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The Electromagnetic Interaction in Chiral Perturbation Theory*

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Abstract

We investigate electromagnetic effects in the framework of chiral perturbation theory. Using a completely independent method, we confirm Urech's results for the divergences of the one–loop functional in the electromagnetic sector. We perform a one–loop analysis of all $P_{\ell 2}$ ($P = \pi, K, \eta$) and the $K_{\ell 3}$ form factors $f_+^{K^+\pi^0}(0)$, $f_+^{K^0\pi^-}(0)$, including a systematic treatment of the $\mathcal{O}(e^2p^2)$ contributions in the mesonic part. We illustrate our results by several numerical estimates.

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1 Introduction

In the standard model of strong and electroweak interactions, the violation of the isospin symmetry has two different origins. First of all, it can be traced back to the mass difference of up and down quark. Secondly, also electromagnetism induces isospin breaking effects.

In the confinement region of the standard model, the usual perturbative methods are not applicable. In order to obtain testable theoretical predictions also in this case one has to resort to a so–called low–energy effective theory. With an appropriately chosen effective Lagrangian, chiral perturbation theory [1, 2, 3] (which is just the effective field theory of the standard model at low energies) is mathematically equivalent [4] to the underlying fundamental theory. Therefore, chiral perturbation theory presents the natural framework for the discussion of isospin breaking effects in the low energy range.

Isospin violating contributions related to $m_u \neq m_d$ are well under control from the theoretical point of view. They are fully described by the effective field theory of the strong interactions. Up to the chiral order p^4, the associated low–energy constants [3] have been determined [5] with rather good accuracy.

In principle, it is also straightforward to establish the theoretical framework for the description of electromagnetic effects. First of all, the photon field has to be included as an additional dynamical degree of freedom. Then one has to construct the most general Lagrangian of the desired order e^2p^{2n} respecting all the symmetries of the standard model. To lowest electromagnetic order e^2p^0, only a single term appears [6]. But already at the next–to–leading order e^2p^2, there are 14 linear independent terms [7, 8] entering the effective Lagrangian . The associated coupling constants K_i absorb the divergences generated by one–loop graphs with a virtual photon or a vertex from the Lagrangian of $\mathcal{O}(e^2p^0)$. The divergent parts of the couplings K_i have been determined in Ref. [7]. However, the finite parts K_i^r of the electromagnetic low–energy constants are remaining as free parameters. At this point one encounters the main difference between the strong and the electromagnetic sector. In contrast to the low–energy constants of the strong interactions, only rough order of magnitude estimates for the K_i^r are presently available [7, 8].

With the methods sketched above, it is possible to obtain the formal expressions of the electromagnetic contributions to $\mathcal{O}(e^2p^2)$ for any mesonic observable. So far, only a small number of applications [7, 8] of this kind has been worked out. It is one of the purposes of the present paper to add some new results to this list.

In Sect. 2, we briefly review the construction of the electromagnetic effective Lagrangian. The one-loop renormalization in the electromagnetic sector is discussed in Sect. 3. There, we also give an alternative determination of eight linear combinations of the renormalization constants K_i^{div} which serves as an independent test of the general results obtained in Ref. [7]. In Sect. 4 we illustrate the power and the limits of simple order of magnitude estimates for the K_i^r in the mass spectrum of the pseudoscalar mesons. A complete list of the $P_{\ell 2}$ form factors including the (mesonic) electromagnetic contributions of $\mathcal{O}(e^2p^2)$ is presented in Sect. 5. The analogous expressions for the $K_{\ell 3}$ form factors $f_+^{K^+\pi^0}(0)$ and $f_+^{K^0\pi^-}(0)$ are given in Sect. 6. In both cases, our results are illustrated by numerical estimates discriminating the pure QCD contributions and the electromagneting ones for certain isospin violating quantities. Finally, our conclusions are summarized in Sect. 7.

2 The Effective Chiral Lagrangian of Electromagnetism

Chiral perturbation theory [1, 2, 3] permits a systematic low–energy expansion of the generating functional $Z[v,a,s,p]$ of QCD. This quantity is defined in terms of the vacuum–to–vacuum amplitude

$$e^{iZ[v,a,s,p]} = \langle 0 \text{ out}|0 \text{ in}\rangle_{v,a,s,p} \tag{2.1}$$

associated with the Lagrangian

$$\mathcal{L} = \mathcal{L}^0_{QCD} + \bar{q}\gamma^\mu(v_\mu + a_\mu\gamma_5)q - \bar{q}(s - ip\gamma_5)q. \tag{2.2}$$

$\mathcal{L}^0_{QCD}$ is the QCD Lagrangian with the masses of the three light quarks $q = (q_u, q_d, q_s)^T$ set to zero. v_μ, a_μ, s, p are external sources represented by hermitian 3×3 matrices in flavour space. The Green functions of the vector, axial–vector, scalar and pseudoscalar quark currents can then be obtained by evaluating the functional derivatives of $Z[v,a,s,p]$ at $v = a = p = 0$, $s = \mathcal{M}_{quark} = \text{ diag}(m_u, m_d, m_s)$.

The effective chiral Lagrangian of QCD consists of a string of terms

$$\mathcal{L}_{eff} = \mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_6 + \ldots \ , \tag{2.3}$$

organized in powers of momenta and meson masses, respectively. The lowest order term $\mathcal{L}_2$ is the nonlinear sigma model Lagrangian in the presence of external fields[1]:

$$\mathcal{L}_2 = \frac{F^2}{4}\langle u_\mu u^\mu + \chi_+\rangle. \tag{2.4}$$

The generating functional $Z[v,a,s,p]$ is given by the expansion of the effective meson field theory in the number of loops,

$$Z = Z_2 + Z_4 + Z_6 + \ldots \ . \tag{2.5}$$

The leading term coincides with the classical action associated with $\mathcal{L}_2$.

At next–to–leading order p^4, the generating functional consists of the following terms: one–loop graphs generated by the vertices of $\mathcal{L}_2$, tree graphs involving one vertex from $\mathcal{L}_4$ and finally a contribution to account for the chiral anomaly.

Also electromagnetic processes where only external photon fields A_μ are present can be treated within this framework. One simply performs the substitution

$$v_\mu = -eQA_\mu, \tag{2.6}$$

where

$$Q = \frac{1}{3}\text{ diag}(2,-1,-1) \tag{2.7}$$

is the electromagnetic charge matrix.

In those cases where virtual photons are involved, the above approach is, of course, not sufficient any more. Now the photon field has to be included as an additional dynamical degree

[1] Our notation is the same as in Refs. [6, 8].

of freedom. In order to construct the pertinent effective Lagrangian of electromagnetic order e^2, one introduces spurion fields $\mathcal{Q}_{L,R}(x)$ [8] transforming as

$$\begin{aligned} \mathcal{Q}_{L,R} &\stackrel{G}{\to} h(\pi)\mathcal{Q}_{L,R}h(\pi)^\dagger, \\ \mathcal{Q}_{L,R} &\stackrel{P}{\to} \mathcal{Q}_{R,L}, \end{aligned} \tag{2.8}$$

under the chiral group $G = SU(3)_L \times SU(3)_R$ and parity P, respectively. The nonlinear realization $h(\pi)$ of G [9] is defined by the action of the chiral group G on the coset space $C = SU(3)_L \times SU(3)_R/SU(3)_V$:

$$\begin{aligned} u(\pi) &\stackrel{G}{\to} g_R u(\pi)h(\pi)^\dagger = h(\pi)u(\pi)g_L^\dagger, \\ u(\pi) &\in C, \\ g_{L,R} &\in SU(3)_{L,R}. \end{aligned} \tag{2.9}$$

The Goldstone fields π_i $(i = 1, \ldots, 8)$ are coordinates of the coset space C. We use the parametrization

$$u = \exp(i\Phi/\sqrt{2}\,F),$$

$$\Phi = \Phi^\dagger = \begin{pmatrix} \frac{\pi_3}{\sqrt{2}} + \frac{\pi_8}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi_3}{\sqrt{2}} + \frac{\pi_8}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\frac{2\pi_8}{\sqrt{6}} \end{pmatrix}. \tag{2.10}$$

Alternatively, one can also define [6] spurions $Q_{L,R}$ with the transformation properties

$$Q_L \stackrel{G}{\to} g_L Q_L g_L^\dagger, \quad Q_R \stackrel{G}{\to} g_R Q_R g_R^\dagger. \tag{2.11}$$

The $\mathcal{Q}_{L,R}$ are related to $Q_{L,R}$ by

$$\begin{aligned} \mathcal{Q}_L &= u Q_L u^\dagger, \\ \mathcal{Q}_R &= u^\dagger Q_R u. \end{aligned} \tag{2.12}$$

At the end $Q_{L,R}$ will be identified with the charge matrix Q.

To lowest order e^2p^0, the electromagnetic effective Lagrangian contains a single term [6]

$$\mathcal{L}|_{\mathcal{O}(e^2p^0)} = F^4 e^2 Z \,\langle \mathcal{Q}_L \mathcal{Q}_R \rangle, \tag{2.13}$$

with a real and dimensionless coupling constant Z. The effective Lagrangians (2.4) and (2.13) generate the lowest–order contributions to the masses of the pseudoscalar mesons from QCD and the electromagnetic interaction, respectively:

$$\begin{aligned} \widehat{M}^2_{\pi^\pm} &= 2B\widehat{m} + 2e^2ZF^2, \\ \widehat{M}^2_{\pi^0} &= 2B\widehat{m}, \\ \widehat{M}^2_{K^\pm} &= B\left[(m_s + \widehat{m}) - \frac{2\varepsilon}{\sqrt{3}}(m_s - \widehat{m})\right] + 2e^2ZF^2, \\ \widehat{M}^2_{\stackrel{(-)}{K}{}^0} &= B\left[(m_s + \widehat{m}) + \frac{2\varepsilon}{\sqrt{3}}(m_s - \widehat{m})\right], \\ \widehat{M}^2_\eta &= \frac{4}{3}B\left(m_s + \frac{\widehat{m}}{2}\right), \end{aligned} \tag{2.14}$$

where $\widehat{m}$ denotes the mean value of the light quark masses,

$$\widehat{m} = \frac{1}{2}(m_u + m_d), \tag{2.15}$$

and B is the vacuum condensate parameter contained in χ_+. The mixing angle

$$\varepsilon = \frac{\sqrt{3}}{4}\,\frac{m_d - m_u}{m_s - \widehat{m}} \tag{2.16}$$

relates π_3, π_8 to the (tree–level) mass eigenfields $\widehat{\pi}_0$, $\widehat{\eta}$:

$$\begin{aligned} \pi_3 &= \widehat{\pi}^0 - \varepsilon\widehat{\eta}, \\ \pi_8 &= \varepsilon\widehat{\pi}^0 + \widehat{\eta}. \end{aligned} \tag{2.17}$$

Terms of higher than linear order in ε have been neglected. In accordance with Dashen's theorem [10], the lowest order electromagnetic Lagrangian (2.13) contributes an equal amount to the squared masses of $\pi^\pm$, $K^\pm$. It does not contribute to the masses of π^0, K^0, $\bar{K}^0$ or η, nor does it generate π^0–η mixing. The relation

$$M^2_{\pi^\pm} - M^2_{\pi^0} = 2e^2ZF^2 + \mathcal{O}(e^2p^2), \tag{2.18}$$

resulting from (2.14) implies $Z \simeq 0.8$ as numerical value.

At next–to–leading order e^2p^2 one finds the following list of local counterterms [7]:

$$\begin{aligned} \mathcal{L}|_{\mathcal{O}(e^2p^2)} &= F^2e^2\{\frac{1}{2}K_1\,\langle\mathcal{Q}_L^2 + \mathcal{Q}_R^2\rangle\,\langle u_\mu u^\mu\rangle \\ &\quad + K_2\,\langle\mathcal{Q}_L\mathcal{Q}_R\rangle\,\langle u_\mu u^\mu\rangle \\ &\quad - K_3\,[\langle\mathcal{Q}_L u_\mu\rangle\,\langle\mathcal{Q}_L u^\mu\rangle + \langle\mathcal{Q}_R u_\mu\rangle\,\langle\mathcal{Q}_R u^\mu\rangle] \\ &\quad + K_4\,\langle\mathcal{Q}_L u_\mu\rangle\,\langle\mathcal{Q}_R u^\mu\rangle \\ &\quad + K_5\,\langle(\mathcal{Q}_L^2 + \mathcal{Q}_R^2)u_\mu u^\mu\rangle \\ &\quad + K_6\,\langle(\mathcal{Q}_L\mathcal{Q}_R + \mathcal{Q}_R\mathcal{Q}_L)u_\mu u^\mu\rangle \\ &\quad + \frac{1}{2}K_7\,\langle\mathcal{Q}_L^2 + \mathcal{Q}_R^2\rangle\,\langle\chi_+\rangle \\ &\quad + K_8\,\langle\mathcal{Q}_L\mathcal{Q}_R\rangle\,\langle\chi_+\rangle \\ &\quad + K_9\,\langle(\mathcal{Q}_L^2 + \mathcal{Q}_R^2)\chi_+\rangle \\ &\quad + K_{10}\,\langle(\mathcal{Q}_L\mathcal{Q}_R + \mathcal{Q}_R\mathcal{Q}_L)\chi_+\rangle \\ &\quad - K_{11}\,\langle(\mathcal{Q}_L\mathcal{Q}_R - \mathcal{Q}_R\mathcal{Q}_L)\chi_-\rangle \\ &\quad - iK_{12}\,\langle(\widehat{\nabla}_\mu\mathcal{Q}_L\mathcal{Q}_L - \mathcal{Q}_L\widehat{\nabla}_\mu\mathcal{Q}_L - \widehat{\nabla}_\mu\mathcal{Q}_R\mathcal{Q}_R + \mathcal{Q}_R\widehat{\nabla}_\mu\mathcal{Q}_R)u^\mu\rangle \\ &\quad + K_{13}\,\langle\widehat{\nabla}_\mu\mathcal{Q}_L\widehat{\nabla}^\mu\mathcal{Q}_R\rangle \\ &\quad + K_{14}\,\langle\widehat{\nabla}_\mu\mathcal{Q}_L\widehat{\nabla}^\mu\mathcal{Q}_L + \widehat{\nabla}_\mu\mathcal{Q}_R\widehat{\nabla}^\mu\mathcal{Q}_R\rangle\}, \end{aligned} \tag{2.19}$$

where

$$\begin{aligned} \widehat{\nabla}_\mu\mathcal{Q}_L &= \nabla_\mu\mathcal{Q}_L + \frac{i}{2}[u_\mu, \mathcal{Q}_L] = uD_\mu Q_L u^\dagger, \\ \widehat{\nabla}_\mu\mathcal{Q}_R &= \nabla_\mu\mathcal{Q}_R - \frac{i}{2}[u_\mu, \mathcal{Q}_R] = u^\dagger D_\mu Q_R u. \end{aligned} \tag{2.20}$$

In order to obtain a linear independent set of terms in (2.19), the Cayley–Hamilton theorem,

$$P_A(A) \equiv 0, \tag{2.21}$$

has been used. The polynomial function P_A is defined by $P_A(\lambda) = \det(A - \lambda 1)$. Explicitly, the identity (2.21) reads:

$$-A^3 + \langle A\rangle A^2 + \frac{1}{2}(\langle A^2\rangle - \langle A\rangle^2)A + \frac{1}{3}[\langle A^3\rangle - \frac{3}{2}\langle A^2\rangle\,\langle A\rangle + \frac{1}{2}\langle A\rangle^3] = 0. \tag{2.22}$$

Replacing A by $A \pm B$ in (2.22) yields identities which can then be used to derive the relations

$$\langle \mathcal{Q}_I u_\mu \mathcal{Q}_I u^\mu\rangle = \frac{1}{2}\langle \mathcal{Q}_I^2\rangle\,\langle u_\mu u^\mu\rangle - 2\langle \mathcal{Q}_I^2 u_\mu u^\mu\rangle + \langle \mathcal{Q}_I u_\mu\rangle\,\langle \mathcal{Q}_I u^\mu\rangle, \qquad I = L, R, \tag{2.23}$$

and

$$\langle \mathcal{Q}_L u_\mu \mathcal{Q}_R u^\mu\rangle = \frac{1}{2}\langle \mathcal{Q}_L\mathcal{Q}_R\rangle\,\langle u_\mu u^\mu\rangle - \langle (\mathcal{Q}_L\mathcal{Q}_R + \mathcal{Q}_R\mathcal{Q}_L)u_\mu u^\mu\rangle + \langle \mathcal{Q}_L u_\mu\rangle\,\langle \mathcal{Q}_R u^\mu\rangle. \tag{2.24}$$

Furthermore, the term $i\langle \mathcal{Q}_L^2 - \mathcal{Q}_R^2\rangle\langle\chi_-\rangle$ vanishes once $Q_L = Q_R = Q$ is inserted. The expression $i\langle(\mathcal{Q}_L^2 - \mathcal{Q}_R^2)\chi_-\rangle$ does not contribute because $[\mathcal{M}, Q] = 0$, and $i\langle \mathcal{Q}_L\mathcal{Q}_R\rangle\langle\chi_-\rangle$ is forbidden by P invariance. Finally, partial integration and the equation of motion allows to relate

$$i\langle(\widehat{\nabla}_\mu \mathcal{Q}_L\mathcal{Q}_R - \mathcal{Q}_R\widehat{\nabla}_\mu\mathcal{Q}_L - \widehat{\nabla}_\mu\mathcal{Q}_R\mathcal{Q}_L + \mathcal{Q}_L\widehat{\nabla}_\mu\mathcal{Q}_R)u^\mu\rangle \tag{2.25}$$

to

$$\langle \mathcal{Q}_L\mathcal{Q}_R\rangle\,\langle u_\mu u^\mu\rangle - 3\langle(\mathcal{Q}_L\mathcal{Q}_R + \mathcal{Q}_R\mathcal{Q}_L)u_\mu u^\mu\rangle + 2\langle \mathcal{Q}_L u_\mu\rangle\,\langle \mathcal{Q}_R u^\mu\rangle + \frac{1}{2}\langle(\mathcal{Q}_L\mathcal{Q}_R - \mathcal{Q}_R\mathcal{Q}_L)\chi_-\rangle + \mathcal{O}(e^2). \tag{2.26}$$

3 One–Loop Renormalization in the Electromagnetic Sector

One–loop graphs with a virtual photon or one vertex from (2.13) are, in general, divergent. These divergences associated with polynomial expressions of order e^2p^2 are absorbed by an appropriate renormalization of the coupling constants in (2.19). To this end, the K_i are decomposed in two parts:

$$K_i = K_i^r(\mu) + \Sigma_i\Lambda(\mu). \tag{3.1}$$

The divergence is contained in the function $\Lambda(\mu)$. In dimensional regularization, this scale dependent term is given by

$$\Lambda(\mu) = \frac{\mu^{d-4}}{(4\pi)^2}\left\{\frac{1}{d-4} - \frac{1}{2}[\ln(4\pi) + \Gamma'(1) + 1]\right\}. \tag{3.2}$$

The renormalized electromagnetic low–energy constants $K_i^r(\mu)$ are, in principle, measurable quantities. The constants Σ_i govern the scale dependence of the $K_i^r(\mu)$,

$$K_i^r(\mu_2) = K_i^r(\mu_1) + \frac{\Sigma_i}{(4\pi)^2}\ln(\frac{\mu_1}{\mu_2}), \tag{3.3}$$

and they also determine the so–called "chiral logs". In any physical amplitude, the scale dependence always cancels between the loop and the counterterm contributions containing the renormalized coupling constants.

A complete list of the renormalization constants Σ_i has been worked out in Ref. [7] by evaluating the divergent part of the generating functional. We have performed an independent check [8] of the values given there by evaluating the (potentially) divergent parts of several observables. The requirement that the divergences associated with these quantities should vanish produces a certain number of conditions to be fulfilled by the Σ_i. We have restricted our analysis to the masses of the pseudoscalar mesons, the axial–vector decay constants F_P and the $P_{\ell 3}$ form factors. In this case, only the following linear combinations of the electromagnetic coupling constants appear:

$$\begin{aligned} &S_1 = K_1 + K_2, && S_2 = K_5 + K_6, && S_3 = -2K_3 + K_4, \\ &S_4 = K_7 + K_8, && S_5 = K_9 + 2K_{10} + K_{11}, && S_6 = K_8, \\ &S_7 = K_{10} + K_{11}, && S_8 = -K_{12}. \end{aligned} \tag{3.4}$$

In analogy to (3.1), the associated renormalization constants Δ_i are defined by

$$S_i = S_i^r(\mu) + \Delta_i \, \Lambda(\mu). \tag{3.5}$$

The finiteness of the electromagnetic contributions to the meson masses implies the relations

$$\begin{aligned} \Delta_3 &= -\frac{2}{3}\,\Delta_2 + 3Z, \\ \Delta_4 &= \Delta_1 + \frac{1}{3}\,\Delta_2 - \frac{1}{2}\,Z, \\ \Delta_5 &= \frac{1}{6}\,\Delta_2 + \frac{3}{4} + \frac{11}{4}\,Z, \\ \Delta_6 &= Z, \\ \Delta_7 &= \frac{1}{6}\,\Delta_2 + \frac{3}{4} + \frac{5}{4}\,Z. \end{aligned} \tag{3.6}$$

The analogous procedure for F_{K^0} yields the relation

$$6\Delta_1 + 2\Delta_2 - 9Z = 0. \tag{3.7}$$

Combined with the expression for Δ_3, (3.7) also renormalizes the electromagnetic contributions to F_{π^0} and F_η. The requirement that $F_{\pi^\pm}$ (or $F_{K^\pm}$) should be finite implies the relation

$$12\Delta_1 + 10\Delta_2 - 18\Delta_8 + 9 - 27Z = 0. \tag{3.8}$$

Finally, an inspection of the divergent terms in the $K_{\ell 3}$ form factor $f_+^{K^0\pi^-}(0)$ gives

$$\Delta_8 = -\frac{1}{4}. \tag{3.9}$$

This provides us with the necessary number of equations for the determination of the eight renormalization constants $\Delta_1, \ldots, \Delta_8$:

$$\Delta_1 = \Sigma_1 + \Sigma_2 = \frac{3}{4} + Z,$$

$$
\begin{aligned}
\Delta_2 &= \Sigma_5 + \Sigma_6 = -\frac{9}{4} + \frac{3}{2}Z, \\
\Delta_3 &= -2\Sigma_3 + \Sigma_4 = \frac{3}{2} + 2Z, \\
\Delta_4 &= \Sigma_7 + \Sigma_8 = Z, \\
\Delta_5 &= \Sigma_9 + 2\Sigma_{10} + \Sigma_{11} = \frac{3}{8} + 3Z, \\
\Delta_6 &= \Sigma_8 = Z, \\
\Delta_7 &= \Sigma_{10} + \Sigma_{11} = \frac{3}{8} + \frac{3}{2}Z, \\
\Delta_8 &= -\Sigma_{12} = -\frac{1}{4}.
\end{aligned} \tag{3.10}
$$

We have also checked the values given in (3.10) by applying them to the $P_{\ell 3}$ form factors $f_+^{K^+\pi^0}$, $f_-^{K^+\pi^0}$, $f_-^{K^0\pi^-}$ and $f_\pm^{\eta\pi}$.

4 Applications of the Electromagnetic Lagrangian

With the methods described in the previous sections, the electromagnetic contributions of order e^2p^2 to any mesonic observable can be calculated. So far, only a few results of this kind have been worked out completely. In Ref. [7], the diagonal elements of the pseudoscalar mass matrix have been calculated to $\mathcal{O}(e^2p^2)$. The remaining off–diagonal term related to $\pi^0 - \eta$ mixing can be found in Ref. [8]. In the same paper, also the $\mathcal{O}(e^2p^2)$ contributions to the ratio of $K_{\ell 3}$ form factors $f_+^{K^+\pi^0}(0)/f_+^{K^0\pi^-}(0)$ and to the $\eta_{\ell 3}$ form factors $f_\pm^{\eta\pi}(t)$ have been given.

For a complete numerical analysis of these results, some information about the electromagnetic low–energy constants $S_i^r(\mu)$ is needed. Unfortunately, our present knowledge of these parameters is restricted to crude order of magnitude estimates. This is in sharp contrast to the $\mathcal{O}(p^4)$ coupling constants $L_i^r(\mu)$ associated with the effective Lagrangian of pure QCD which have been determined [5] rather accurately by using experimental input and large N_c arguments [3]. But even with our limited knowledge about the couplings of the $\mathcal{O}(e^2p^2)$ Lagrangian, non–trivial results about the possible size of the electromagnetic contributions can be obtained.

This can be seen, for instance, in the mass spectrum of the pseudoscalars: The "magic" combination [3] of kaon and pion masses

$$
(M_{K^0}^2 - M_{K^\pm}^2 + M_{\pi^\pm}^2 - M_{\pi^0}^2) \cdot \frac{M_\pi^2}{(M_K^2 - M_\pi^2)M_K^2}, \tag{4.1}
$$

can be expressed through the masses of the three light quarks and an electromagnetic term of $\mathcal{O}(e^2p^2)$:

$$
\begin{aligned}
\frac{m_d^2 - m_u^2}{m_s^2 - \widehat{m}^2} &= [(M_{K^0}^2 - M_{K^\pm}^2 + M_{\pi^\pm}^2 - M_{\pi^0}^2)_{exp} - (M_{K^0}^2 - M_{K^\pm}^2 + M_{\pi^\pm}^2 - M_{\pi^0}^2)_{EM}] \\
&\quad \cdot \frac{M_\pi^2}{(M_K^2 - M_\pi^2)M_K^2}.
\end{aligned} \tag{4.2}
$$

The purely electromagnetic quantity [7, 8]

$$
(M_{K^0}^2 - M_{K^\pm}^2 + M_{\pi^\pm}^2 - M_{\pi^0}^2)_{EM} = e^2 M_K^2 \left[\frac{1}{(4\pi)^2} \left(3 \ln \frac{M_K^2}{\mu^2} - 4 + 2Z \ln \frac{M_K^2}{\mu^2} \right) \right.
$$

$$+\ \frac{4}{3}S_2^r(\mu) - 8S_7^r(\mu) + 16ZL_5^r(\mu)\Big] + \mathcal{O}(e^2M_\pi^2) \tag{4.3}$$

gives the deviation from Dashen's limit [10]. The unknown combination of low–energy constants $S_2^r(\mu) - 6S_7^r(\mu)$ determines the size of this deviation. Chiral dimensional analysis [1, 11] suggests the upper bound

$$|S_i^r(M_\rho)| \lesssim \frac{1}{(4\pi)^2} = 6.3 \cdot 10^{-3} \tag{4.4}$$

for the coupling constants of the effective Lagrangian. The resulting bounds

$$-\frac{7}{(4\pi)^2} \leq S_2^r(M_\rho) - 6S_7^r(M_\rho) \leq \frac{7}{(4\pi)^2}. \tag{4.5}$$

imply the range

$$1.5 \cdot 10^{-3} \lesssim 1/Q^2 := \frac{m_d^2 - m_u^2}{m_s^2 - \widehat{m}^2} \lesssim 2.4 \cdot 10^{-3}. \tag{4.6}$$

for the combination of quark masses occuring in (4.2). This estimate has to be compared with the value for $1/Q^2$ in Dashen's limit (corresponding to a vanishing electromagnetic contribution (4.3)):

$$1/Q^2|_{Dashen} = 1.72 \cdot 10^{-3}. \tag{4.7}$$

Values for $1/Q^2$ rather close to the upper bound of (4.6) have been obtained by certain model calculations [12, 13] which might also be supported by the present experimental data on $\eta \to 3\pi$ decays.

The size of the parameter Q constitutes an important ingredient for the determination of m_u/m_d and m_s/m_d [14]. The potentially large deviation of Q from its value in the Dashen limit led to some doubts [15] about the validity of the standard results [14] for these quark mass ratios. However, taking into account also the additional constraints from the mass splitting of the baryons [16, 17] and from an analysis of $\eta - \eta'$ mixing [3], the possible effects [8, 18] on the determination of m_u/m_d and m_s/m_d are not too dramatic.

5 $P_{\ell 2}$ Form Factors

In this section we investigate the contributions of order e^2p^2 to the $P_{\ell 2}$ form factors $F_\alpha(X)$. These quantities are defined by the hadronic matrix elements

$$\langle 0|\bar{q}(0)\gamma^\mu\gamma_5 X^\dagger q(0)|\alpha, p\rangle = i\sqrt{2}p^\mu F_\alpha(X), \tag{5.1}$$

where we have used a covariant normalization of one–particle states,

$$\langle p'|p\rangle = (2\pi)^3\, 2p^0\, \delta^{(3)}(\vec{p}' - \vec{p}). \tag{5.2}$$

The index α denotes a pseudoscalar mass eigenstate and the 3×3 matrix X picks out the desired component of the axial vector current. For the form factors associated with the non–vanishing

matrix elements we find the following expressions[2]:

$$
\begin{aligned}
F_{\pi^\pm} \;:=\; & F_{\pi^+}\left(\frac{\lambda_1+i\lambda_2}{2}\right) = F_{\pi^-}\left(\frac{\lambda_1-i\lambda_2}{2}\right) \\
=\; & F\left\{1+\frac{4}{F^2}[L_4^r(\mu)(M_\pi^2+2M_K^2)+L_5^r(\mu)M_\pi^2]\right. \\
& -\frac{1}{4(4\pi)^2F^2}\left[2M_{\pi^\pm}^2\ln\frac{M_{\pi^\pm}^2}{\mu^2}+2M_{\pi^0}^2\ln\frac{M_{\pi^0}^2}{\mu^2}+M_{K^\pm}^2\ln\frac{M_{K^\pm}^2}{\mu^2}+M_{K^0}^2\ln\frac{M_{K^0}^2}{\mu^2}\right] \\
& \left.+\frac{2e^2}{9}[6S_1^r(\mu)+5S_2^r(\mu)-9S_8^r(\mu)]+\frac{e^2}{2(4\pi)^2}\left[3\ln\frac{M_\pi^2}{\mu^2}-6-2\ln\frac{m_\gamma^2}{\mu^2}\right]\right\}, \qquad (5.3)
\end{aligned}
$$

$$
\begin{aligned}
F_{K^\pm} \;:=\; & F_{K^+}\left(\frac{\lambda_4+i\lambda_5}{2}\right) = F_{K^-}\left(\frac{\lambda_4-i\lambda_5}{2}\right) \\
=\; & F\left\{1+\frac{4}{F^2}[L_4^r(\mu)(M_\pi^2+2M_K^2)+L_5^r(\mu)M_K^2]\right. \\
& -\frac{1}{8(4\pi)^2F^2}\left[2M_{\pi^\pm}^2\ln\frac{M_{\pi^\pm}^2}{\mu^2}+M_{\pi^0}^2\ln\frac{M_{\pi^0}^2}{\mu^2}\right. \\
& \left.+4M_{K^\pm}^2\ln\frac{M_{K^\pm}^2}{\mu^2}+2M_{K^0}^2\ln\frac{M_{K^0}^2}{\mu^2}+3M_\eta^2\ln\frac{M_\eta^2}{\mu^2}\right] \\
& +\frac{8\sqrt{3}\,\varepsilon}{3F^2}\,L_5^r(\mu)(M_\pi^2-M_K^2)-\frac{\sqrt{3}\,\varepsilon}{4(4\pi)^2F^2}\left[M_\pi^2\ln\frac{M_\pi^2}{\mu^2}-M_\eta^2\ln\frac{M_\eta^2}{\mu^2}\right] \\
& \left.+\frac{2e^2}{9}[6S_1^r(\mu)+5S_2^r(\mu)-9S_8^r(\mu)]+\frac{e^2}{2(4\pi)^2}\left[3\ln\frac{M_K^2}{\mu^2}-6-2\ln\frac{m_\gamma^2}{\mu^2}\right]\right\}, \qquad (5.4)
\end{aligned}
$$

$$
\begin{aligned}
F_{K^0} \;:=\; & F_{K^0}\left(\frac{\lambda_6+i\lambda_7}{2}\right) = F_{\bar{K}^0}\left(\frac{\lambda_6-i\lambda_7}{2}\right) \\
=\; & F\left\{1+\frac{4}{F^2}[L_4^r(\mu)(M_\pi^2+2M_K^2)+L_5^r(\mu)M_K^2]\right. \\
& -\frac{1}{8(4\pi)^2F^2}\left[2M_{\pi^\pm}^2\ln\frac{M_{\pi^\pm}^2}{\mu^2}+M_{\pi^0}^2\ln\frac{M_{\pi^0}^2}{\mu^2}\right. \\
& \left.+2M_{K^\pm}^2\ln\frac{M_{K^\pm}^2}{\mu^2}+4M_{K^0}^2\ln\frac{M_{K^0}^2}{\mu^2}+3M_\eta^2\ln\frac{M_\eta^2}{\mu^2}\right] \\
& -\frac{8\sqrt{3}\,\varepsilon}{3F^2}\,L_5^r(\mu)(M_\pi^2-M_K^2)+\frac{\sqrt{3}\,\varepsilon}{4(4\pi)^2F^2}\left[M_\pi^2\ln\frac{M_\pi^2}{\mu^2}-M_\eta^2\ln\frac{M_\eta^2}{\mu^2}\right] \\
& \left.+\frac{4e^2}{9}[3S_1^r(\mu)+S_2^r(\mu)]\right\}, \qquad (5.5)
\end{aligned}
$$

$$
F_{\pi^0}\left(\frac{\lambda_3}{\sqrt{2}}\right) \;=\; F\left\{1+\frac{4}{F^2}[L_4^r(\mu)(M_\pi^2+2M_K^2)+L_5^r(\mu)M_\pi^2]\right.
$$

[2]See also Ref. [3] for the results in the limit $e=0$.

$$- \frac{1}{4(4\pi)^2F^2}\left[4M_{\pi^\pm}^2 \ln\frac{M_{\pi^\pm}^2}{\mu^2} + M_{K^\pm}^2 \ln\frac{M_{K^\pm}^2}{\mu^2} + M_{K^0}^2 \ln\frac{M_{K^0}^2}{\mu^2}\right]$$
$$+ \frac{e^2}{9}[12S_1^r(\mu) + 10S_2^r(\mu) + 9S_3^r(\mu)]\Bigg\}, \qquad (5.6)$$

$$F_{\pi^0}\left(\frac{\lambda_8}{\sqrt{2}}\right) = F\Bigg\{\varepsilon - \frac{M_{\hat\pi^0\hat\eta}^2}{M_\eta^2 - M_\pi^2} + \frac{4\varepsilon}{F^2}[L_4^r(\mu)(M_\pi^2 + 2M_K^2) + L_5^r(\mu)M_\pi^2]$$
$$- \frac{\varepsilon}{2(4\pi)^2F^2}\left[2(M_\pi^2 - M_K^2) + (2M_\pi^2 + M_K^2)\ln\frac{M_K^2}{\mu^2}\right]$$
$$+ \frac{\sqrt{3}\,e^2}{9}[2S_2^r(\mu) + 3S_3^r(\mu)] - \frac{\sqrt{3}\,e^2}{2(4\pi)^2}Z\left[1 + \ln\frac{M_K^2}{\mu^2}\right]\Bigg\}, \qquad (5.7)$$

$$F_\eta\left(\frac{\lambda_3}{\sqrt{2}}\right) = F\Bigg\{-\varepsilon + \frac{M_{\hat\pi^0\hat\eta}^2}{M_\eta^2 - M_\pi^2} - \frac{4\varepsilon}{F^2}[L_4^r(\mu)(M_\pi^2 + 2M_K^2) + L_5^r(\mu)M_\eta^2]$$
$$- \frac{\varepsilon}{2(4\pi)^2F^2}\left[2(M_\pi^2 - M_K^2) - 2M_\pi^2\ln\frac{M_\pi^2}{\mu^2} + (2M_\pi^2 - 3M_K^2)\ln\frac{M_K^2}{\mu^2}\right]$$
$$+ \frac{\sqrt{3}\,e^2}{9}[2S_2^r(\mu) + 3S_3^r(\mu)] - \frac{\sqrt{3}\,e^2}{2(4\pi)^2}Z\left[1 + \ln\frac{M_K^2}{\mu^2}\right]\Bigg\}, \qquad (5.8)$$

$$F_\eta\left(\frac{\lambda_8}{\sqrt{2}}\right) = F\Bigg\{1 + \frac{4}{F^2}[L_4^r(\mu)(M_\pi^2 + 2M_K^2) + L_5^r(\mu)M_\eta^2]$$
$$- \frac{3}{4(4\pi)^2F^2}\left[M_{K^\pm}^2 \ln\frac{M_{K^\pm}^2}{\mu^2} + M_{K^0}^2 \ln\frac{M_{K^0}^2}{\mu^2}\right]$$
$$+ \frac{e^2}{3}[4S_1^r(\mu) + 2S_2^r(\mu) + S_3^r(\mu)]\Bigg\}. \qquad (5.9)$$

The quantity $M_{\hat\pi^0\hat\eta}^2$ in (5.7) and (5.8) is the off–diagonal element of the $\pi^0 - \eta$ mass matrix in the basis of the tree–level mass eigenfields $\hat\pi^0, \hat\eta$. Its explicit form can be found in Ref. [8]. In all our formulas, terms of higher than linear order in the isospin breaking parameters ε, e^2 have been neglected. The electromagnetic infrared divergence occuring in (5.3) and (5.4) has been taken into account by introducing the small photon mass m_γ. Taken for themselves, the expressions given above are not observable quantities but only (major) parts in a full analysis of $P_{\ell 2}$ decays. The infrared divergences are absorbed by adding the corresponding $P_{\ell 2\gamma}$ contributions [19]. Furthermore, the leptonic part together with the associated electromagnetic corrections has to be included [20]. However, for our present purposes, the information contained in (5.3–5.9) will be sufficient.

To get a feeling for the possible size of the electromagnetic contributions to isospin violating quantities we build the ratio

$$R := \frac{F_{K^0}F_{\pi^\pm}}{F_{K^\pm}F_{\pi^0}(\lambda_3/\sqrt{2})} = 1 + \frac{4\varepsilon}{\sqrt{3}}\left\{\frac{F_K}{F_\pi} - 1 + \frac{1}{4(4\pi)^2F^2}\left[M_\pi^2 - M_K^2 + M_\pi^2\ln\frac{M_K^2}{M_\pi^2}\right]\right\}$$
$$- \frac{e^2}{3}[2S_2^r(\mu) + 3S_3^r(\mu)] + \frac{3e^2}{2(4\pi)^2}\left[Z\left(1 + \ln\frac{M_K^2}{\mu^2}\right) - \ln\frac{M_K^2}{M_\pi^2}\right]. \qquad (5.10)$$

In this specific combination of the form factors (5.3–5.6), the infrared divergent terms cancel. The only remaining uncertainty in (5.10) is the electromagnetic low–energy constant $2S_2^r(\mu) + 3S_3^r(\mu)$. At this point, we completely disregard the question if it will ever be possible to determine the quantity R with a sufficient experimental accuracy. We just want to compare the size of the electromagnetic and the QCD part contained in (5.10). With

$$\varepsilon = (1.00 \pm 0.07) \cdot 10^{-2}, \tag{5.11}$$

extracted from the mass splitting in the baryon octet [14, 16, 17] and $F_K/F_\pi = 1.22$ [21], we find

$$(R-1)_{QCD} = 4.4 \cdot 10^{-3}. \tag{5.12}$$

Assuming the validity of (4.4), we expect an electromagnetic contribution within the range

$$-3.2 \cdot 10^{-3} \lesssim (R-1)_{EM} \lesssim -1.2 \cdot 10^{-3}, \tag{5.13}$$

where the lower (upper) bound corresponds to $2S_2^r(M_\rho) + 3S_3^r(M_\rho) = {}^{+}_{(-)}5/(4\pi)^2$. We conclude from (5.12) and (5.13) that, in general, isospin violating terms of electromagnetic origin can be of equal importance as the corresponding QCD pieces proportional to the quark mass difference $m_d - m_u$.

6 $K_{\ell 3}$ Form Factors

Finally, we discuss the $K_{\ell 3}$ form factors $f_+^{K^+\pi^0}(0)$ and $f_+^{K^0\pi^-}(0)$ including the electromagnetic contributions of $\mathcal{O}(e^2p^2)$. Our results are given by

$$\begin{aligned} f_+^{K^+\pi^0}(0) &= 1 + \frac{1}{2}H_{K^\pm\pi^0}(0) + \frac{3}{2}H_{K^\pm\eta}(0) + H_{K^0\pi^\pm}(0) \\ &\quad + \sqrt{3}\left(\varepsilon - \frac{M^2_{\hat\pi^0\hat\eta}}{M_\eta^2 - M_\pi^2}\right) + \sqrt{3}\,\varepsilon\left[\frac{5}{2}H_{K\pi}(0) + \frac{1}{2}H_{K\eta}(0)\right] \\ &\quad - \frac{e^2}{(4\pi)^2}\left[2 + \ln\frac{m_\gamma^2}{M_K^2} + \frac{1}{4}\ln\frac{M_K^2}{\mu^2} + 2(4\pi)^2 S_8^r(\mu)\right], \end{aligned} \tag{6.1}$$

and

$$\begin{aligned} f_+^{K^0\pi^-}(0) &= 1 + H_{K^0\pi^\pm}(0) + \frac{1}{2}H_{K^\pm\pi^0}(0) + \frac{3}{2}H_{K^\pm\eta}(0) \\ &\quad + \sqrt{3}\,\varepsilon\,[H_{K\pi}(0) - H_{K\eta}(0)] \\ &\quad - \frac{e^2}{(4\pi)^2}\left[2 + \ln\frac{m_\gamma^2}{M_\pi^2} + \frac{1}{4}\ln\frac{M_\pi^2}{\mu^2} + 2(4\pi)^2 S_8^r(\mu)\right]. \end{aligned} \tag{6.2}$$

The function $H_{PQ}(t)$ was defined in [22], where also $f_+^{K^+\pi^0}(t)$, $f_+^{K^0\pi^-}(t)$ in the limit $e = 0$ were presented.

In the ratio

$$r_{K\pi} = \frac{f_+^{K^+\pi^0}(0)}{f_+^{K^0\pi^-}(0)} = 1 + \sqrt{3}\left(\varepsilon - \frac{M^2_{\hat\pi^0\hat\eta}}{M_\eta^2 - M_\pi^2}\right) + \frac{3e^2}{4(4\pi)^2}\ln\frac{M_K^2}{M_\pi^2}, \tag{6.3}$$

the infrared divergent terms cancel. As in the previous example (5.10), only $2S_2^r(\mu) + 3S_3^r(\mu)$ (contained in $M^2_{\pi^0\hat{\eta}}$) remains as an unknown parameter. Disentangling the QCD and the electromagnetic contribution to (6.3) one finds [8]

$$(r_{K\pi} - 1)_{QCD} = 2.1 \cdot 10^{-2}, \tag{6.4}$$

and

$$0 \lesssim (r_{K\pi} - 1)_{EM} \lesssim 0.2 \cdot 10^{-2}, \tag{6.5}$$

respectively, where (6.5) is again based on (4.4). In spite of our ignorance of the exact values of the electromagnetic coupling constants, we have obtained a rather precise result: The electromagnetic contribution to $r_{K\pi} - 1$ can increase the pure QCD value by at most 10 %.

Let us also compare the theoretical results (6.4) and (6.5) with the present experimental data. Dividing the rates of $K^+ \to \pi^0 e^+ \nu_e$ and $K^0 \to \pi^- e^+ \nu_e$ by the relevant phase space integrals (including those electromagnetic corrections which are sensitive to the lepton kinematics [23]) one finds [22]

$$|\frac{f_+^{K^+\pi^0}(0)}{f_+^{K^0\pi^-}(0)}|^2 = 1.057 \pm 0.019, \tag{6.6}$$

which implies

$$(r_{K\pi} - 1)_{exp} = (2.8 \pm 0.9) \cdot 10^{-2}. \tag{6.7}$$

This means that the error in the present data is still much larger than the theoretical uncertainty due to electromagnetism.

7 Conclusions

We have used the machinery of chiral perturbation theory including a systematic treatment of the electromagnetic interaction [7]. Within this theoretical framework, a one–loop–analysis allows the computation of the pure QCD contributions to $\mathcal{O}(p^4)$ and of the electromagnetic part to $\mathcal{O}(e^2p^2)$ for any observable in the sector of pseudoscalar mesons. The low–energy constants associated with the $\mathcal{O}(p^4)$ effective Lagrangian of strong interactions are well known parameters. For the coupling constants of the $\mathcal{O}(e^2p^2)$ electromagnetic Lagrangian, only order of magnitude estimates based on chiral dimensional analysis are presently available.

This situation might change by future precision measurements of isospin breaking observables or, on the theoretical side, by a determination of the relevant low–energy constants using chiral models or even lattice calculations (for examples in the strong sector see Ref. [24] and the citations therein). Such an improvement of our knowledge about the $\mathcal{O}(e^2p^2)$ coupling constants would also drastically increase the value of our formal expressions for the electromagnetic contributions to several isospin breaking quantities.

We have performed a one–loop analysis of all $P_{\ell 2}$ and the $K_{\ell 3}$ form factors $f_+^{K^+\pi^0}(0)$ and $f_+^{K^0\pi^-}(0)$. Our results allow a discussion of the magnitude of isospin violating effects due to pure QCD, that is the difference of the up and down quark masses, and those originating from QED isospin violation. There is no general feature, the size of the respective contributions depends strongly on the observed quantity.

For example, in the specific combination of $P_{\ell 2}$ form factors $R - 1$ (defined in (5.10)), the isospin violating effects of electromagnetic origin can be of equal size as the QCD ones. Similarly, it has been found [7, 8] that sizable deviations from Dashen's limit [10] for the pseudoscalar

meson masses cannot be excluded. On the other hand, for the ratio $r_{K\pi} = f_+^{K^+\pi^0}(0)/f_+^{K^0\pi^-}(0)$ of $K_{\ell 3}$ form factors we have obtained the rather precise result that the electromagnetic contribution to $r_{K\pi} - 1$ can at most be 10 % of the corresponding QCD value, which is quite similar in the case of the $\eta_{\ell 3}$ form factors $f_\pm^{\eta\pi}(t)$ [8].

At present, the experimental errors are still much larger than the uncertainties induced by electromagnetic isospin violating contributions. But our analysis shows quite clearly that if isospin violating effects due to $m_u \neq m_d$ are considered with one–loop accuracy, one also has to take into account electromagnetic effects up to $\mathcal{O}(e^2p^2)$.

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References

[1] S. Weinberg, Physics 96A (1979) 327

[2] J. Gasser and H. Leutwyler, Ann. Phys. 158 (1984) 142

[3] J. Gasser and H. Leutwyler, Nucl. Phys. B250 (1985) 465

[4] H. Leutwyler, Ann. Phys. 235 (1994) 165

[5] J. Bijnens, G. Ecker and J. Gasser, in: The Second DAΦNE Physics Handbook, eds. L. Maiani, G. Pancheri, N. Paver (INFN, Frascati, 1995)

[6] G. Ecker, J. Gasser, A. Pich and E. de Rafael, Nucl. Phys. B321 (1989) 311

[7] A. Urech, Nucl. Phys. B433 (1995) 234

[8] H. Neufeld and H. Rupertsberger, Isospin Breaking in Chiral Perturbation Theory and the Decays $\eta \to \pi\ell\nu$ and $\tau \to \eta\pi\nu$, University of Vienna preprint UWThPh–1994–15, to be published in Z. Phys. C

[9] S. Coleman, J. Wess and B. Zumino, Phys. Rev. 177 (1969) 2239; C. Callan, S. Coleman, J. Wess and B. Zumino, Phys. Rev. 177 (1969) 2247

[10] R. Dashen, Phys. Rev. 183 (1969) 1245

[11] A.V. Manohar and H. Georgi, Nucl. Phys. B234 (1984) 189; M. Soldate and R. Sundrum, Nucl. Phys. B340 (1990) 1; R.S. Chivukula, M.J. Dugan and M. Golden, Phys. Rev. D47 (1993) 2930

[12] J.F. Donoghue, B.R. Holstein and D. Wyler, Phys. Rev. D47 (1993) 2089

[13] J. Bijnens, Phys. Lett. B306 (1993) 343

[14] H. Leutwyler, Nucl. Phys. B337 (1990) 108

[15] K. Maltman, T. Goldman and G.L. Stephenson Jr., Phys. Lett B234 (1990) 158

[16] J. Gasser, Ann. Phys. 136 (1981) 62

[17] J. Gasser and H. Leutwyler, Phys. Rep. 87 (1982) 77

[18] H. Neufeld, Isospin Violation in Chiral Perturbation Theory and the Decays $\eta \to \pi\ell\nu$ and $\tau \to \eta\pi\nu$, Univ. Wien preprint UWThPh-1994-42, to appear in: Proc. of the 7th Adriatic Meeting on Particle Physics (Brijuni, Croatia, September 1994)

[19] J. Bijnens, G. Ecker and J. Gasser, Nucl. Phys. B396 (1993) 81

[20] S.M. Berman, Phys. Rev. Lett. 1 (1958) 468; T. Kinoshita, Phys. Rev. Lett. 2 (1959) 477; M.V. Terent'ev, Yad. Fiz. 18 (1973) 870 [Sov. J. Nucl. Phys. 18 (1974) 449]; W.J. Marciano and A. Sirlin, Phys. Rev. Lett. 36 (1976) 1425; T. Goldman and W.J. Wilson, Phys. Rev. D15 (1977) 709; B.R. Holstein, Phys. Lett. B244 (1990) 83; W.J. Marciano and A. Sirlin, Phys. Rev. Lett. 71 (1993) 3629; M. Finkemeier, Radiative Corrections to $\pi_{\ell 2}$ and $K_{\ell 2}$ decays, Frascati preprint LNF-94/076(IR)

[21] Particle Data Group, L. Montanet et al., Phys. Rev. D50 (1994), Number 3, Part I

[22] J. Gasser and H. Leutwyler, Nucl. Phys. B250 (1985) 517

[23] H. Leutwyler and M. Roos, Z. Phys. C25 (1984) 91

[24] G. Ecker, Estimates of Low–Energy Parameters, Univ. Wien preprint UWThPh-1994-34, to appear in: Proc. of the Workshop on Chiral Dynamics: Theory and Experiment (MIT, Cambridge, July 1994)