i}{2} \int_M \Bigr[\widetilde{\psi}^{A'} \Bigr(\nabla_{AA'} \psi^A \Bigr) - \Bigr(\nabla_{AA'} \widetilde{\psi}^{A'} \Bigr) \psi^A \Bigr] \sqrt{\det g} \; d^4x + \widehat{I}_B \quad . \tag{10}$$

This action is *real*, and the factor i occurs by virtue of the convention for Infeld-van der Waerden symbols used in [2,3]. In (10) $\widehat{I}_B$ is a suitable boundary term, to be added to ensure that I_E is stationary under the boundary conditions chosen at the various components of the boundary (e.g. initial and final surfaces, as in [1]). Of course, the variation δI_E of I_E is linear in the variations $\delta \psi^A$ and $\delta \widetilde{\psi}^{A'}$. Defining $\kappa \equiv \frac{2}{i}$ and $\kappa \widehat{I}_B \equiv I_B$, variational rules for anticommuting spinor fields lead to

$$\begin{aligned} \kappa \Bigr(\delta I_E \Bigr) = & \int_M \Bigr[2 \delta \widetilde{\psi}^{A'} \Bigr(\nabla_{AA'} \psi^A \Bigr) \Bigr] \sqrt{\det g} \; d^4x - \int_M \Bigr[\Bigr(\nabla_{AA'} \widetilde{\psi}^{A'} \Bigr) 2 \delta \psi^A \Bigr] \sqrt{\det g} \; d^4x \\ & - \int_{\partial M} \Bigr[{}_e n_{AA'} \Bigr(\delta \widetilde{\psi}^{A'} \Bigr) \psi^A \Bigr] \sqrt{\det h} \; d^3x + \int_{\partial M} \Bigr[{}_e n_{AA'} \widetilde{\psi}^{A'} \Bigr(\delta \psi^A \Bigr) \Bigr] \sqrt{\det h} \; d^3x \\ & + \delta I_B \end{aligned} \tag{11}$$

where I_B should be chosen in such a way that its variation δI_B combines with the sum of the two terms on the second line of (11) so as to specify what is fixed on the boundary (see below). Indeed, setting $\epsilon = \pm 1$ and using the boundary conditions (5) one finds

$$ {}_e n_{AA'} \widetilde{\psi}^{A'} = \frac{\epsilon}{\sqrt{2}} \psi_A - \epsilon \; {}_e n_{AA'} \Phi^{A'} \text{ on } S^3 \quad . \tag{12} $$

Thus, anticommutation rules for spinor fields [1] show that the second line of equation (11) reads

$$ \delta I_{\partial M} \equiv - \int_{\partial M} \Bigr[\Bigr(\delta \widetilde{\psi}^{A'} \Bigr) {}_e n_{AA'} \psi^A \Bigr] \sqrt{\det h} \; d^3x + \int_{\partial M} \Bigr[{}_e n_{AA'} \widetilde{\psi}^{A'} \Bigr(\delta \psi^A \Bigr) \Bigr] \sqrt{\det h} \; d^3x $$

$$ = \epsilon \int_{\partial M} {}_e n_{AA'} \Bigr[\Bigr(\delta \Phi^{A'} \Bigr) \psi^A - \Phi^{A'} \Bigr(\delta \psi^A \Bigr) \Bigr] \sqrt{\det h} \; d^3x \quad . \tag{13} $$

Now it is clear that setting

$$ I_B \equiv \epsilon \int_{\partial M} \Phi^{A'} {}_e n_{AA'} \; \psi^A \sqrt{\det h} \; d^3x \quad , \tag{14} $$

enables one to specify $\Phi^{A'}$ on the boundary, since

$$ \delta \Bigr[I_{\partial M} + I_B \Bigr] = 2\epsilon \int_{\partial M} {}_e n_{AA'} \Bigr(\delta \Phi^{A'} \Bigr) \psi^A \sqrt{\det h} \; d^3x \quad . \tag{15} $$

Hence the action integral (10) appropriate for our boundary-value problem is

$$ I_E = \frac{i}{2} \int_M \Bigr[\widetilde{\psi}^{A'} \Bigr(\nabla_{AA'} \psi^A \Bigr) - \Bigr(\nabla_{AA'} \widetilde{\psi}^{A'} \Bigr) \psi^A \Bigr] \sqrt{\det g} \; d^4x $$

$$ + \frac{i\epsilon}{2} \int_{\partial M} \Phi^{A'} {}_e n_{AA'} \; \psi^A \sqrt{\det h} \; d^3x \quad . \tag{16} $$

Note that, by virtue of (5), equation (13) may also be cast in the form

$$ \delta I_{\partial M} = \frac{1}{\sqrt{2}} \int_{\partial M} \Bigr[\widetilde{\psi}^{A'} \Bigr(\delta \Phi_{A'} \Bigr) - \Bigr(\delta \widetilde{\psi}^{A'} \Bigr) \Phi_{A'} \Bigr] \sqrt{\det h} \; d^3x \quad , \tag{17} $$

which implies that an equivalent form of I_B is

$$I_B \equiv \frac{1}{\sqrt{2}} \int_{\partial M} \widetilde{\psi}^{A'} \; \Phi_{A'} \sqrt{\det h} \; d^3 x \quad . \tag{18}$$

The local boundary conditions studied at the classical level in this paper, have been applied to one-loop quantum cosmology in [2-4]. Interestingly, our work seems to add evidence in favour of quantum amplitudes having to respect the properties of the classical boundary-value problem. In other words, if fermionic fields are massless, their one-loop properties in the presence of boundaries coincide in the case of spectral [1,3,5] or local boundary conditions [2-4], while we find that classical modes for a regular solution of the Weyl equation obey the same conditions on a 3-sphere boundary with spectral or local boundary conditions, provided the spinor field $\Phi^{A'}$ of (5) is set to zero on S^3. We also hope that the analysis presented in Eqs. (10)-(18) may clarify the spin-$\frac{1}{2}$ variational problem in the case of local boundary conditions on a 3-sphere (cf. the analysis in [6] for pure gravity).

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