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Geometrical Well Posed Systems for the Einstein Equations

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Abstract

We show that, given an arbitrary shift, the lapse N can be chosen so that the extrinsic curvature K of the space slices with metric $\overline{g}$ in arbitrary coordinates of a solution of Einstein's equations satisfies a quasi-linear wave equation. We give a geometric first order symmetric hyperbolic system verified in vacuum by $\overline{g}$, K and N. We show that one can also obtain a quasi-linear wave equation for K by requiring N to satisfy at each time an elliptic equation which fixes the value of the mean extrinsic curvature of the space slices.

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Résumé Nous donnons deux conditions différentes sur le lapse pour que la courbure extrinsèque K des sections d'espace d'une solution des équations d'Einstein satisfasse une équation d'onde quasi-linéaire, le shift étant arbitraire.

Version française abrégée

Nous considérons les équations d'Einstein sur un espace temps de dimension 4, de topologie $M \times R$ dont nous écrivons la métrique

$$ds^2 = -N^2(\theta^0)^2 + g_{ij}\theta^i\theta^j$$

dans le corepère donné par (x^i coordonnées locales sur M, $t \in R$)

$$\theta^0 = dt, \qquad \theta^i = dx^i + \beta^i dt.$$

Nous définissons l'opérateur $\hat{\partial}_0$ sur les tenseurs d'espace dépendant du temps par ($\mathcal{L}$ est la dérivée de Lie, ∂_0 la dérivée pfaffienne)

$$\hat{\partial}_0 = \partial/\partial t - \mathcal{L}_\beta, \quad alors \;\; que \quad \partial_0 = \partial/\partial t - \beta^i\partial/\partial x^i.$$

La courbure extrinséque K des sections d'espace $M \times \{t\}$ est telle que

$$\hat{\partial}_0 g_{ij} = -2NK_{ij}.$$

Nous utilisons la décomposition 3+1 du tenseur de Ricci de l'espace temps pour obtenir l'identité, (2.1) donnée au §2, généralisation de celle donnée dans [2] dans le cas $\beta = 0$, où on a posé $H = K^i_i$ et $(ij) = ij + ji$. Cette identité donne une équation d'ondes pour K dans les cas suivants:

1. N est choisi tel que $\partial_0 N + N^2 H = 0$.

2. N est choisi tel que $H = h$, une fonction donnée.

Nous donnons dans le cas 1 un système symétrique hyperbolique du premier ordre satisfait par les espaces temps einsteiniens.

Introduction

We examine the Cauchy problem[1] for general relativity as the time history of the geometry of a spacelike hypersurface. Constraints on the initial data, $\overline{g}$, the space metric, and K, the extrinsic curvature, can be posed and solved as an elliptic system by known methods. However, the equations of motion for $\overline{g}$ and K, which are essentially the Arnowitt-Deser-Misner canonical equations, do not, despite their usefulness, manifest mathematically the physical propagation of gravitational effects along the light cone. These equations, of course, contain gauge effects and cannot, therefore, yield directly a physical wave equation, that is, a hyperbolic system with suitable characteristics. (For a review of the relevant geometry, see, for example,[5].)

In this paper we give two different methods for obtaining exact nonlinear physical wave equations from the equations of motion of $\overline{g}$ and K. One of these, which relies on a "harmonic" time-slicing condition, gives equations of motion completely equivalent to a first-order symmetric hyperbolic system with only physically appropriate characteristics. We construct this system explicitly. Among the propagated quantities is, in effect, the Riemann curvature. The space coordinates and shift vector are arbitrary; and, in this sense, the system is gauge-invariant.

Our gauge invariant nonlinear hyperbolic system, being *exact and always on the physical light cone*, is well suited for use in a number of problems that now confront gravity theorists. These include large-scale computations of astrophysically significant processes (such as black hole collisions) that require efficient stable numerical integration, extraction of gravitational radiation with arbitrarily high accuracy from Cauchy data, gauge-invariant perturbation and approximation methods, and posing boundary conditions compatibly with the causal structure of spacetime.

1. NOTATIONS AND 3+1 DECOMPOSITION

We consider a manifold V of dimension 4, which has the topology $M \times R$, with M a $\mathcal{C}^\infty$, 3 dimensional manifold. We denote by $(x,t) \in M \times R$ a point of V. When we take local coordinates they will always be adapted to the product structure: x^i, $i = 1,2,3$ are coordinates on M and $x^0 \equiv t \in R$. We consider on V a pseudo-riemannian metric g of lorentzian signature $(- + ++)$. It induces a properly riemannian metric $\bar{g}_t$ on each space submanifold $M_t \equiv M \times \{t\}$. We denote by β and N respectively the shift and the lapse of the foliation. We define a coframe θ^α by

$$\theta^0 \equiv dt, \qquad \theta^i \equiv dx^i + \beta^i dt.$$

The corresponding Pfaff derivatives ∂_α are

$$\partial_0 \equiv \frac{\partial}{\partial t} - \beta^i \partial_i, \qquad \partial_i \equiv \frac{\partial}{\partial x^i}.$$

In this coframe the metric reads

$$ds^2 \equiv g_{\alpha\beta}\theta^\alpha\theta^\beta \equiv -N^2(\theta^0)^2 + g_{ij}\theta^i\theta^j.$$

We define the operator $\hat{\partial}_0$ on t dependent space tensors by ($\mathcal{L}_\beta$ is the Lie derivative with respect to β)

$$\hat{\partial}_0 \equiv \frac{\partial}{\partial t} - \mathcal{L}_\beta.$$

The extrinsic curvature K of the space manifold is

$$K_{ij} \equiv -\frac{1}{2}N^{-1}\hat{\partial}_0 g_{ij}.$$

The <u>Ricci curvature</u> of space time admits the 3+1 decomposition, with $H = K^h{}_h$

$$R_{ij} = -N^{-1}\hat{\partial}_0 K_{ij} + HK_{ij} - 2K_{im}K^m{}_j - N^{-1}\bar{\nabla}_j\partial_i N + \bar{R}_{ij},$$

$$R^0{}_j = N^{-1}(\bar{\nabla}_h K^h{}_j - \partial_j H),$$

$$R^0{}_0 = -(N^{-1}\bar{\nabla}^i\partial_i N - K_{ij}K^{ij} + N^{-1}\partial_0 H),$$

where an overbar denotes a quantity relative to the space metric $\bar{g} \equiv (g_{ij})$.

2. SECOND ORDER EQUATION FOR K

In the formula above R_{ij}, like the right hand side giving its decomposition, is a t dependent space tensor, the projection on space of the Ricci tensor of g. We compute its $\hat{\partial}_0$ derivative. First we compute $\hat{\partial}_0 \bar{R}_{ij}$. The infinitesimal variation of the Ricci curvature corresponding to an infinitesimal $\delta\bar{g}$ variation of the space metric is, with $(ij) = ij + ji$ (no factor of $1/2$)

$$\delta \bar{R}_{ij} = \frac{1}{2}\{\bar{\nabla}^h \bar{\nabla}_{(i}\delta g_{j)h} - \bar{\nabla}_h \bar{\nabla}^h \delta g_{ij} - \bar{\nabla}_j \partial_i (g^{hk}\delta g_{hk})\}.$$

This expression applies to $(\partial/\partial t)\bar{R}_{ij}$ with $\delta g_{ij} = (\partial/\partial t) g_{ij}$ and to $\mathcal{L}_\beta \bar{R}_{ij}$ with $\delta g_{ij} = \mathcal{L}_\beta g_{ij}$. Therefore, using the relation between $\hat{\partial}_0 g_{ij}$ and K_{ij}, we obtain

$$\begin{aligned}\hat{\partial}_0 \bar{R}_{ij} &= -\bar{\nabla}^h \bar{\nabla}_{(i}(N K_{j)h}) + \bar{\nabla}_h \bar{\nabla}^h (N K_{ij}) + \bar{\nabla}_j \partial_i (N H)\\ &\equiv -\bar{\nabla}_{(i} \bar{\nabla}^h (N K_{j)h}) + \bar{\nabla}_h \bar{\nabla}^h (N K_{ij}) + \bar{\nabla}_j \partial_i (N H)\\ &\qquad -2N \bar{R}^h{}_{ijm} K^m{}_h - N \bar{R}_{m(i} K_{j)}{}^m.\end{aligned}$$

We now use the expressions for $R^0{}_i$ and R_{ij} to obtain the identity

$$\begin{aligned}\Omega_{ij} \equiv \hat{\partial}_0 R_{ij} + \bar{\nabla}_{(i}(N^2 R_{j)}{}^0) \equiv\\ &-\hat{\partial}_0 (N^{-1}\hat{\partial}_0 K_{ij}) + \bar{\nabla}^h \bar{\nabla}_h (N K_{ij}) + \hat{\partial}_0 (H K_{ij} - 2K_{im}K^m{}_j)\\ &\qquad -\hat{\partial}_0 (N^{-1}\bar{\nabla}_j \partial_i N) - N \bar{\nabla}_i \partial_j H\\ &-\bar{\nabla}_{(i}(K_{j)h}\partial^h N) - 2N \bar{R}^h{}_{ijm} K^m{}_h - N \bar{R}_{m(i} K_{j)}{}^m + H \bar{\nabla}_j \partial_i N.\end{aligned}$$

3. HYPERBOLIC SYSTEM FOR $\bar{g}$, K, N

We shall eliminate at the same time the third derivatives of N and the second derivatives of H as follows (cf. this elimination in the case of zero shift in[2]). We compute

$$\hat{\partial}_0 \bar{\nabla}_j \partial_i N \equiv (\frac{\partial}{\partial t} - \mathcal{L}_\beta)\bar{\nabla}_j \partial_i N.$$

We find at once

$$\frac{\partial}{\partial t}\bar{\nabla}_j \partial_i N \equiv \bar{\nabla}_j \partial_i \frac{\partial N}{\partial t} - \frac{1}{2} g^{kl}(\bar{\nabla}_{(i}\frac{\partial}{\partial t} g_{j)l} - \bar{\nabla}_l \frac{\partial}{\partial t} g_{ij})\partial_k N,$$

and we find an analogous formula when the operator $\partial/\partial t$ is replaced by $\mathcal{L}_\beta$. Finally

$$\hat{\partial}_0 \bar{\nabla}_j \partial_i N \equiv \bar{\nabla}_j \partial_i \hat{\partial}_0 N + \{\bar{\nabla}_{(i}(N K_{j)l}) - \bar{\nabla}_l (N K_{ij})\}\partial^l N.$$

The third order terms in N and second order terms in H in the second order equation for K can therefore be written in terms of

$$C_{ij} \equiv N^{-1} \bar{\nabla}_j \partial_i (\hat{\partial}_0 N + N^2 H).$$

We satisfy the condition $C_{ij} = 0$ by requiring N to satisfy the differential equation (note that $\hat{\partial}_0 N \equiv \partial_0 N$)

$$(N') \qquad \partial_0 N + N^2 H = 0.$$

Using the expression for H, the equation (N') reads

$$\hat{\partial}_0 \log\{N (\det \bar{g})^{-1/2}\} = 0.$$

We find, generalizing the result obtained in[2] in the case of zero shift, that the general solution of this equation is

$$N = \alpha^{-1} \det(\bar{g})^{1/2}, \qquad \hat{\partial}_0 \alpha = 0.$$

The second equation is a linear differential equation for the scalar density α, depending only on β. The above choice of N is called an <u>algebraic gauge</u>. A possible choice if β does not depend on t is to take α independent of t and such that $\mathcal{L}_\beta \alpha = 0$. In practice, one may simply regard (N$'$) as a differential equation to be solved simultaneously with (K$'$) below.

Taking into account the equation (N$'$) satsified by N we see that the Einstein equations $R_{\alpha\beta} = \rho_{\alpha\beta}$ imply the wave equation for K

$$(K') \qquad N K_{ij} = Q_{ij} + \Theta_{ij},$$

where we now set

$$K_{ij} = -N^{-2} \hat{\partial}_0 \hat{\partial}_0 K_{ij} + \bar{\nabla}^h \bar{\nabla}_h K_{ij},$$

$$Q_{ij} \equiv -K_{ij}\partial_0 H + 2g^{hm} K_{m(i}\hat{\partial}_0 K_{j)h} + 4N g^{hl} g^{mk} K_{lk} K_{im} K_{jh}$$

$$+(2\bar{\nabla}_{(i}K_{j)l})\partial^l N - 2HN^{-1}\partial_i N \partial_j N - 2\partial_{(i} N \partial_{j)} H - 3\partial_h N \bar{\nabla}^h K_{ij}$$

$$-K_{ij}\bar{\nabla}^h \bar{\nabla}_h N - N^{-1} K_{ij}(\bar{\nabla}^h N)\partial_h N + N^{-1} K_{h(i}\bar{\nabla}_{j)} N \partial^h N$$

$$+(\bar{\nabla}_{(i}\partial^h N) K_{j)h} + 2N \bar{R}^h{}_{ijm} K^m{}_h + N \bar{R}_{m(i} K_{j)}{}^m - 2H \bar{\nabla}_j \partial_i N,$$

$$\Theta_{ij} \equiv \hat{\partial}_0 \rho_{ij} + \bar{\nabla}_{(i}(N^2 \rho_{j)}{}^0).$$

An immediate consequence is the following theorem.

Theorem *In algebraic gauge and with the definition $K_{ij} \equiv -(2N)^{-1}\hat{\partial}_0 g_{ij}$, the system $\Omega_{ij} - \Theta_{ij} = 0$ is a quasi-diagonal third order system for g_{ij} with principal operator ∂_0. This system is hyperbolic if $\bar{g}$ is properly riemannian and $N^2 > 0$. (In practice, one can regard the definition of K_{ij} as an equation (g') for $\hat{\partial}_0 g_{ij}$, and then consider the differential equations (N'), (K') and (g') to be solved simultaneously.)*

The usual local in time existence theorem for a solution of the Cauchy problem results from the Leray theory of hyperbolic systems[4].

4. VERIFICATION OF THE ORIGINAL EINSTEIN EQUATIONS

We set

$$\Sigma_{\alpha\beta} \equiv (R_{\alpha\beta} - \rho_{\alpha\beta}) - \frac{1}{2} g_{\alpha\beta}(R - \rho).$$

Suppose we have solved the equations

$$\Omega_{ij} \equiv \hat{\partial}_0 R_{ij} + \bar{\nabla}_{(i}(N^2 R^0_{j)}) = \Theta_{ij}, \tag{4.1}$$

that is, equivalently, doing a few computations,

$$\nabla_0 \Sigma_{ij} + \nabla_{(i}(N^2 \Sigma^0_{j)}) - g_{ij}\nabla_h(N^2 \Sigma^{0h}) + f_{ij} = 0, \tag{4.2}$$

where f_{ij} is linear and homogeneous in $\Sigma_{\alpha\beta}$.

We suppose that the stress energy tensor $T_{\alpha\beta} \equiv \rho_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta}\rho$ satisfies the conservation laws $\nabla_\alpha T^{\alpha\beta} = 0$. We deduce then from the Bianchi identities the equations

$$\nabla^\alpha \Sigma_{\alpha\beta} = 0. \tag{4.3}$$

From (4.2) and (4.3) we obtain for $\Sigma_{\alpha\beta}$ a quasi diagonal third order system with principal operator the hyperbolic operator $\hat{\partial}_0$. The vanishing for $t = 0$ of $\Sigma_{\alpha\beta}$ results from the constraints satisfied by the original initial data

$$\Sigma^0_j|_{t=0} = 0, \qquad\qquad \Sigma^{00}|_{t=0} = 0,$$

and the determination of $\hat{\partial}_0 K|_{t=0}$ by the equation

$$\Sigma_{ij}|_{t=0} = 0.$$

We then deduce from (4.2) and (4.3) the vanishing for $t = 0$ of the derivatives of order ≤ 2 of $\Sigma_{\alpha\beta}$. It results therefore, from the uniqueness theorem of Leray[4] for hyperbolic systems, that a solution of (4.1) with initial data satisfying the constraints satisfies the full Einstein equations $\Sigma_{\alpha\beta} = 0$.

5. FIRST ORDER SYSTEM (vacuum)

The preceding results can be extended without major change to dimensions greater than 4. We will show that in dimension 4 a solution of the vacuum Einstein equations, together with the gauge condition (N'), satisfies a first order symmetric system, hyperbolic if $\bar{g}$ is properly Riemannian and $N^2 > 0$. Such a system could be useful to establish *a priori* estimates relevant to global problems, as well as to the physical applications mentioned in the Introduction.

We have obtained for the unknowns $\bar{g}$, K, and N the equations

$$\hat{\partial}_0 g_{ij} = -2NK_{ij}, \tag{5.1}$$

$$\hat{\partial}_0(N^2) = -2NH(N^2), \tag{5.2}$$

$$NK_{ij} = Q_{ij}. \tag{5.3}$$

To obtain a first order system we take as additional unknowns:

$$N^{-1}\hat{\partial}_0 K_{ij} \equiv L_{ij}, \quad \bar{\nabla}_h K_{ij} \equiv M_{hij}, \quad \partial_i \log N \equiv a_i,$$

$$N^{-1}\hat{\partial}_0 \partial_i \log N \equiv a_{0i}, \quad a_{ij} = \bar{\nabla}_i \partial_j logN.$$

We take as equation (5.3′)

$$\hat{\partial}_0 K_{ij} = NL_{ij}. \tag{5.3′}$$

The equation (5.3) gives

$$\hat{\partial}_0 L_{ij} - N\bar{\nabla}^h M_{hij} = N(HL_{ij} - Q_{ij}). \tag{5.4}$$

In three space dimensions, the Riemann tensor is a linear function of the Ricci tensor. Using the equation $R_{ij} = 0$ to express $\bar{R}_{ij}$, we write Q_{ij} as a polynomial in the unknowns and in g^{ij}. (L_{ij} is the essential piece of the spacetime Riemann tensor $R^o{}_{ioj}$.)

We have the following lemma proved for instance by using the commutativity $\hat{\partial}_0\partial_i = \partial_i\hat{\partial}_0$ on components of tensors

Lemma *For an arbitrary covariant vector* u_i *we have*

$$\hat{\partial}_0 \bar{\nabla}_h u_i = \bar{\nabla}_h \hat{\partial}_0 u_i + u_l\{\bar{\nabla}_{(h}(NK_{i)}{}^l) - \bar{\nabla}^l(NK_{ih})\}.$$

and an analogous formula for tensors with additional terms for each index.

Using this lemma we see that L_{ij} and M_{hij} must satisfy the equation

$$\begin{aligned}\hat{\partial}_0 M_{kij} - N\overline{\nabla}_k L_{ij} = N\Big(&a_k L_{ij} + M_{k(i}{}^l K_{j)l} + K_{l(i} M_{j)k}{}^l - K_{l(i} M^l{}_{j)k} \\ &+ K_{l(i}(K^l{}_{j)} a_k + a_{j)} K^l{}_k - a^l K_{j)k})\Big).\end{aligned} \tag{5.5}$$

On the other hand, (5.2) implies

$$\hat{\partial}_0 a_i = -N(Ha_i + M_{ik}{}^k) \tag{5.6}$$

while (5.2) and the lemma yield

$$\hat{\partial}_0 a_{ji} - N\overline{\nabla}_j a_{i0} = Na_l\Big(M_{(ij)}{}^l - M^l{}_{ij} + a_{(i}K_{j)}{}^l - a^l K_{ij}\Big) + Na_j a_{i0}. \tag{5.7}$$

Now we deduce the value of $\hat{\partial}_0 a_{0i}$ from (5.2) and the Einstein equation

$$R^0{}_0 \equiv -\{N^{-1}\bar{\nabla}^h\bar{\nabla}_h N - K_{ij}K^{ij} + N^{-1}\partial_0 H\} = 0.$$

Indeed these two equations imply

$$\partial_0\partial_0 N = N^2\bar{\nabla}^h\bar{\nabla}_h N - N^3 K_{ij}K^{ij} + 2N^3H^2.$$

Hence, by differentiation and use of the Ricci formula and the definitions of a_{hi} and a_{0i}, we obtain

$$\begin{aligned}\hat{\partial}_0 a_{0i} - N\overline{\nabla}^k a_{ki} = N\Big(&-\overline{R}^k{}_i a_k + a_i(H^2 - 2K_{kl}K^{kl} + 2a^k a_k + 2a^k{}_k)\\ &+2a_k a^k{}_i + HM_{ik}{}^k - 2K^{kl}M_{ikl}\Big).\end{aligned} \tag{5.8}$$

We use again the equation $R_{ij} = 0$ in (5.8) to replace $\bar{R}_{ij}$ by its values in terms of the unknowns.

The right hand sides of the equations (5.1), (5.2), (5.3′), (5.4), (5.5), (5.6), (5.7), and (5.8) are polynomial in the unknowns and g^{ij}; they do not depend on their derivatives. We eliminate the first derivatives of $\bar{g}$ appearing on the left hand sides for instance by introducing on M an *a priori* given metric e which may depend on t but is such that

$$\hat{\partial}_0 e_{ij} = 0.$$

We denote by D the covariant derivative in the metric e. We deduce then from (5.1) the following equation for $G_{hij} \equiv D_h g_{ij}$:

$$\hat{\partial}_0 G_{hij} = -2N\{a_h K_{ij} + M_{hij} + S^m{}_{h(i}K_{j)m}\}. \tag{5.9}$$

The tensor S, the difference of the connections of $\bar{g}$ and e, is given by

$$S^m{}_{ij} = \frac{1}{2}g^{mh}(G_{(ij)h} - G_{hij}).$$

We thus obtain a <u>first order differential system,</u> equivalent to a <u>symmetric</u> system, <u>hyperbolic if $N^2 > 0$ and g_{ij} is properly riemannian. Its characteristics are the physical light cone and the "time" axis ∂_0.</u>

6. MIXED HYPERBOLIC ELLIPTIC SYSTEM FOR K, $\bar{g}$, N WHEN H IS GIVEN

Another procedure to reduce the second order equation for K obtained above to a quasi diagonal system with principal part the wave operator is to replace in the term $\bar{\nabla}_i\partial_j H$ the mean curvature H by an a priori given function h; this method was used in [3], with $h = 0$, in the asymptotically euclidean case. With this replacement the equations $\Omega_{ij} = \Theta_{ij}$ take the form

$$NK_{ij} = P_{ij} + \Theta_{ij} \tag{6.1}$$

where P_{ij} depends on K and its first derivatives and on $\bar{g}$, N and $\partial_0 N$ together with their space derivatives of order ≤ 2. When N and Θ_{ij} are known the above equations together with (5.1) are again a third order quasi-diagonal system for $\bar{g}$, hyperbolic if $N > 0$ and $\bar{g}$ properly riemannian. On the other hand, the equation $R^0{}_0 = \rho^0{}_0$ together with $H = h$ implies the equation

$$\bar{\nabla}^i\partial_i N = (K_{ij}K^{ij} - \rho^0{}_0)N = -\partial_0 h. \tag{6.2}$$

This equation is an elliptic equation for N when $\bar{g}$, K and ρ are known. Note that for energy sources satisfying the "strong" energy condition we have $-\rho^0{}_0 \geq 0$ as well as $|K|^2 \equiv K_{ij}K^{ij} \geq 0$. These properties are important for the solution of the elliptic equation.

The hyperbolic system that we have constructed (for given N and Θ) has local in time solutions for Cauchy data in local Sobolev spaces, like the one obtained in algebraic gauge. But now N is determined by an elliptic equation which has to be solved globally on each space slice M_t at time t. An iteration procedure leading to a local in time solution has been used in the vacuum asymptotically flat case with $h = 0$ [3]. We give below a theorem which applies to the case of a compact M and which can also be proven by iteration. (Further theorems with $h \neq 0$ on asymptotically flat spaces can also be given.)

Theorem. Let (M, e) be a given smooth compact riemannian manifold. Let there be given on $M \times I$, $I \equiv [0, T]$, a "pure space" smooth vector field β and a function h such that

(H_k is the usual Sobolev space relative to (M, e))

$$h \in \bigcap_{2 \leq k \leq 3} C^{3-k}(I, H_k), \quad \partial_0 h \geq 0, \quad \partial_0 h \not\equiv 0, \quad h(0,.) \geq 0, or \quad h(0,.) \leq 0, \quad h \not\equiv 0.$$

There exists an interval $J \equiv [0, \ell]$, $\ell \leq T$ such that the system (6.1), with $\Theta_{ij} = 0$, (5.1), (6.2) with ${\rho^0}_0 = 0$, has one and only one solution on $M \times J$,

$$\bar{g} \in \bigcap_{1 \leq k \leq 3} C^{3-k}(J, H_k), \quad K \in \bigcap_{0 \leq k \leq 2} C^{2-k}(J, H_k), \quad N \in \bigcap_{0 \leq k \leq 2} C^{2-k}(J, H_{2+k})$$

with $N > 0$ and $\bar{g}$ uniformly equivalent to e, taking the initial data

$$g_{ij}(0,.) = \gamma_{ij} \in H_3, \quad K_{ij}(0,.) = k_{ij} \in H_2$$

if γ is a properly riemannian metric uniformly equivalent to e and $k.k \not\equiv 0$.

It can be proved by using the Bianchi identities that if the initial data satisfy the constraints the constructed solution satisfies the original Einstein equations.

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