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Affine Lie Algebra Symmetry of Sine-Gordon Theory at Reflectionless Points

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The quantum affine symmetry of the sine-Gordon theory at $q^2 = 1$, which occurs at the reflectionless points, is studied. Conserved currents that correspond to the closure of simple root generators are considered, and shown to be local. We argue that they satisfy the $\widehat{sl(2)}$ algebra. Examples of these currents are explicitly constructed.

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1. Introduction

It is known that many massive integrable quantum field theories possess dynamical symmetries corresponding to q-deformations of affine Lie algebras, and that these symmetries place powerful constraints on the S-matrices. In [1], the conserved currents which correspond to the finite number of simple roots of the q-deformed affine Lie algebra were constructed using conformal perturbation theory, as developed by Zamolodchikov[2]. The q-deformed affine Lie algebra has an infinite number of generators, and a presentation of the complete set of algebraic relations was obtained by Drinfel'd[3]. It is therefore interesting to investigate the properties of the conserved currents for the infinite number of additional generators which arise upon closure of the simple-root generators. For generic q this is expected to be very complicated, since the currents for the simple roots have fractional Lorentz spin, and q^2 is a braiding phase. This implies that in general the braided commutators of the simple root generators do not close on charges which are integrals of local currents. Thus, for generic q, the currents for the higher Drinfel'd charges are expected to be highly non-local.

In this paper we study the sine-Gordon (SG) theory when $q^2 = 1$, which occurs at the so-called reflectionless points of the SG coupling β. At these points, the locality properties of the currents for the simple roots are such that the conserved currents for the non-simple roots are local. Our main results are the following. We conjecture the existence of an affine $\widehat{sl(2)}$ symmetry at all of the reflectionless points of the SG theory. This does not follow directly from the $q^2 = 1$ specialization of the results in [1], since, as we will see, the complete relations of the affine Lie algebra necessarily arise at higher order in conformal perturbation theory. The difficulty in carrying out explicitly higher order conformal perturbation theory is what limits us to only making a conjecture about the existence of the $\widehat{sl(2)}$ symmetry. We also construct explicitly some of the higher currents and show that they are products of the currents for the simple roots with local operators.

2. $\widehat{sl(2)}$ Symmetry at Reflectionless Points

We consider the SG theory defined by the Euclidean action

$$S = \frac{1}{4\pi} \int d^2 z \left(\partial_z \Phi \partial_{\overline{z}} \Phi + 4\lambda \cos(\widehat{\beta}\Phi) \right), \tag{2.1}$$

where $z, \overline{z}$ are the usual Euclidean light-cone coordinates, and $\widehat{\beta}$ is related to the conventional SG coupling β as $\widehat{\beta} = \beta/\sqrt{4\pi}$. The conserved $U(1)$ topological charge normalized to be ± 1 on the solitons is

$$T_0 = \frac{\widehat{\beta}}{2\pi} \int_{-\infty}^{\infty} dx \partial_x \Phi. \tag{2.2}$$

In [1], four additional charges corresponding to the simple root generators of the $q - \widehat{sl(2)}$ were constructed in conformal perturbation theory:

$$\begin{aligned} Q^{\pm}_{-1} &= \int dz \, J^{\pm}_{(-1)} + \int d\overline{z} \, \overline{J}^{\pm}_{(-1)} \\ Q^{\pm}_{1} &= \int dz \, J^{\pm}_{(1)} + \int d\overline{z} \, \overline{J}^{\pm}_{(1)} \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} J^{\pm}_{(-1)} &= \exp\left(\pm \frac{2i}{\widehat{\beta}} \varphi\right) \qquad & \overline{J}^{\pm}_{(-1)} &= \lambda \frac{\widehat{\beta}^2}{\widehat{\beta}^2 - 2} \exp\left[\pm i (\frac{2}{\widehat{\beta}} - \widehat{\beta}) \varphi \mp i \widehat{\beta} \overline{\varphi}\right] \\ \overline{J}^{\pm}_{(1)} &= \exp\left(\mp \frac{2i}{\widehat{\beta}} \overline{\varphi}\right) \qquad & J^{\pm}_{(1)} &= \lambda \frac{\widehat{\beta}^2}{\widehat{\beta}^2 - 2} \exp\left[\mp i (\frac{2}{\widehat{\beta}} - \widehat{\beta}) \overline{\varphi} \pm i \widehat{\beta} \varphi\right] \end{aligned} \tag{2.4}$$

and φ, $\overline{\varphi}$ are quasi-chiral components of Φ defined in conformal perturbation theory: when $\lambda = 0$, $\Phi = \varphi(z) + \overline{\varphi}(\overline{z})$. For generic $\widehat{\beta}$, these currents are exact to first order in λ. Together with the topological charge T_0, $Q^{\pm}_{1}$, $Q^{\pm}_{-1}$ satisfy the defining relations for the simple roots generators of the $q - \widehat{sl(2)}$ algebra, where $q = \exp(-2\pi i/\widehat{\beta}^2)$. In particular one has

$$Q^{\pm}_{-1} Q^{\mp}_{1} - q^{-2} Q^{\mp}_{1} Q^{\pm}_{-1} = \frac{\lambda}{2\pi i} \left(\frac{\widehat{\beta}^2}{2 - \widehat{\beta}^2}\right)^2 \left(1 - q^{\pm 2 T_0}\right). \tag{2.5}$$

The q deformation parameter arises as a braiding phase for the above currents. This is related to the fact that the charges have fractional Lorentz spin in general. Let L denote the generator of Euclidean rotations. Then,

$$\left[L, Q^{\pm}_{-1}\right] = \left(\frac{2}{\widehat{\beta}^2} - 1\right) Q^{\pm}_{-1}, \qquad \left[L, Q^{\pm}_{1}\right] = -\left(\frac{2}{\widehat{\beta}^2} - 1\right) Q^{\pm}_{1} \tag{2.6}$$

We now specialize to the reflectionless points:

$$\widehat{\beta}^2 = \frac{2}{N+1} \tag{2.7}$$

where N is a positive integer. The point $N = 1$ corresponds to the free fermion point. For each of these couplings $q^2 = 1$ and the Lorentz spin of the above charges is $\pm N$.

Since $q^2 = 1$, one is led to consider whether an undeformed affine $\widehat{sl(2)}$ symmetry exists in these models. The affine $\widehat{sl(2)}$ algebra in the principal gradation is generated by $\widetilde{Q}^\pm_n$, for n and odd integer, and $\widetilde{T}_n$ for n even, satisfying

$$\left[\widetilde{T}_n, \widetilde{T}_m\right] = 0, \qquad \left[\widetilde{T}_n, \widetilde{Q}^\pm_m\right] = \pm 2\ \widetilde{Q}^\pm_{n+m}, \qquad \left[\widetilde{Q}^+_n, \widetilde{Q}^-_m\right] = \widetilde{T}_{n+m}. \tag{2.8}$$

To understand how this structure can arise at the reflectionless points, consider first the conformal limit $\lambda = 0$, where the charges $Q^\pm_{-1}$ become left-moving and $Q^\pm_1$ become right-moving. When $q^2 = 1$, $Q^\pm_{-1}$ satisfy the undeformed Serre relations for $\widehat{sl(2)}$, and similarly for $Q^{\pm 1}_1$. The commutator algebra of $Q^\pm_{-1}$ closes on a set of left-moving charges $Q^\pm_{-n},\ T_{-n},\ n > 0$, and similarly $Q^\pm_1$ closes on right-moving charges $Q^\pm_n,\ T_n,\ n > 0$. Due to the well-behaved locality properties at the reflectionless points, one can in principal explicitly compute the currents for these higher charges. When $\lambda = 0$, since the left and right moving charges commute, one thus obtains two decoupled Borel subalgebras of $\widehat{sl(2)}$.

Consider now turning on the perturbation $\lambda \neq 0$. Since $Q^\pm_1,\ Q^\pm_{-1}$ are conserved to first order in perturbation theory, then so are the higher charges. Therefore one obtains

$$\begin{aligned} Q^\pm_n &= \int dz\ J^\pm_{(n)} + \int d\overline{z}\ \overline{J}^\pm_{(n)} \\ T_n &= \int dz\ J_{(n)} + \int d\overline{z}\ \overline{J}_{(n)} \end{aligned} \tag{2.9}$$

where when $\lambda = 0$, $\overline{J}^\pm_{n<0} = \overline{J}_{n<0} = 0$, and $J^\pm_{n>0} = J_{n>0} = 0$. When $\lambda \neq 0$, the two Borel subalgebras no longer commute, and the interesting question is whether they do indeed satisfy the $\widehat{sl(2)}$ relations. Consider for example the relation (2.5). For generic values of $\widehat{\beta}$, it was shown in [1] that the RHS of (2.5) is exact to first order in λ. When $q^2 = 1$, the RHS of (2.5) actually vanishes on states of integer topological charge.

A simple scaling argument shows that the complete $\widehat{sl(2)}$ relations must arise at higher order in perturbation theory. From the Lorentz spin of $Q^\pm_1,\ Q^\pm_{-1}$, one deduces that

$$\left[L, Q^\pm_n\right] = -nN\ Q^\pm_n, \qquad \left[L, T_n\right] = -nN\ T_n. \tag{2.10}$$

In the conformal limit, $L = L_0 - \overline{L}_0$ where L_0 and $\overline{L}_0$ are the Virasoro zero modes, and the scaling dimensions of operators are given by $L_0 + \overline{L}_0$. Since in the conformal limit

[1] The Serre relations were proven for arbitrary q in [4]

the negative (positive) frequency modes of $\widehat{sl(2)}$ are left (right) moving, one deduces the following scaling dimension of the charges:

$$\dim\left(Q_n^{\pm}\right) = \dim\left(T_n\right) = N|n|. \tag{2.11}$$

The parameter λ has scaling dimension $2-\widehat{\beta}^2 = 2N/(N+1)$, thus $\widetilde{Q}_n^{\pm} = (c\lambda)^{-|n|(N+1)/2} Q_n^{\pm}$ and $\widetilde{T}_n = (c\lambda)^{-|n|(N+1)/2} T_n$ are dimensionless operators, where c is a dimensionless constant. Taking the latter operators to satisfy the algebra (2.8), one obtains

$$\begin{aligned}
[T_n, T_m] &= 0 \\
\left[T_n, Q_m^{\pm}\right] &= \pm 2\ \ (c\lambda)^{(N+1)(|n|+|m|-|n+m|)/2}\ \ Q_{n+m}^{\pm} \\
\left[Q_n^{+}, Q_m^{-}\right] &= (c\lambda)^{(N+1)(|n|+|m|-|n+m|)/2}\ \ T_{n+m}.
\end{aligned} \tag{2.12}$$

The powers of λ on the RHS of (2.12) are always integers, and as λ goes to zero, one obtains two decoupled Borel subalgebras.

At the free fermion point $N = 1$, the conserved charges satisfying (2.12) were constructed explicitly[5]. In the next section we will provide evidence for the structure (2.12) by verifying the existence of some higher dimensional currents to first order in λ.

Though we are unable to verify explicitly the algebra (2.12) since it involves higher order conformal perturbation theory, we can at least verify that this structure is allowed in perturbation theory. Consider the commutator $[Q_{-1}^{+}, Q_1^{-}]$. In general one has the following perturbative expansion:

$$\left[Q_{-1}^{+}, Q_1^{-}\right] = \sum_n \lambda^n \left(\int dz \mathcal{O}_n + \int d\overline{z}\overline{\mathcal{O}}_n \right). \tag{2.13}$$

In conformal perturbation theory $\mathcal{O}_n$ must be a product of $\exp(2i\Phi/\widehat{\beta})$ with some power of the perturbation $\cos(\widehat{\beta}\Phi)$. Thus, $\mathcal{O}_n$ must be the operator $\exp((i(2/\widehat{\beta} + k\widehat{\beta})\Phi)$ or its derivatives for $k \in \mathcal{Z}$. If $\mathcal{O}_n$ involves m derivatives, then the dimension of $\mathcal{O}_n$ is $(2/\widehat{\beta} + k\widehat{\beta})^2 + m$. Since the dimension of $Q_{-1}^{+},\ Q_1^{-}$ is $2/\widehat{\beta}^2 - 1$ and that of λ is $2 - \widehat{\beta}^2$, simple dimensional analysis requires

$$2\left(\frac{2}{\widehat{\beta}^2} - 1\right) = n(2-\widehat{\beta}^2) + \left(\frac{2}{\widehat{\beta}} + k\widehat{\beta}\right)^2 + m - 1. \tag{2.14}$$

For generic irrational $\widehat{\beta}$, the above equation has no solution except for $n = 1$. However, at the reflectionless points, one has the additional solution $n = N+1, k = -(N+1), m = 1$. This corresponds to the relation

$$\left[Q_{-1}^{+}, Q_1^{-}\right] = (c\lambda)^{N+1}\ T_0, \tag{2.15}$$

where T_0 is the $U(1)$ charge.

3. Higher Conserved Currents

In this section we construct explicitly some of the higher conserved currents at the reflectionless points. One way of verifying the structure proposed in the last section is as follows. We focus on the charges Q^+_{-n} for $n > 0$. When $\lambda = 0$, the current has one chiral component $J^+_{(-n)}$. Given that J^+_{-n} must have $U(1)$ charge $+2$, one considers the following form for this current:

$$J^+_{(-n)} = \exp\left(\frac{2i}{\widehat{\beta}}\varphi\right)\ \mathcal{O}_{-n}(z) \tag{3.1}$$

where $\mathcal{O}_{-n}$ is a local operator depending on powers of φ and its derivatives. At the reflectionless points, since the spin of Q^+_{-n} is nN, and that of $\exp(2i\varphi/\widehat{\beta})$ is $N+1$, the spin of $\mathcal{O}_{-n}$ must be $N(n-1)$.

The specific computation we carried out is the following. Suppose one takes a current of the form $J = \exp(2i\varphi/\widehat{\beta})\mathcal{O}$ where $\mathcal{O}$ is the most general operator of a fixed dimension m. To first order in conformal perturbation theory, one has[2]

$$\partial_{\overline{z}} J = \lambda \oint_z \frac{dw}{2\pi i}\ \cos(\widehat{\beta}\Phi(w,\overline{z}))\ J(z). \tag{3.2}$$

Having fixed m, based on the hypothesis of the last section one expects that J is conserved only for $\widehat{\beta}^2 = 2/(N+1)$, where $m = N(n-1)$. Since n is odd one has the following possibilities for the lowest dimensions of $\mathcal{O}$:

$$\begin{aligned}
&\dim(\mathcal{O}) = 2: && (N=1, n=3)\\
&\dim(\mathcal{O}) = 4: && (N=1, n=5),\ (N=2, n=3)\\
&\dim(\mathcal{O}) = 6: && (N=1, n=7),\ (N=3, n=3).\\
&\dim(\mathcal{O}) = 8: && (N=1, n=9),\ (N=2, n=5),\ (N=4, n=3).\\
&\dim(\mathcal{O}) = 10: && (N=1, n=11),\ (N=5, n=3).\\
&\dim(\mathcal{O}) = 12: && (N=1, n=13),\ (N=2, n=7),\ (N=3, n=5),\ (N=6, n=3).
\end{aligned} \tag{3.3}$$

The cases above at $N=1$ correspond to the bosonic form of the lowest six higher charges at the free fermion point. The additional solutions at $N = 2,3,4,5,6$ correspond to the first higher charge at grade 3, and the next higher charges at grades 5,7.

Consider first the simplest example where $\mathcal{O}$ is of dimension 2. The most general such operator is

$$\mathcal{O}(z) = a\partial_z^2\phi + b(\partial_z\phi)^2. \tag{3.4}$$

Since the two terms above can be related by a total derivative of the form $\partial_z(\partial_z\phi\exp(2i\phi/\widehat{\beta}))$, it is sufficient to consider just one of them. Take $\mathcal{O}=\partial_z^2\phi$. According to (3.2), J is conserved if the $1/(z-w)$ term in the operator product expansion (OPE) $\cos(\widehat{\beta}\Phi(w,\overline{z}))J(z)$ is a total derivative. For $\mathcal{O}$ of dimension 2, only the $\exp(-i\widehat{\beta}\Phi)$ piece of $\cos\widehat{\beta}\Phi$ can give a pole, however as we will see for higher dimension $\mathcal{O}$ one must consider both pieces. The relevant OPE is

$$e^{-i\widehat{\beta}\phi(z)}J(w)\sim\frac{1}{z-w}\left(-\frac{\widehat{\beta}^2}{6}\phi'''+i(\widehat{\beta}^3/2-\widehat{\beta})\phi'\phi''+\frac{\widehat{\beta}^4}{6}(\phi')^3\right)e^{i(2/\widehat{\beta}-\widehat{\beta})\phi(w)}+\ldots \quad (3.5)$$

where $\phi'=\partial_w\phi$ etc. Above we have only written down the simple pole term since it is the only term that is relevant for our analysis. Integrating by parts, one finds that the operator coefficient of the simple pole is

$$\partial_w\left[\left(-\frac{\widehat{\beta}^2}{6}\phi''+\frac{i}{6}(\widehat{\beta}^3-2\widehat{\beta})(\phi')^2\right)e^{i(2/\widehat{\beta}-\widehat{\beta})\phi(w)}\right]+\frac{2}{3}(\widehat{\beta}^2-1)(\phi')^3e^{i(2/\widehat{\beta}-\widehat{\beta})\phi(w)}. \quad (3.6)$$

Thus, we only have a conserved current with $\mathcal{O}$ of dimension 2 if $\widehat{\beta}^2=1$, which is the case $(N=1,n=3)$ listed in (3.3).

Consider next the case when $\mathcal{O}$ is dimension 4. After taking into account total derivatives, the most general $\mathcal{O}$ is

$$\mathcal{O}=\partial_z^4\phi+A(\partial_z\phi)^4. \quad (3.7)$$

One needs the following OPE's:

$$e^{-i\widehat{\beta}\phi(z)}\partial_w^4\phi(w)e^{2i\phi(w)/\widehat{\beta}}\sim\frac{1}{z-w}[-\frac{\widehat{\beta}^2}{20}\phi'''''+i(\frac{\widehat{\beta}^3}{4}-\widehat{\beta})\phi'\phi''''+\frac{i}{2}\widehat{\beta}^3\phi''\phi'''+\frac{\widehat{\beta}^4}{2}(\phi')^2\phi'''$$
$$+\frac{3\widehat{\beta}^4}{4}\phi'(\phi'')^2-\frac{i}{2}\widehat{\beta}^5(\phi')^3\phi''-\frac{\widehat{\beta}^6}{20}(\phi')^5]e^{i(2/\widehat{\beta}-\widehat{\beta})\phi(w)}+.... \quad (3.8)$$

$$e^{-i\widehat{\beta}\phi(z)}(\partial_w\phi)^4e^{2i\phi(w)/\widehat{\beta}}\sim\frac{1}{z-w}[-\frac{i}{120}\widehat{\beta}^5\phi'''''+\left(\frac{\widehat{\beta}^4}{6}-\frac{\widehat{\beta}^6}{24}\right)\phi'\phi''''-\frac{\widehat{\beta}^6}{12}\phi'''\phi''$$
$$+i\left(\widehat{\beta}^3-\frac{2}{3}\widehat{\beta}^5+\frac{\widehat{\beta}^7}{12}\right)(\phi')^2\phi'''+i\left(-\frac{\widehat{\beta}^5}{2}+\frac{\widehat{\beta}^7}{8}\right)\phi'(\phi'')^2$$
$$+\left(-2\widehat{\beta}^2+3\widehat{\beta}^4-\widehat{\beta}^6+\frac{\widehat{\beta}^8}{12}\right)(\phi')^3\phi''+$$
$$i\left(-\widehat{\beta}+2\widehat{\beta}^3-\widehat{\beta}^5+\frac{1}{6}\widehat{\beta}^7-\frac{1}{120}\widehat{\beta}^9\right)(\phi')^5]e^{i(2/\widehat{\beta}-\widehat{\beta})\phi(w)}+.... \quad (3.9)$$

One finds that if

$$A = -i\frac{\widehat{\beta}^3(3\widehat{\beta}^2 - 10)}{2(18 - 11\widehat{\beta}^2)}, \tag{3.10}$$

then the current is conserved, so long as

$$\widehat{\beta}^2(\widehat{\beta}^2 - 1)(\widehat{\beta}^2 + 1)(\widehat{\beta}^2 - 3)(3\widehat{\beta}^2 - 2) = 0. \tag{3.11}$$

This yields three solutions for positive $\widehat{\beta}^2 = 2/3, 1, 3$. Since conformal perturbation theory is only meaningful when the perturbation is a relevant operator, which corresponds to $\widehat{\beta}^2 < 2$, we are left with the only two solutions listed in (3.3).

With increasing dimension of $\mathcal{O}$, the above computations rapidly become very complicated. With the aid of Mathematica, we have confirmed the predictions in (3.3) for dim$\mathcal{O} = 6$. Repeating the above procedure, one finds a conserved current provided $\widehat{\beta}^2$ is a solution of the equation:

$$-840 + 4978\widehat{\beta}^2 - 11397\widehat{\beta}^4 + 12862\widehat{\beta}^6 - 7590\widehat{\beta}^8 + 2302\widehat{\beta}^{10} - 333\widehat{\beta}^{12} + 18\widehat{\beta}^{14} = 0. \tag{3.12}$$

This polynomial can be factorized:

$$(\widehat{\beta}^2 - 7)(\widehat{\beta}^2 - 5)(\widehat{\beta}^2 - 3)(\widehat{\beta}^2 - 1)(2\widehat{\beta}^2 - 1)(3\widehat{\beta}^2 - 4)(3\widehat{\beta}^2 - 2) = 0. \tag{3.13}$$

After considering the OPE of the current with the other piece $\exp(i\widehat{\beta}\Phi)$ of the $\cos\widehat{\beta}\Phi$ perturbation, one finds that only $\widehat{\beta}^2 = 1/2, 1$ (in the allowed range of $\widehat{\beta}^2$) remain as solutions, which again are the two solutions listed in (3.3).

4. Concluding Remarks

We have provided evidence for an $\widehat{sl(2)}$ affine Lie algebra symmetry in the sine-Gordon theory at the reflectionless points. The main difficulty which remains toward proving this result is that many of the relations in (2.12) arise in higher order conformal perturbation theory.

Assuming this affine symmetry exists, the interesting question is to understand to what extent the symmetry characterizes the main dynamical properties, such as the form factors and correlation functions. For example, it would be very interesting to obtain the form factors from ordinary affine Lie algebraic vertex operators.

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