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SELF-DUAL ACTION FOR FERMIONIC FIELDS AND GRAVITATION

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Summary. - This paper studies the self-dual Einstein-Dirac theory. A generalization is obtained of the Jacobson-Smolin proof of the equivalence between the self-dual and Palatini purely gravitational actions. Hence one proves equivalence of self-dual Einstein-Dirac theory to the Einstein-Cartan-Sciama-Kibble-Dirac theory. The Bianchi symmetry of the curvature, core of the proof, now contains a non-vanishing torsion. Thus, in the self-dual framework, the "extra" terms entering the equations of motion with respect to the standard Einstein-Dirac field equations, are neatly associated with torsion.

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1. - Introduction.

Over the last few years, many efforts have been produced in the literature to provide a new mathematical framework for the analysis of canonical gravity (e.g. [1-2] and references therein). Thus, instead of the standard geometrodynamical variables, one now deals with a new set of variables involving soldering forms and connections. Remarkably, in terms of these geometrical objects proposed by Ashtekar [1], the constraints of general relativity take a polynomial form, and this has motivated the introduction of yet new (loop) variables to solve the quantum version of the constraint equations [1].

So far, any testable property of gravitation involves matter, and, in fact, matter could help to come to terms with the issue of defining physical observables and time in quantum gravity [3]. Hence, coupling matter fields to gravitation becomes of primary interest. Spin-$\frac{1}{2}$ fields coupled to gravity within the framework of Ashtekar's variables were studied in [4] and [5] (the latter included, besides, scalar and Yang-Mills fields). Since the gravitational part of the action using these variables is first-order, it is natural to consider a gravitational connection admitting torsion. There are several kinds of matter which can support a non-vanishing torsion when coupled to gravity [5-6]. Attention is here focused on the spin-$\frac{1}{2}$ Dirac field minimally coupled to gravity. A key question is whether the simplifying Ashtekar form of the theory contains the whole information of the standard well-known theories, at the classical level. In the pure-gravity case, Jacobson and Smolin [2] showed that the Palatini action

$$S_P[e,\omega] = \int d^4x \;\; e \; e^{a\hat{a}} e^{b\hat{b}} R_{ab\hat{a}\hat{b}}(\omega) \; , \tag{1.1}$$

with e^{-1} the determinant of the tetrad $e^{a\hat{a}}$, a,b world indices and $\hat{a},\hat{b}$ frame indices, can be recovered from the self-dual action

$$S_{SD}[e,{}^{+}\omega] = \int d^4x \;\; e \; e^{a\hat{a}} e^{b\hat{b}} R_{ab\hat{a}\hat{b}}({}^{+}\omega) \; , \tag{1.2}$$

with ${}^+\omega := \frac{1}{2}(\omega - i * \omega)$ the self-dual (on frame indices) part of ω. This follows on substituting the equation of motion $\delta S_{SD}/\delta^+\omega = 0$ in (1.2) [2]. That is to say

$$S_{SD}[e, {}^+\omega(e)] = \frac{1}{2} S_P[e, \omega(e)] - \frac{i}{4} \int d^4x \;\; e \, e^{a\hat{a}} \epsilon_{\hat{a}}{}^{\hat{c}\hat{d}\hat{e}} R_{a\hat{c}\hat{d}\hat{e}}(\omega(e)) \; . \qquad (1.3)$$

Thus, by virtue of the Bianchi symmetry of the curvature for vanishing torsion, $R_{a[\hat{c}\hat{d}\hat{e}]} = 0$, the second term in (1.3) vanishes.

Our work aims to complement the results of [4-5], on spin-$\frac{1}{2}$ fields, in two respects:

(i) The equivalence (analogue of (1.3)) is *explicitly* given for the ECSKD (Einstein-Cartan-Sciama-Kibble-Dirac) [6-7] and the sd-ED (self-dual Einstein-Dirac) actions [4-5].

(ii) The origin of some extra terms in the equations of motion derived from the self-dual action [5], w.r.t. standard Einstein-Dirac ones [8], is traced back to the non-vanishing torsion. A result considered previously, in a different framework, in the literature [9].

Moreover, our proof of the equivalence (i) is necessarily a generalization of that of Jacobson and Smolin [2] since, for non-vanishing torsion, $R_{a[cde]} \neq 0$. Nevertheless, the Bianchi symmetry of the curvature including torsion already exists [10] and we make use of it below.

Sect. **2** establishes the equivalence between the sd-ED and ECSKD actions. Sect. **3** studies the field equations for both theories. Concluding remarks are presented in sect. **4**.

2. - ECSKD theory and self-dual Einstein-Dirac theory.

In coupling fermionic fields to gravity the introduction of orthonormal tetrads is natural because spinors are defined w.r.t. orthonormal frames [10]. Furthermore, whenever tetrads are adopted, connections also enter the description of gravity. In building up an action from which to obtain the equations of motion, one has the possibility of considering tetrads and connections as independent fields or not. If they are not one gets the Einstein-Dirac (second-order) action. Instead, by taking them as independent, one gets the ECSKD (first-order) action. The corresponding variational problems differ on what should be fixed at the boundary. One finds it is necessary to add boundary terms to get a well-posed problem only in the second-order case [11]. In the first-order case, on the other hand, one is left with an equation of motion associating a non-vanishing torsion to the connection [6-7].

In a four-dimensional Lorentzian space-time, a massive Dirac field is represented by the spinor fields $\left(\kappa^A, \mu^A\right)$, say, jointly with their complex conjugates, hereafter denoted by overbars. Thus, using two-component spinor notation, the corresponding action functional for ECSKD theory is

$$\begin{aligned} S_{ECSKD} = \int_M d^4x \Big\{ & \sigma \left[\sigma^{aMA'} \sigma^b{}_{AA'} R_{abM}{}^A[{}^+\omega] + \sigma^{aAM'} \sigma^b{}_{AA'} \overline{R}_{abM'}{}^{A'}[{}^-\omega] \right] \\ & - \sqrt{2}\,\sigma\,\sigma^a{}_{AA'} \left[\overline{\kappa}^{A'} \left(\nabla_a \kappa^A \right) - \left(\nabla_a \mu^A \right) \overline{\mu}^{A'} \right] \\ & + \sqrt{2}\,\sigma\,\sigma^a{}_{AA'} \left[\left(\nabla_a \overline{\kappa}^{A'} \right) \kappa^A - \mu^A \left(\nabla_a \overline{\mu}^{A'} \right) \right] \\ & - 2im\sigma \left[\mu_A \kappa^A - \overline{\kappa}^{A'} \overline{\mu}_{A'} \right] \Big\}, \end{aligned} \tag{2.1}$$

where $\sigma \equiv \det(\sigma_a{}^{AA'})$ and $\sigma_a{}^{AA'}$ is the soldering form (i.e. the two-spinor version of the tetrad in curved space-time). Note that the connection, here splitted into self-dual and anti-self-dual parts, develops a torsion contribution supported by the fermionic fields, since we use a first-order formalism with connection and soldering forms taken to be independent fields instead of adopting *a priori* the relation between them [8].

By contrast, the authors of [5] studied the coupling of fermions to gravity in terms of an action containing the self-dual part of a connection only, disregarding the anti-self-dual part. They assumed, though, this connection to be torsion-free. In this paper the analysis is carried out by extending the connection to admit torsion. This simplifies the proof of equivalence between sd-ED and ECSKD, as shown below, and makes it easier to interpret the corresponding equations of motion in section 3. Let the self-dual action [5] be

$$\begin{aligned} S_{SD} = \int_M d^4x \Big\{ & -\sigma\,\sigma^{aMA'}\,\sigma^{bJ}{}_{A'}\,F_{abMJ} \\ & - \sqrt{2}\,\sigma\,\sigma^a{}_{AA'} \left[\overline{\kappa}^{A'} \left(\mathcal{D}_a \kappa^A \right) - \left(\mathcal{D}_a \mu^A \right) \overline{\mu}^{A'} \right] \\ & - im\sigma \left[\mu_A \kappa^A - \overline{\kappa}^{A'} \overline{\mu}_{A'} \right] \Big\}, \end{aligned} \tag{2.2}$$

where F_{abMN} is the curvature of the connection $\mathcal{D}$ defined at this stage to act only on unprimed spinor indices. Clearly, (2.2) is obtained from (2.1) by taking only the contribution of the *self-dual* piece of the connection, ${}^{+}\omega$ and half of the mass term. The name *self-dual* hence accounts for this. Thus, (2.2) is manifestly not real. Nevertheless, it will be shown below it reproduces (2.1) modulo the equations of motion for $\mathcal{D}$ and via the Bianchi symmetry of the curvature for a "metric-compatible" connection having torsion, so that no spurious equations of motion are picked up.

The goal here is to determine $\mathcal{D}$ dynamically. The variation of (2.2) with respect to $\mathcal{D}$ can be carried out by introducing the auxiliary forms $Q_a{}^M{}_N$ and $P^c{}_{ab}$ so as to define $\mathcal{D}$ with respect to ∇, the connection compatible with the soldering form, i.e. such that $\nabla_a \sigma_b^{AA'} = 0$, and having associated a non-vanishing torsion $T_{ab}{}^c$

$$2\nabla_{[a}\nabla_{b]}f \equiv T_{ab}{}^c\ \nabla_c f\ ,\ f\ \text{a zero} - \text{form}. \tag{2.3}$$

Namely,

$$\mathcal{D}_a \lambda_b^A = \nabla_a \lambda_b^A + Q_a{}^A{}_B\ \lambda_b^B + P^c{}_{ab}\ \lambda_c^A, \tag{2.4}$$

with associated torsion $\mathcal{T}_{ab}{}^c$

$$2\mathcal{D}_{[a}\mathcal{D}_{b]}f \equiv \mathcal{T}_{ab}{}^c\ \nabla_c f\ ,\ f\ \text{a zero} - \text{form}, \tag{2.5}$$

$$\mathcal{T}_{ab}{}^c \equiv T_{ab}{}^c - 2P^c{}_{[ab]}. \tag{2.6}$$

By requiring the annihilation of the symplectic form ϵ_{AB} one gets a restriction on Q_{aAB} above:

$$\mathcal{D}_a \epsilon_{AB} = \nabla_a \epsilon_{AB} = 0 \quad \Rightarrow Q_{aAB} = Q_{a(AB)}\ \ (\text{trace} - \text{free}). \tag{2.7}$$

Concerning the action on space-time indices, and thus $P^c{}_{ab}$, it is known that to control both metricity condition and torsion it is necessary to include kinetic terms for them in the corresponding action [12-13]; otherwise one should impose either of them and get the other as an equation of motion [12-13]. We follow the latter possibility by imposing

$$P^c{}_{ab} = 0. \tag{2.8}$$

This amounts to specify that the torsion of ∇, $T_{ab}{}^{c}$, is exactly that of $\mathcal{D}$, $\mathcal{T}_{ab}{}^{c}$ (cf. (2.6)), to be determined dynamically. Also, from (2.4), the action on space-time indices of both $\mathcal{D}$ and ∇ is identified.

Varying $\mathcal{D}$ is equivalent to varying Q so the action (2.2) should be re-expressed in terms of ∇, Q_{aMN} and $T_{ab}{}^{c}$. The curvatures, F_{abMN} and R_{abMN} of $\mathcal{D}$ and ∇ (on unprimed indices), respectively, are such that [10]

$$F_{abMN} = R_{abMN} - 2\nabla_{[a}Q_{b]MN} + 2Q_{[aM}{}^{P}\, Q_{b]PN} + T_{ab}{}^{c}\, Q_{cMN}. \tag{2.9}$$

On inserting (2.9) into (2.2), an integration by parts is necessary to deal with the $\nabla_{[a}Q_{b]MN}$ term. This gives a total divergence and a term containing the derivative of products of soldering forms

$$\begin{aligned}
&\int_M d^4x \left\{-2\sigma\ \sigma^{aMA'}\sigma^{bN}{}_{A'}\nabla_{[a}Q_{b]MN}\right\} \\
&= \int_M d^4x \left\{-2\nabla_a\left[\sigma\ \sigma^{[aMA'}\sigma^{b]N}{}_{A'}\ Q_{bMN}\right] + 2\left[\nabla_a\left(\sigma\ \sigma^{[aMA'}\sigma^{b]N}{}_{A'}\right)\right] Q_{bMN}\right\} \\
&= -2\int_{\partial M} dS_a\ \sigma^{[aMA'}\sigma^{b]N}{}_{A'}\ Q_{bMN} - 2\int_M d^4x\ T_{am}{}^{m}\ \sigma\ \sigma^{[aMA'}\sigma^{b]N}{}_{A'}\ Q_{bMN}.
\end{aligned} \tag{2.10}$$

The second term on the second line above drops by virtue of the metricity condition, whereas the total divergence turns into a sum of a boundary and a volume term, the latter containing torsion.

Using the above results in varying (2.2) w.r.t. Q_{gMN} yields the equation

$$\sigma^{[a(MA'}\sigma^{b]N)}{}_{A'}\left(2T_{am}{}^{m}\ \delta_b{}^{g} - T_{ab}{}^{g}\right) + 4\sigma^{[g(MA'}\sigma^{a]}{}_{AA'}\ Q_a{}^{N)A} + i\sigma^{g(M}{}_{A'}\ k^{N)A'} = 0. \tag{2.11}$$

Here $[a(MA'\ \ b]N)$ means antisymmetrization in a, b and symmetrization in M, N, and similarly for the other terms. $k^{AA'}$ and k^m are defined by

$$k^{AA'} \equiv -i\sqrt{2}\left(\overline{\kappa}^{A'}\kappa^{A} - \mu^{A}\overline{\mu}^{A'}\right), \tag{2.12a}$$

$$k^m \equiv \sigma^{m}{}_{AA'}\ k^{AA'}. \tag{2.12b}$$

One readily solves (2.11) for Q_{aMN} pointing out that $i\sigma^{g(M}{}_{A'}\ k^{N)A'} = i\sigma^{g(M}{}_{A'}\ \epsilon_R{}^{N)} k^{RA'}$, whose r.h.s., in turn, obeys the identity

$$2i\sigma^{g(M}{}_{A'}\ \epsilon_R{}^{N)} = \sigma^{p(MB'}\sigma^{qN)}{}_{A'}\ \sigma^m{}_{RB'}\ \epsilon_{pqm}{}^g. \tag{2.13}$$

Note that there is an implicit antisymmetrization in p, q on the r.h.s. of this identity owed to the contraction with the volume four-form. It is possible now to factor out the soldering-form factors in (2.11). This leads to

$$\sigma^{pRA'}\sigma^{qS}{}_{A'} \left\{ \left(2T_{[pm}{}^m\ \delta_{q]}{}^g - T_{pq}{}^g \right) \epsilon_R{}^{(M}\epsilon_S{}^{N)} \right.$$
$$\left. +4\epsilon_{SA}\delta_{[p}{}^g Q_{q]}{}^{A(N}\epsilon_R{}^{M)} + \frac{1}{2}\epsilon_{pqm}{}^g\ k^m \epsilon_R{}^{(M}\epsilon_S{}^{N)} \right\} = 0. \tag{2.14}$$

Assuming $\sigma^{aAB'}$ is non-degenerate enables one to set to zero the factor in braces. Tracing of such a factor over R, M then yields

$$\epsilon_S{}^N \left(2T_{[pm}{}^m\ \delta_{q]}{}^g - T_{pq}{}^g \right) - 4Q_{[qS}{}^N\ \delta_{p]}{}^g + \frac{1}{2}\epsilon_{pqm}{}^g\ k^m\ \epsilon_S{}^N = 0. \tag{2.15}$$

Since Q is traceless, taking the traces over S, N and q, g, one finds

$$T_{am}{}^m = 0, \tag{2.16}$$

so that torsion takes the value

$$T_{pq}{}^g = \frac{1}{2}\epsilon_{mpq}{}^g\ k^m. \tag{2.17}$$

Moreover, on inserting this value of torsion in (2.15) one finds

$$Q_{aSN} = 0. \tag{2.18}$$

Hence, according to (2.9), $\mathcal{D}$ is the self-dual part of the connection ∇, and the corresponding torsion is (2.17) by virtue of (2.6) and (2.8).

Reproducing (2.1) from (2.2) is easy at this stage. Recall that the Bianchi symmetry of the curvature of a connection ∇ having torsion T (see e.g. [10])

$$R_{[abc]}{}^d - T_{[ab}{}^e\ T_{c]e}{}^d - \nabla_{[a}T_{bc]}{}^d = 0, \tag{2.19}$$

where antisymmetrization is understood on all three indices a, b, c, can be related to the self-dual Riemann tensor [1]

$$^{+}R_{abc}{}^{d} = \frac{1}{2}\left(\delta^{m}_{c}\delta^{d}_{n} - \frac{i}{2}\epsilon_{c}{}^{dm}{}_{n}\right) R_{abm}{}^{n} = R_{abA}{}^{B}\,\sigma_{c}{}^{AM'}\sigma^{d}{}_{BM'}. \tag{2.20}$$

Such a relation is as follows. The self-dual scalar curvature providing the total pure-gravity contribution to the self-dual action can be written as

$$^{+}R \equiv \delta_{d}{}^{b} g^{ac}\, {}^{+}R_{abc}{}^{d} = \frac{1}{2}R - \frac{i}{4}\epsilon^{abm}{}_{n}\, R_{abm}{}^{n}. \tag{2.21}$$

The second term of the last equality can be obtained by means of the Bianchi symmetry (2.19) and of the torsion (2.17) as

$$\epsilon_{d}{}^{abc}\, R_{abc}{}^{d} = \epsilon_{d}{}^{abc}\, \nabla_{[a}T_{bc]}{}^{d} = 3\nabla_{a}k^{a}, \tag{2.22}$$

since the term quadratic in torsion drops out in view of the form of the torsion (2.17). Finally, one obtains

$$^{+}R = \frac{1}{2}R + \frac{3i}{4}\nabla_{a}k^{a}. \tag{2.23}$$

Correspondingly, the terms containing derivatives of the fermionic fields in the self-dual action (2.2) can be re-written in terms of ∇ and k^{m} as follows:

$$\begin{aligned} -\sqrt{2}\,\sigma\,\sigma^{a}{}_{AA'}\left[\overline{\kappa}^{A'}\left(\mathcal{D}_{a}\kappa^{A}\right) - \left(\mathcal{D}_{a}\mu^{A}\right)\overline{\mu}^{A'}\right] &= -\frac{\sigma}{\sqrt{2}}\sigma^{a}{}_{AA'}\left[\overline{\kappa}^{A'}\left(\nabla_{a}\kappa^{A}\right) - \left(\nabla_{a}\mu^{A}\right)\overline{\mu}^{A'}\right] \\ &+ \frac{\sigma}{\sqrt{2}}\sigma^{a}{}_{AA'}\left[\left(\nabla_{a}\overline{\kappa}^{A'}\right)\kappa^{A} - \mu^{A}\left(\nabla_{a}\overline{\mu}^{A'}\right)\right] \\ &- \frac{i}{2}\sigma\nabla_{a}k^{a}. \end{aligned} \tag{2.24}$$

In the light of (2.23)-(2.24) we have shown that the ECSK-Dirac action (2.1) and the self-dual action (2.2) are equivalent modulo total divergences and the equation of motion for $\mathcal{D}$ (i.e. $\mathcal{D}$ is the self-dual part of ∇). Note that the mass terms in the actions differ by a factor of 2. Because of the non-vanishing torsion of ∇ these divergences give, apart from the boundary terms, volume terms involving the trace of the torsion. However, for the Einstein-Dirac system, torsion is traceless (see (2.16)) and hence we get, indeed, a complete dynamical equivalence. Explicitly,

$$S_{SD}\left[{}^{+}\omega(\sigma,T),\sigma,\kappa,\mu\right] = \frac{1}{2}S_{ECSKD}\left[\omega(\sigma,T),\sigma,\kappa,\mu\right] + \frac{i}{4}\int_{\partial M} dS^a k_a, \tag{2.25}$$

${}^{+}\omega(\sigma,T)$ being the self-dual part of the connection $\omega(\sigma,T)$; the arguments, soldering form and torsion, indicating their equations of motion (cf. (2.4), (2.8) and (2.18)), have been used. For real general relativity, it is then evident that, although S_{SD} is not real, its imaginary part is a boundary term. This is a non-trivial generalization to spin-$\frac{1}{2}$ fields coupled to gravity of the results obtained in [2] for pure gravity (cf. (1.3)).

3. - Field equations.

The form of the action (2.2) makes it easy to get the equations of motion for the remaining fields. By varying with respect to $\mu^A, \overline{\mu}^{A'}, \kappa^A, \overline{\kappa}^{A'}$ and using $\widetilde{\sigma}^a{}_{AA'} \equiv \sigma\ \sigma^a{}_{AA'}$, the equations of motion for the Dirac field are

$$\widetilde{\sigma}^a{}_{AA'}\mathcal{D}_a\kappa^A = \frac{im}{\sqrt{2}}\sigma\ \overline{\mu}_{A'} \qquad \mathcal{D}_a\left(\widetilde{\sigma}^a{}_{AA'}\overline{\kappa}^{A'}\right) = \frac{im}{\sqrt{2}}\sigma\ \mu_A, \tag{3.1a}$$

$$\mathcal{D}_a\left(\widetilde{\sigma}^a{}_{AA'}\overline{\mu}^{A'}\right) = \frac{im}{\sqrt{2}}\sigma\ \kappa_A \qquad \widetilde{\sigma}^a_{AA'}\mathcal{D}_a\mu^A = \frac{im}{\sqrt{2}}\sigma\ \overline{\kappa}_{A'}. \tag{3.1b}$$

Note that $\mathcal{D}$ does not act on primed indices (hence its compatibility with the soldering form is undefined) but one needs to know its action on space-time indices, whereas in the pure-gravity case it is independent of its extension to act on space-time indices because of the torsion-free condition [1]. Here, however, it develops a non-vanishing torsion. This problem is automatically solved by our request that the connection ∇ should coincide with $\mathcal{D}$ when acting on space-time indices and hence should have identical torsion (see (2.4), (2.8)).

To compare with the standard Dirac equations of motion we simply replace $\mathcal{D}$ with ∇. This leads to

$$\sigma^a{}_{AA'}\nabla_a\kappa^A = \frac{im}{\sqrt{2}}\overline{\mu}_{A'} \qquad \sigma^a{}_{AA'}\nabla_a\overline{\kappa}^{A'} = \frac{im}{\sqrt{2}}\mu_A, \tag{3.2a}$$

$$\sigma^a{}_{AA'}\nabla_a\overline{\mu}^{A'} = \frac{im}{\sqrt{2}}\kappa_A \qquad \sigma^a{}_{AA'}\nabla_a\mu^A = \frac{im}{\sqrt{2}}\overline{\kappa}_{A'}. \tag{3.2b}$$

With our conventions $\sigma^a{}_{AA'}$ are taken to be antihermitian. Although equations (3.2a)-(3.2b) resemble ordinary Dirac equations in curved space-time, one should bear in mind ∇ is *not* torsion-free. These are the equations for a Dirac field minimally coupled to gravity with torsion (see e.g. [9]). The rest of the field equations requires varying w.r.t. $\mathcal{D}$ and $\sigma^a{}_{AA'}$. As usual in the Palatini formalism, the former variation yields the value of the torsion (section 2) whereas the latter leads to the Einstein (-Cartan) equations with source the spin-$\frac{1}{2}$ field [7,9], i.e.

$$G_{ab} = \frac{1}{\sqrt{2}}\sigma_{bAA'}\left[\overline{\kappa}^{A'}\left(\nabla_a\kappa^A\right) - \left(\nabla_a\overline{\kappa}^{A'}\right)\kappa^A + \mu^A\left(\nabla_a\overline{\mu}^{A'}\right) - \left(\nabla_a\mu^A\right)\overline{\mu}^{A'}\right]. \tag{3.3}$$

It is now possible to make contact with the results of [5]. The authors of this reference found a cubic term in fermionic fields in their Dirac equation and stressed it has its origin in the kind of theory they started with, i.e. the torsion in the first-order theory (2.2) by analogy with the ECSKD action (2.1). This is explicitly shown below by splitting out the torsion contribution from the connection ∇ introduced above. Let $\widetilde{\nabla}$ be the unique torsion-free connection compatible with the metric $g_{ab} = \sigma_{aAA'}\sigma_b^{AA'}$. Hence, there exists a tensor $Q_{ab}{}^c$, and its spinor version Θ_{aBC}, $\overline{\Theta}_{aB'C'}$, relating $\widetilde{\nabla}$ and ∇ through [10]

$$\left(\widetilde{\nabla}_a - \nabla_a\right)v^b = Q_{ac}{}^b\, v^c, \tag{3.4}$$

$$\left(\widetilde{\nabla}_a - \nabla_a\right)\kappa^A = \Theta_a{}^A{}_B\, \kappa^B, \tag{3.5}$$

$$\left(\widetilde{\nabla}_a - \nabla_a\right)\lambda^{A'} = \overline{\Theta}_a{}^{A'}{}_{B'}\, \lambda^{B'}, \tag{3.6}$$

where the spinor decomposition

$$Q_{ab}{}^c = \left[\Theta_a{}^C{}_B\, \epsilon_{B'}^{C'} + \overline{\Theta}_a{}^{C'}{}_{B'}\, \epsilon_B^C\right]\, \sigma_b^{BB'}\, \sigma^c{}_{CC'}, \tag{3.7}$$

$$\Theta_{aBC} = \frac{1}{2}\sigma^b_{BB'}\, \sigma^c{}_C{}^{B'} Q_{abc}, \tag{3.8}$$

is implied. With our notation, Θ_{aBD} corresponds to C_{aBD} appearing in [5]. Furthermore

$$T_{ab}{}^{c} = 2Q_{[ab]}{}^{c}, \tag{3.9}$$

$$Q_{abc} = T_{a[bc]} - \frac{1}{2}T_{bca}, \tag{3.10}$$

the first of which states that $\widetilde{\nabla}$ is torsion-free and the second is a result of the metricity condition [10]. The torsion information is hence contained in Θ_{aBC}. In the case of Dirac fields one gets, inserting (2.17) into the above relations,

$$\Theta_{aBC} = \frac{i}{4}k_{(BA'}\ \sigma_{aC)}{}^{A'}. \tag{3.11}$$

Hereby the modification to the Dirac equation found in [5] is explicitly determined, its origin being the non-vanishing torsion; by virtue of (3.5) and (3.11), the first of the Dirac equations (3.2a) takes the form

$$\sigma^{a}{}_{AA'}\nabla_{a}\kappa^{A} = \sigma^{a}{}_{AA'}\left(\widetilde{\nabla}_{a} - \frac{3i}{8}k_{a}\right)\kappa^{A} \ = \frac{im}{\sqrt{2}}\overline{\mu}_{A'}. \tag{3.12}$$

This result extends to the primed-indices spinor equations (3.2) through $\overline{\Theta}_{aB'C'}$, and, similarly, to the Einstein equations (3.3). Such a modification was discussed previously in a U_4-theory [9]. The reduced action of [5] is thereby obtained. In particular, the four-Fermi interaction term, $\sigma k_m k^m$, is brought into the action; in other words, using the space-time-indices version of the identity (2.9) (cf. [10]) for the curvatures of ∇ and $\widetilde{\nabla}$, one gets the following relation between the corresponding scalars: $R = \widetilde{R} - \frac{3}{8}k_m k^m$.

4. - Concluding remarks.

Our analysis proves the equivalence between the self-dual and the ECSK forms of the action coupling Dirac fields to gravity, by introducing a connection with non-vanishing torsion. The key steps are the use of the Bianchi symmetry of the curvature of such a connection, here including torsion, and the result that the torsion for this system is totally antisymmetric. Thus, the actions differ by total divergences. They lead to boundary terms only, since the volume terms they involve are proportional to the trace of the torsion, and hence vanish. This can be considered the explicit version of an observation first made by Dolan [14]. He studied the canonical transformation, in pure gravity, from tetrad and connection variables to Ashtekar's variables. According to [14], the generating function, when torsion is present, has the same structure, whenever torsion is totally antisymmetric; this is the case for ED theory and supergravity. On the other hand, Jacobson [4] used another approach to prove the above equivalence of the actions. The boundary terms he finds, can thus be traced back to the Bianchi identity by using the present results (see the second term in (2.25)).

Moreover, we have shown explicitly that the extra term entering the Dirac equations obtained in [1,5] from the self-dual action is a torsion term. By splitting the self-dual connection into its torsion-free and torsion parts, the standard four-Fermi interaction in the action is obtained [4-5]. These results completely agree with [9], where the authors investigated the four-Fermi interaction using a certain anholonomic basis and the corresponding connection.

We are currently investigating the holomorphic version of the ECSKD theory, motivated by the complex space-time program of Penrose [15]. The corresponding theory appears to be more rich than the usual ECSKD theory studied in canonical gravity and in our paper, and it deserves further study to shed new light on complex general relativity and quantum gravity.

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