LOCAL SUPERSYMMETRY IN ONE-LOOP QUANTUM COSMOLOGY

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Abstract. The contribution of physical degrees of freedom to the one-loop amplitudes of Euclidean supergravity is here evaluated in the case of flat Euclidean backgrounds bounded by a three-sphere, recently considered in perturbative quantum cosmology. In Euclidean supergravity, the spin-$\frac{3}{2}$ potential has the pair of independent spatial components $\left(\psi_i^A, \ \widetilde{\psi}_i^{A'}\right)$. Massless gravitinos are here subject to the following local boundary conditions on S^3: $\sqrt{2} \ {}_e n_A^{\ A'} \psi_i^A = \pm \widetilde{\psi}_i^{A'}$, where ${}_e n_A^{\ A'}$ is the Euclidean normal to the three-sphere boundary. The physical degrees of freedom (denoted by PDF) are picked out imposing the supersymmetry constraints and choosing the gauge condition $e_{AA'}^{\ \ \ \ i} \psi_i^A = 0$, $e_{AA'}^{\ \ \ \ i} \widetilde{\psi}_i^{A'} = 0$. These local boundary conditions are then found to imply the eigenvalue condition $\Big[J_{n+2}(E)\Big]^2 - \Big[J_{n+3}(E)\Big]^2 = 0, \ \forall n \geq 0$, with degeneracy $(n+4)(n+1)$. One can thus apply again a zeta-function technique previously used for massless spin-$\frac{1}{2}$ fields.

The PDF contribution to the full $\zeta(0)$ value is found to be $= -\frac{289}{360}$. Remarkably, for the massless gravitino field the PDF method and local boundary conditions lead to a result for $\zeta(0)$ which is equal to the PDF value one obtains using spectral boundary conditions on S^3.

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1. Introduction

The problem of one-loop finiteness of supergravity theories in the presence of boundaries is still receiving careful consideration in the current literature.[1−10] As emphasized in Refs. 9,11-12, one can perform one-loop calculations paying attention to: (1) S-matrix elements; (2) topological invariants; (3) presence of boundaries. For example, in the case of pure gravity with vanishing cosmological constant: $\Lambda = 0$, it is known that one-loop on-shell S-matrix elements are finite. This property is shared by $N = 1$ supergravity when $\Lambda = 0$, and in that theory two-loop on-shell finiteness also holds. However, when $\Lambda \neq 0$, both pure gravity and $N = 1$ supergravity are no longer one-loop finite in the sense (1) and (2), because the non-vanishing on-shell one-loop counterterm[11] is given by

$$S_{(1)} = \frac{1}{\epsilon}\left[A\chi - \frac{2BG\Lambda S}{3\pi}\right] \quad . \tag{1.1}$$

In equation (1.1), $\epsilon = n - 4$ is the dimensional-regularization parameter, χ is the Euler number, S is the classical on-shell action, and one finds :[9,11] $A = \frac{106}{45}, B = -\frac{87}{10}$ for pure gravity, and $A = \frac{41}{24}, B = -\frac{77}{12}$ for $N = 1$ supergravity. Thus, $B \neq 0$ is responsible for lack of S-matrix one-loop finiteness, and $A \neq 0$ does not yield topological one-loop finiteness.

If any theory of quantum gravity can be studied from a perturbative point of view, boundary effects play a key role in understanding whether it has interesting and useful finiteness properties. It is therefore necessary to analyze in detail the structure of the one-loop boundary counterterms for fields of various spins. This problem has been recently studied within the framework of one-loop quantum cosmology, where the boundary is usually taken to be a three-sphere, and the background is flat Euclidean space or a de Sitter four-sphere or a more general curved four-geometry.[2−10]

Our paper describes one-loop properties of spin-$\frac{3}{2}$ fields to present a calculation which was previously studied in research books[5,9] but not in physics journals (see, however, remarks at the end of Ref. 10). In the Euclidean-time regime, the spin-$\frac{3}{2}$ field is represented, using two-component spinor notation, by a pair of independent spinor-valued one-forms $\left(\psi_\mu^A, \widetilde{\psi}_\mu^{A'}\right)$ with spatial components $\left(\psi_i^A, \widetilde{\psi}_i^{A'}\right)$.[4,9] After imposing the gauge conditions (hereafter $e_{AA'}^{\ \ \ \mu}$ is the tetrad)

$$e_{AA'}^{\ \ \ i} \ \psi_i^A = 0 \quad , \quad e_{AA'}^{\ \ \ i} \widetilde{\psi}_i^{A'} = 0 \quad . \tag{1.2}$$

and the linearized supersymmetry constraints, the expansion of $\left(\psi_i^A, \widetilde{\psi}_i^{A'}\right)$ on a family of three-spheres centred on the origin takes the form[4,9]

$$\psi_i^A = \frac{\tau^{-\frac{3}{2}}}{2\pi} \sum_{n=0}^{\infty} \sum_{p,q=1}^{(n+1)(n+4)} \alpha_n^{pq} \left[m_{np}(\tau) \beta^{nqABB'} + \widetilde{r}_{np}(\tau) \overline{\mu}^{nqABB'} \right] e_{BB'i} \ , \tag{1.3}$$

$$\widetilde{\psi}_i^{A'} = \frac{\tau^{-\frac{3}{2}}}{2\pi} \sum_{n=0}^{\infty} \sum_{p,q=1}^{(n+1)(n+4)} \alpha_n^{pq} \left[\widetilde{m}_{np}(\tau) \overline{\beta}^{nqBA'B'} + r_{np}(\tau) \mu^{nqBA'B'} \right] e_{BB'i} \ . \tag{1.4}$$

With our notation, τ is the radial distance from the origin in flat Euclidean four-space, the matrix α_n^{pq} is block-diagonal in the indices pq, with blocks $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Note also that the modes $\widetilde{m}_{np}(\tau), \widetilde{r}_{np}(\tau)$ are not the complex conjugates of $m_{np}(\tau), r_{np}(\tau)$ respectively. Moreover, one has[4,9] $\beta^{nqABB'} = -\rho^{nq(ABC)} n_C^{\ \ B'}$, $\mu^{nqBA'B'} = -\sigma^{nq(A'B'C')} n^B_{\ \ C'}$ where the harmonics $\rho^{nq(ABC)}$ and $\sigma^{nq(A'B'C')}$ are symmetric in their three spinor indices and have positive eigenvalues $\frac{1}{2}\left(n+\frac{5}{2}\right)$ of the intrinsic three-dimensional Dirac operator on S^3,[4] and $n^{CB'}$ is the Lorentzian normal to S^3.[3,9]

Sec. 2 studies locally supersymmetric boundary conditions on S^3 for the spin-$\frac{3}{2}$ potential, and the equation obeyed by the eigenvalues by virtue of these boundary conditions is derived. Sec. 3 uses zeta-function regularization and obtains the contribution of physical degrees of freedom (hereafter referred to as PDF) to the full $\zeta(0)$ value. Concluding remarks and open research problems are presented in Sec. 4.

2. Local Boundary Conditions for the Spin-$\frac{3}{2}$ Potential

In Euclidean supergravity, the mathematical description of the gravitino leads to the introduction of the independent spinor-valued one-forms $\left(\psi_{\mu}^{A},\ \widetilde{\psi}_{\mu}^{A'}\right)$ with spatial components $\left(\psi_{i}^{A},\ \widetilde{\psi}_{i}^{A'}\right)$. We are here interested in a generalization to simple supergravity of the calculations in Ref. 3 for the spin-$\frac{1}{2}$ field. Thus, we consider a flat Euclidean background, requiring on the bounding S^3 that

$$\sqrt{2}\ {}_{e}n_{A}^{\ \ A'}\psi_{i}^{A}=\epsilon\ \widetilde{\psi}_{i}^{A'} \quad , \tag{2.1}$$

where $\epsilon=\pm 1$. The consideration of (2.1) is suggested by the work in Ref. 1, where it is shown that the spatial tetrad $e^{AA'}_{\ \ \ \ i}$ and the projection $\left(\pm\ \widetilde{\psi}_{i}^{A'}-\sqrt{2}\ {}_{e}n_{A}^{\ \ A'}\psi_{i}^{A}\right)$ transform into each other under half of the local supersymmetry transformations at the boundary, and that after adding a suitable boundary term, the supergravity action is invariant under these local supersymmetry transformations.[3,9]

Indeed, from Sec. 1 we already know that, imposing the supersymmetry constraints and choosing the gauge condition (1.2), the spin-$\frac{3}{2}$ potential finally assumes the form (1.3)-(1.4). It is therefore useful to derive identities relating barred to unbarred harmonics, generalizing the technique in Ref. 13. This is achieved by using the relations

$$\int d\mu \ \rho_{ABC}^{np} n^{AA'} n^{BB'} n^{CC'} \overline{\rho}_{A'B'C'}^{mq} = \delta^{nm} H_n^{pq} \quad , \tag{2.2}$$

$$\int d\mu \ \rho_{ABC}^{np} \epsilon^{AD} \epsilon^{BE} \epsilon^{CF} \rho_{DEF}^{mq} = \delta^{nm} A_n^{pq} \quad , \tag{2.3}$$

and the expansion of the totally symmetric field strength

$$\phi_{ABC}(x) = \sum_{n=0}^{\infty} \sum_{p=1}^{(n+1)(n+4)} \Bigl(\widehat{a}_{np} \rho_{ABC}^{np}(x) + \widehat{b}_{np} \overline{\sigma}_{ABC}^{np}(x) \Bigr) \quad . \tag{2.4}$$

Thus, we can express the $\widehat{a}_{np}$ coefficients in two equivalent ways using (2.4), and (2.2) or (2.3). The equality of the two resulting formulae leads to

$$n^{AA'} n^{BB'} n^{CC'} \sum_{q=1}^{(n+1)(n+4)} \overline{\rho}_{A'B'C'}^{nq} \bigl(H_n^{-1} \bigr)^{qp} = \epsilon^{AD} \epsilon^{BE} \epsilon^{CF} \sum_{q=1}^{(n+1)(n+4)} \rho_{DEF}^{nq} \bigl(A_n^{-1} \bigr)^{qp} \ , \tag{2.5}$$

which is finally cast in the form

$$\overline{\rho}_{D'E'F'}^{np} = -8 n^{D}{}_{D'} n^{E}{}_{E'} n^{F}{}_{F'} \sum_{q=1}^{(n+1)(n+4)} \rho_{DEF}^{nq} \bigl(A_n^{-1} H_n \bigr)^{qp} \quad . \tag{2.6}$$

In a similar way, we obtain

$$\overline{\sigma}_{DEF}^{np} = -8 n_D^{\ D'} n_E^{\ E'} n_F^{\ F'} \sum_{q=1}^{(n+1)(n+4)} \sigma_{D'E'F'}^{nq} \left(A_n^{-1} H_n\right)^{qp} \quad . \tag{2.7}$$

The form of the matrices A_n^{pq} and H_n^{pq} is obtained taking the complex conjugate of (2.6), and then inserting the form of ρ_{DEF}^{np} so obtained into the right-hand side of (2.6). This yields the consistency condition

$$A_n^{-1} H_n A_n^{-1} H_n = -\frac{1}{8} 1_n \quad , \tag{2.8}$$

which is solved by $A_n^{-1} = \frac{1}{2\sqrt{2}} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $H_n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, so that $A_n = 2\sqrt{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

We can now remark that (1.3)-(1.4) and (2.1) imply

$$-i\sqrt{2} \sum_{p,q=1}^{(n+1)(n+4)} \alpha_n^{pq} m_{np}^{(\beta)}(a) \rho^{nqABD} n_A^{\ A'} n_D^{\ B'} = \epsilon \sum_{p,q=1}^{(n+1)(n+4)} \alpha_n^{pq} \widetilde{m}_{np}^{(\beta)}(a) \overline{\rho}^{nqA'B'D'} n^B_{\ D'} \; , \tag{2.9}$$

$$-i\sqrt{2} \sum_{p,q=1}^{(n+1)(n+4)} \alpha_n^{pq} \widetilde{r}_{np}^{(\mu)}(a) \overline{\sigma}^{nqABD} n_A^{\ A'} n_D^{\ B'} = \epsilon \sum_{p,q=1}^{(n+1)(n+4)} \alpha_n^{pq} r_{np}^{(\mu)}(a) \sigma^{nqA'B'D'} n^B_{\ D'} \quad . \tag{2.10}$$

This is why Eqs. (2.6)-(2.7), (2.9)-(2.10) and the formulae for $A_n^{-1} H_n$ lead to the boundary conditions

$$i \sum_{pq} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{pq} m_{np}^{(\beta)}(a) \rho^{nqABC} = \epsilon \sum_{pq} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{pq} \widetilde{m}_{np}^{(\beta)}(a) \cdot$$

$$\cdot \sum_d \rho^{ndABC} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{dq} \quad , \tag{2.11}$$

$$-\epsilon \sum_{pq} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{pq} r_{np}^{(\mu)}(a) \sigma^{nqA'B'C'} = i \sum_{pq} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{pq} \widetilde{r}_{np}^{(\mu)}(a) \cdot$$

$$\cdot \sum_{d} \sigma^{ndA'B'C'} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{dq} \quad . \tag{2.12}$$

Since the ρ- and σ-harmonics on the bounding three-sphere of radius a are linearly independent, the typical case of the indices $p, q = 1, 2$ yields[3,9]

$$i m_{n1}^{(\beta)}(a) = \epsilon \, \widetilde{m}_{n2}^{(\beta)}(a) \quad , \tag{2.13}$$

$$-i m_{n2}^{(\beta)}(a) = \epsilon \, \widetilde{m}_{n1}^{(\beta)}(a) \quad , \tag{2.14}$$

$$i \widetilde{r}_{n1}^{(\mu)}(a) = \epsilon \, r_{n2}^{(\mu)}(a) \quad , \tag{2.15}$$

$$-i \widetilde{r}_{n2}^{(\mu)}(a) = \epsilon \, r_{n1}^{(\mu)}(a) \quad . \tag{2.16}$$

If we now set $\kappa_n \equiv n + \frac{5}{2}$ and define, $\forall n \geq 0$, the operators

$$L_n \equiv \frac{d}{d\tau} - \frac{\kappa_n}{\tau} \quad , \tag{2.17}$$

$$M_n \equiv \frac{d}{d\tau} + \frac{\kappa_n}{\tau} \quad , \tag{2.18}$$

the coupled eigenvalue equations take, in light of the mode-by-mode expansion of the action integral,[4,9] the form

$$L_n x = E \widetilde{x} \quad , \quad M_n \widetilde{x} = -E x \quad , \tag{2.19}$$

$$L_n y = E \widetilde{y} \quad , \quad M_n \widetilde{y} = -E y \quad , \tag{2.20}$$

$$L_n X = E \widetilde{X} \quad , \quad M_n \widetilde{X} = -E X \quad , \tag{2.21}$$

$$L_nY = E\widetilde{Y} \quad , \quad M_n\widetilde{Y} = -EY \quad , \tag{2.22}$$

where

$$x \equiv m_{n1}^{(\beta)} \quad , \quad X \equiv m_{n2}^{(\beta)} \quad , \tag{2.23}$$

$$\widetilde{x} \equiv \widetilde{m}_{n1}^{(\beta)} \quad , \quad \widetilde{X} \equiv \widetilde{m}_{n2}^{(\beta)} \quad , \tag{2.24}$$

$$y \equiv r_{n1}^{(\mu)} \quad , \quad Y \equiv r_{n2}^{(\mu)} \quad , \tag{2.25}$$

$$\widetilde{y} \equiv \widetilde{r}_{n1}^{(\mu)} \quad , \quad \widetilde{Y} \equiv \widetilde{r}_{n2}^{(\mu)} \quad . \tag{2.26}$$

We now define $\forall n \geq 0$ the differential operators

$$P_n \equiv \frac{d^2}{d\tau^2} + \left[E^2 - \frac{\left((n+3)^2 - \frac{1}{4}\right)}{\tau^2}\right] \quad , \tag{2.27}$$

$$Q_n \equiv \frac{d^2}{d\tau^2} + \left[E^2 - \frac{\left((n+2)^2 - \frac{1}{4}\right)}{\tau^2}\right] \quad . \tag{2.28}$$

Eqs. (2.19)-(2.22) lead to the following second-order equations, $\forall n \geq 0$:

$$P_n\widetilde{x} = P_n\widetilde{X} = P_n\widetilde{y} = P_n\widetilde{Y} = 0 \quad , \tag{2.29}$$

$$Q_ny = Q_nY = Q_nx = Q_nX = 0 \quad . \tag{2.30}$$

The solutions of (2.29)-(2.30) regular at the origin are

$$\widetilde{x} = C_1\sqrt{\tau}J_{n+3}(E\tau) \quad , \quad \widetilde{X} = C_2\sqrt{\tau}J_{n+3}(E\tau) \quad , \tag{2.31}$$

$$x = C_3\sqrt{\tau}J_{n+2}(E\tau) \quad , \quad X = C_4\sqrt{\tau}J_{n+2}(E\tau) \quad , \tag{2.32}$$

$$\widetilde{y} = C_5 \sqrt{\tau} J_{n+3}(E\tau) \quad , \quad \widetilde{Y} = C_6 \sqrt{\tau} J_{n+3}(E\tau) \quad , \tag{2.33}$$

$$y = C_7 \sqrt{\tau} J_{n+2}(E\tau) \quad , \quad Y = C_8 \sqrt{\tau} J_{n+2}(E\tau) \quad . \tag{2.34}$$

To find the condition obeyed by the eigenvalues E, we now insert (2.31)-(2.34) into the boundary conditions (2.13)-(2.16), taking into account also the first-order system given by (2.19)-(2.22). This gives the eight equations

$$iC_3 J_{n+2}(Ea) = \epsilon \, C_2 J_{n+3}(Ea) \quad , \tag{2.35}$$

$$iC_4 J_{n+2}(Ea) = -\epsilon \, C_1 J_{n+3}(Ea) \quad , \tag{2.36}$$

$$iC_5 J_{n+3}(Ea) = \epsilon \, C_8 J_{n+2}(Ea) \quad , \tag{2.37}$$

$$iC_6 J_{n+3}(Ea) = -\epsilon \, C_7 J_{n+2}(Ea) \quad , \tag{2.38}$$

$$C_1 = - \frac{E C_3 J_{n+2}(Ea)}{\left[E \dot{J}_{n+3}(Ea) + (n+3) \frac{J_{n+3}(Ea)}{a} \right]} \quad , \tag{2.39}$$

$$C_2 = - \frac{E C_4 J_{n+2}(Ea)}{\left[E \dot{J}_{n+3}(Ea) + (n+3) \frac{J_{n+3}(Ea)}{a} \right]} \quad , \tag{2.40}$$

$$C_7 = \frac{E C_5 J_{n+3}(Ea)}{\left[E \dot{J}_{n+2}(Ea) - (n+2) \frac{J_{n+2}(Ea)}{a} \right]} \quad , \tag{2.41}$$

$$C_8 = \frac{E C_6 J_{n+3}(Ea)}{\left[E \dot{J}_{n+2}(Ea) - (n+2) \frac{J_{n+2}(Ea)}{a} \right]} \quad . \tag{2.42}$$

Interestingly, these give separate relations among the constants C_1, C_2, C_3, C_4 and among C_5, C_6, C_7, C_8 [3,9]. For example, eliminating C_1, C_2, C_3, C_4, using (2.35)-(2.36), (2.39)-(2.40) and the useful identities[14]

$$Ea\dot{J}_{n+2}(Ea) - (n+2)J_{n+2}(Ea) = -EaJ_{n+3}(Ea) \quad , \tag{2.43}$$

$$Ea\dot{J}_{n+3}(Ea) + (n+3)J_{n+3}(Ea) = EaJ_{n+2}(Ea) \quad , \tag{2.44}$$

one finds

$$i\epsilon \frac{J_{n+2}(Ea)}{J_{n+3}(Ea)} = \epsilon^2 \frac{C_2}{C_3} = \epsilon^2 \frac{C_4}{C_1} = i\epsilon^3 \frac{J_{n+3}(Ea)}{J_{n+2}(Ea)} \quad , \tag{2.45}$$

which implies (since $\epsilon = \pm 1$)

$$\Big[J_{n+2}(E)\Big]^2 - \Big[J_{n+3}(E)\Big]^2 = 0 \quad , \quad \forall n \geq 0 \quad , \tag{2.46}$$

where we set $a = 1$ for simplicity.

3. Physical-Degrees-of-Freedom Contribution to $\zeta(0)$

The eigenvalue condition (2.46) is very similar to the formula found in Refs. 3,9 for spin $\frac{1}{2}$, i.e. $\Big[J_{n+1}(E)\Big]^2 - \Big[J_{n+2}(E)\Big]^2 = 0, \forall n \geq 0$. Thus, the same technique can be now applied to derive the PDF contribution to $\zeta(0)$ in the case of gravitinos. As we know from Refs. 4,9, the completely symmetric harmonics have degeneracy $d(n) = (n+4)(n+1)$, $\forall n \geq 0$. This is the full degeneracy in the case of local boundary conditions (2.1), since we need twice as many modes to get the same number of eigenvalue conditions as in the spectral case.[3−4,9] The $\zeta(0)$ calculation is now performed using ideas first described in Ref. 15, and then used in Refs. 3,9. Given the zeta-function at large x

$$\zeta(s,x^2) \equiv \sum_{j=1}^{\infty} \Bigl(\lambda_j + x^2\Bigr)^{-s} \quad , \tag{3.1}$$

where $\lambda_j = E^2$ are the squared eigenvalues of the Dirac operator in our case,[3,9] one has in four dimensions

$$\Gamma(3)\zeta(3,x^2) = \int_0^{\infty} T^2 e^{-x^2 T} G(T)\, dT \sim \sum_{l=0}^{\infty} C_l \Gamma\Bigl(1+\frac{l}{2}\Bigr) x^{-l-2} \quad , \tag{3.2}$$

where we have used the asymptotic expansion[9] of the heat kernel for $T \rightarrow 0^+$

$$G(T) \sim \sum_{l=0}^{\infty} C_l T^{\frac{l}{2}-2} \quad . \tag{3.3}$$

On the other hand, defining $m \equiv n+3$, we find[3,9]

$$\Gamma(3)\zeta(3,x^2) = \sum_{m=3}^{\infty} (m+1)(m-2)\biggl(\frac{1}{2x}\frac{d}{dx}\biggr)^3 \log\Bigl[(ix)^{-2(m-1)}\bigl(J_{m-1}^2 - J_m^2\bigr)(ix)\Bigr]$$

$$\sim \sum_{m=0}^{\infty} \bigl(m^2 - m\bigr)\biggl(\frac{1}{2x}\frac{d}{dx}\biggr)^3 \Biggl[\sum_{i=1}^{5} S_i(m,\alpha_m(x))\Biggr]$$

$$+ Z_1 + Z_2 + \sum_{n=5}^{\infty} q_n x^{-2-n} \quad , \tag{3.4}$$

where[3,9]

$$\alpha_m(x) \equiv \sqrt{m^2 + x^2} \quad , \tag{3.5}$$

$$S_1(m,\alpha_m(x)) \equiv -\log(\pi) + 2\alpha_m \quad , \tag{3.6}$$

$$S_2(m,\alpha_m(x)) \equiv -(2m-1)\log(m+\alpha_m) \quad , \tag{3.7}$$

$$S_3(m,\alpha_m(x)) \equiv \sum_{r=0}^{2} k_{1r} m^r \alpha_m^{-r-1} \quad , \tag{3.8}$$

$$S_4(m,\alpha_m(x)) \equiv \sum_{r=0}^{4} k_{2r} m^r \alpha_m^{-r-2} \quad , \tag{3.9}$$

$$S_5(m,\alpha_m(x)) \equiv \sum_{r=0}^{6} k_{3r} m^r \alpha_m^{-r-3} \quad , \tag{3.10}$$

$$Z_1 \equiv -2\sum_{m=0}^{\infty}\left(\frac{1}{2x}\frac{d}{dx}\right)^3\left[\sum_{i=1}^{5} S_i(m,\alpha_m(x))\right] = \sum_{i=1}^{5} X_\infty^{(i)} \quad , \tag{3.11}$$

$$Z_2 \equiv 2\sum_{m=0}^{1}\left(\frac{1}{2x}\frac{d}{dx}\right)^3\left[\sum_{i=1}^{5} S_i(m,\alpha_m(x))\right] = \sum_{i=1}^{5} Y_\infty^{(i)} \quad . \tag{3.12}$$

One can thus obtain $\zeta(0) = C_4$ as half the coefficient of x^{-6} in the asymptotic expansion of the right-hand side of (3.4), by comparison of (3.2) and (3.4), and bearing in mind that[3,9]

$$k_{10} = -\frac{1}{4} \quad , \quad k_{11} = 0 \quad , \quad k_{12} = \frac{1}{12} \quad , \tag{3.13}$$

$$k_{20} = 0 \ , \ k_{21} = -\frac{1}{8} \ , \ k_{22} = k_{23} = \frac{1}{8} \ , \ k_{24} = -\frac{1}{8} \quad , \tag{3.14}$$

$$k_{30} = \frac{5}{192} \ , \ k_{31} = -\frac{1}{8} \ , \ k_{32} = \frac{9}{320} \ , \ k_{33} = \frac{1}{2} \quad , \tag{3.15a}$$

$$k_{34} = -\frac{23}{64} \quad , \quad k_{35} = -\frac{3}{8} \quad , \quad k_{36} = \frac{179}{576} \quad . \tag{3.15b}$$

The PDF $\zeta(0)$ value for spin $\frac{3}{2}$ is thus given by the spin-$\frac{1}{2}$ value first found in Ref. 3 plus the contributions of Z_1 and Z_2. For this purpose, we also use the identities[3,9,15]

$$\left(\frac{1}{2x}\frac{d}{dx}\right)^3 \log\left(\frac{1}{m+\alpha_m}\right) = (m+\alpha_m)^{-3}\left[-\alpha_m^{-3} - \frac{9}{8}m\alpha_m^{-4} - \frac{3}{8}m^2\alpha_m^{-5}\right] \ , \tag{3.16}$$

$$(m+\alpha_m)^{-3} = \frac{(\alpha_m - m)^3}{x^6} \quad . \tag{3.17}$$

The insertion of (3.17) into (3.16) yields[3,9,15]

$$\left(\frac{1}{2x}\frac{d}{dx}\right)^3 \Bigl[-m\log(m+\alpha_m)\Bigr] = -mx^{-6} + m^2x^{-6}\alpha_m^{-1} + \frac{m^2}{2}x^{-4}\alpha_m^{-3} + \frac{3}{8}m^2x^{-2}\alpha_m^{-5} \ . \tag{3.18}$$

This further identity leads to divergences in the calculation, but these are only *fictitious* in light of (3.16). Such fictitious divergences are regularized dividing by α_m^{2s}, summing using the contour formulae[3,9,15]

$$\sum_{m=0}^{\infty} m^{2k}\alpha_m^{-2k-q} = \frac{\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(\frac{q}{2}-\frac{1}{2}\right)}{2\Gamma\left(k+\frac{q}{2}\right)} x^{1-q} \ , \ \forall k = 1,2,3,... \tag{3.19}$$

$$\sum_{m=0}^{\infty} m\alpha_m^{-1-q} \sim \frac{x^{1-q}}{\sqrt{\pi}} \sum_{r=0}^{\infty} \frac{2^r}{r!} B_r x^{-r} \frac{\Gamma\left(\frac{r}{2}+\frac{1}{2}\right)\Gamma\left(\frac{q}{2}-\frac{1}{2}+\frac{r}{2}\right)}{2\Gamma\left(\frac{1}{2}+\frac{q}{2}\right)} \cos\left(\frac{r\pi}{2}\right) \ , \tag{3.20}$$

where B_r are Bernoulli numbers, and then taking the limit $s \rightarrow 0$.[3,9,15]

Indeed, from (3.11) we find

$$X_{\infty}^{(1)} = -\frac{3}{2}\sum_{m=0}^{\infty} \alpha_m^{-5} \quad , \tag{3.21}$$

which does not contain x^{-6} and hence does not contribute to $\zeta(0)$. However, (3.18) and (3.7) imply

$$X_{\infty}^{(2)} = 4x^{-6}\beta_1 - 4x^{-6}\beta_2 - 2x^{-4}\beta_3 - \frac{3}{2}x^{-2}\beta_4 - 2x^{-6}\beta_5 + 2x^{-6}\beta_6 + x^{-4}\beta_7 + \frac{3}{4}x^{-2}\beta_8 \ , \quad (3.22)$$

where

$$\beta_1 \equiv \sum_{m=0}^{\infty} m \quad , \quad (3.23)$$

$$\beta_2 \equiv \sum_{m=0}^{\infty} m^2 \alpha_m^{-1} \quad , \quad (3.24)$$

$$\beta_3 \equiv \sum_{m=0}^{\infty} m^2 \alpha_m^{-3} \quad , \quad (3.25)$$

$$\beta_4 \equiv \sum_{m=0}^{\infty} m^2 \alpha_m^{-5} \quad , \quad (3.26)$$

$$\beta_5 \equiv \lim_{s \to 0} \sum_{m=0}^{\infty} \alpha_m^{-2s} \quad , \quad (3.27)$$

$$\beta_6 \equiv \lim_{s \to 0} \sum_{m=0}^{\infty} m \alpha_m^{-1-2s} \quad , \quad (3.28)$$

$$\beta_7 \equiv \sum_{m=0}^{\infty} m \alpha_m^{-3} \quad , \quad (3.29)$$

$$\beta_8 \equiv \sum_{m=0}^{\infty} m \alpha_m^{-5} \quad . \quad (3.30)$$

Note that only β_1 and β_5 contribute to $\zeta(0)$. This is proved using (3.19)-(3.20) and the Euler-Maclaurin formula. According to this algorithm, if f is a real- or complex-valued function defined on $0 \leq t \leq \infty$, and if $f^{(2m)}(t)$ is absolutely integrable on $(0, \infty)$ then, for $u = 1, 2, ...$ [3,9,16]

$$\sum_{i=0}^{u} f(i) - \int_0^u f(x)\, dx = \frac{1}{2}\Big[f(0) + f(u)\Big] + \sum_{s=1}^{m-1} \frac{B_{2s}}{(2s)!}\Big[f^{(2s-1)}(u) - f^{(2s-1)}(0)\Big] + R_m(u) \; , \tag{3.31}$$

where the remainder R_m satisfies the inequality

$$\mid R_m(u) \mid \leq \Big(2 - 2^{1-2m}\Big)\frac{\mid B_{2m} \mid}{(2m)!} \int_0^u \mid f^{(2m)}(x) \mid \; dx \quad . \tag{3.32}$$

The asymptotic expansion (3.20) implies that β_1 gives the contribution

$$\delta^{(a)} = 2\cos(\pi)\frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} B_2 = -\frac{1}{6} \quad , \tag{3.33}$$

and the Euler-Maclaurin formula shows that β_5 contributes

$$\delta^{(b)} = -\frac{1}{2} \quad . \tag{3.34}$$

By virtue of (3.13), (3.8) and (3.11), we also find that

$$X_{\infty}^{(3)} = \frac{15}{4} k_{10} \sum_{m=0}^{\infty} \alpha_m^{-7} + \frac{105}{4} k_{12} \sum_{m=0}^{\infty} m^2 \alpha_m^{-9} \quad . \tag{3.35}$$

Thus, using (3.19) and (3.13), we derive the following contribution to $\zeta(0)$:

$$\delta^{(c)} = (k_{10} + k_{12}) = -\frac{1}{6} \quad . \tag{3.36}$$

Finally, using (3.9)-(3.11) we obtain

$$X_{\infty}^{(4)} = \frac{1}{4}\sum_{r=0}^{4} k_{2r}(r+2)(r+4)(r+6)\left[\sum_{m=0}^{\infty} m^r \alpha_m^{-r-8}\right] \quad , \tag{3.37}$$

$$X_{\infty}^{(5)} = \frac{1}{4}\sum_{r=0}^{6} k_{3r}(r+3)(r+5)(r+7)\left[\sum_{m=0}^{\infty} m^r \alpha_m^{-r-9}\right] \quad , \tag{3.38}$$

and in light of (3.19)-(3.20) we derive that the asymptotic behaviour of $X_{\infty}^{(4)}$ is $\mathrm{O}(x^{-7})$, and the asymptotic form of $X_{\infty}^{(5)}$ is $\mathrm{O}(x^{-8})$. Thus, they do not affect the $\zeta(0)$ value.

Moreover, the whole of Z_2 (cf (3.12)) does not affect $\zeta(0)$. In fact one finds

$$Y_{\infty}^{(1)} = \frac{3}{2}x^{-5}\left[1+\left(1+x^{-2}\right)^{-\frac{5}{2}}\right] \quad , \tag{3.39}$$

$$Y_{\infty}^{(2)} = 2x^{-7}\left(1+x^{-2}\right)^{-\frac{1}{2}} + x^{-7}\left(1+x^{-2}\right)^{-\frac{3}{2}} + \frac{3}{4}x^{-7}\left(1+x^{-2}\right)^{-\frac{5}{2}} \quad , \tag{3.40}$$

$$Y_{\infty}^{(3)} = -\frac{15}{4}k_{10}x^{-7}\left[1+\left(1+x^{-2}\right)^{-\frac{7}{2}}\right] - \frac{105}{4}k_{12}x^{-9}\left(1+x^{-2}\right)^{-\frac{9}{2}} \quad , \tag{3.41}$$

$$Y_{\infty}^{(4)} = -\frac{1}{4}\sum_{r=1}^{4} x^{-r-8}k_{2r}(r+2)(r+4)(r+6)\left(1+x^{-2}\right)^{-\frac{r}{2}-4} \quad , \tag{3.42}$$

$$Y_{\infty}^{(5)} = -\frac{105}{4}k_{30}x^{-9} - \frac{1}{4}\sum_{r=0}^{6} x^{-r-9}k_{3r}(r+3)(r+5)(r+7)\left(1+x^{-2}\right)^{-\frac{r}{2}-\frac{9}{2}} \quad , \tag{3.43}$$

and the reader can now easily see that the formulae (3.39)-(3.43) do not contain terms proportional to x^{-6}.

At the end, we have to consider more carefully the effect of higher-order terms in the asymptotic expansion of $\log\Bigr[(ix)^{-2(m-1)}\Bigr(J_{m-1}^2 - J_m^2\Bigr)(ix)\Bigr]$. In light of Refs. 3,9 and of Eqs. (3.4)-(3.12) we study, $\forall n > 3$

$$\widetilde{H}_{\infty}^{n,A} \equiv -\frac{1}{4}\sum_{p=1}^{l} h_{np} \sum_{m=0}^{\infty} \Bigr[a_{np}\alpha_m^{p-n-6}(m+\alpha_m)^{-p} + b_{np}\alpha_m^{p-n-5}(m+\alpha_m)^{-p-1} + c_{np}\alpha_m^{p-n-4}(m+\alpha_m)^{-p-2} + d_{np}\alpha_m^{p-n-3}(m+\alpha_m)^{-p-3}\Bigr] \quad , \tag{3.44}$$

$$\widetilde{H}_{\infty}^{n,B} \equiv \frac{1}{4}\sum_{r=0}^{2n} k_{nr}(r+n)(r+n+2)(r+n+4) \sum_{m=0}^{\infty} m^r \alpha_m^{-r-n-6} \quad , \tag{3.45}$$

$$\widetilde{H}_{\infty}^{n,C} \equiv \frac{1}{4}\sum_{p=1}^{l} h_{np} \sum_{m=0}^{1} \Bigr[a_{np}\alpha_m^{p-n-6}(m+\alpha_m)^{-p} + b_{np}\alpha_m^{p-n-5}(m+\alpha_m)^{-p-1} + c_{np}\alpha_m^{p-n-4}(m+\alpha_m)^{-p-2} + d_{np}\alpha_m^{p-n-3}(m+\alpha_m)^{-p-3}\Bigr] \quad , \tag{3.46}$$

$$\widetilde{H}_{\infty}^{n,D} \equiv -\frac{1}{4}\sum_{r=0}^{2n} k_{nr}(r+n)(r+n+2)(r+n+4) \sum_{m=0}^{1} m^r \alpha_m^{-r-n-6} \quad , \tag{3.47}$$

where $a_{np}, b_{np}, c_{np}, d_{np}, h_{np}$ are constant coefficients. In (3.44)-(3.47), n should not be confused with the integer appearing in (2.46) and in the definition of m. Again, the Euler-Maclaurin formula is very useful in studying $\widetilde{H}_{\infty}^{n,A}$. The equivalent of $f(0)$ in (3.31) gives a contribution proportional to x^{-n-6}. Bernoulli numbers and derivatives of odd order give a contribution proportional to x^{-n-7} plus higher-order terms. The conversion of (3.44) into an integral yields a term proportional to x^{-n-5}, as it is evident studying the integrals

$$\widetilde{I}_1^{(np)} \equiv \int_0^{\infty} \Bigr(y + \sqrt{x^2+y^2}\Bigr)^{-p} \Bigr(x^2+y^2\Bigr)^{\frac{p}{2}-\frac{n}{2}-3} \, dy \quad , \tag{3.48}$$

$$\widetilde{I}_2^{(np)} \equiv \int_0^\infty \left(y+\sqrt{x^2+y^2}\right)^{-p-1} \left(x^2+y^2\right)^{\frac{p}{2}-\frac{n}{2}-\frac{5}{2}} dy \quad , \tag{3.49}$$

$$\widetilde{I}_3^{(np)} \equiv \int_0^\infty \left(y+\sqrt{x^2+y^2}\right)^{-p-2} \left(x^2+y^2\right)^{\frac{p}{2}-\frac{n}{2}-2} dy \quad , \tag{3.50}$$

$$\widetilde{I}_4^{(np)} \equiv \int_0^\infty \left(y+\sqrt{x^2+y^2}\right)^{-p-3} \left(x^2+y^2\right)^{\frac{p}{2}-\frac{n}{2}-\frac{3}{2}} dy \quad . \tag{3.51}$$

The effect of $\widetilde{H}_\infty^{n,B}$ is derived by using (3.19)-(3.20). When $r=0$ we have to consider $\sum_{m=0}^\infty \alpha_m^{-n-6}$, which does not contain x^{-6}. When $r=2k>0$, (3.19) leads to a contribution proportional to x^{-n-5}, and when $r=2k+1$, (3.20) leads to a contribution proportional to x^{-n-5} plus higher-order terms. One also finds that

$$\begin{aligned}\widetilde{H}_\infty^{n,C} = \frac{x^{-n-6}}{4} \sum_{p=1}^{l} h_{np} \Big[& (a_{np}+b_{np}+c_{np}+d_{np}) \\ & + a_{np}\left(1+x^{-2}\right)^{\frac{p}{2}-\frac{n}{2}-3} \left(x^{-1}+\sqrt{1+x^{-2}}\right)^{-p} \\ & + b_{np}\left(1+x^{-2}\right)^{\frac{p}{2}-\frac{n}{2}-\frac{5}{2}} \left(x^{-1}+\sqrt{1+x^{-2}}\right)^{-p-1} \\ & + c_{np}\left(1+x^{-2}\right)^{\frac{p}{2}-\frac{n}{2}-2} \left(x^{-1}+\sqrt{1+x^{-2}}\right)^{-p-2} \\ & + d_{np}\left(1+x^{-2}\right)^{\frac{p}{2}-\frac{n}{2}-\frac{3}{2}} \left(x^{-1}+\sqrt{1+x^{-2}}\right)^{-p-3} \Big] \quad ,\end{aligned} \tag{3.52}$$

$$\begin{aligned}\widetilde{H}_\infty^{n,D} = & -\frac{1}{4}k_{n0}\; n(n+2)(n+4)x^{-n-6} \left[1+\left(1+x^{-2}\right)^{-\frac{n}{2}-3}\right] \\ & -\frac{1}{4}\sum_{r=1}^{2n} k_{nr}(r+n)(r+n+2)(r+n+4)x^{-r-n-6}\left(1+x^{-2}\right)^{-\frac{r}{2}-\frac{n}{2}-3} .\end{aligned} \tag{3.53}$$

This is why $\widetilde{H}_{\infty}^{n,A}$, $\widetilde{H}_{\infty}^{n,B}$, $\widetilde{H}_{\infty}^{n,C}$ and $\widetilde{H}_{\infty}^{n,D}$ do not contain terms proportional to x^{-6}, and hence do not contribute to $\zeta(0)$.

To sum up, in light of (3.4), (3.33)-(3.34), (3.36), (3.44)-(3.47), and using the $\zeta(0)$ value obtained in Ref. 3, we find

$$\zeta(0) = \frac{11}{360} - \frac{5}{6} = -\frac{289}{360} \quad , \tag{3.54}$$

which is equal to the PDF value found in Ref. 4 when one sets to zero on S^3 all untwiddled coefficients of ψ_i^A and $\widetilde{\psi}_i^{A'}$. However, as shown in Ref. 10, $\zeta(0)$ values depend on the boundary conditions if Majorana fermions and gravitinos are massive.

4. Concluding Remarks

The calculation appearing in our paper was not performed explicitly in Refs. 5,10, and was only available in Ref. 9. We have therefore tried to present it in a self-contained way in this journal, to make it accessible to a wider audience. Interestingly, if the gauge constraints (1.2) and supersymmetry constraints are imposed *before* quantization, the PDF value is found to be $\zeta^{(PDF)}(0) = -\frac{289}{360}$. However, Becchi-Rouet-Stora-Tyutin-invariant quantization techniques might lead to different $\zeta(0)$ values. This is indeed what happens in Ref. 2, where, studying the effect of ghost fields and gauge degrees of freedom, the author finds $\zeta_{\frac{3}{2}}(0) = \frac{197}{180}$. In this case the difference with respect to the PDF value (3.54) is substantial, at least because the signs are opposite. However, one should bear in mind that the discrepancy found in Ref. 3 for the spin-$\frac{1}{2}$ result also affects the spin-$\frac{3}{2}$ calculation.

Moreover, it is also worth remarking that in Ref. 2 the gravitino contribution to $\zeta(0)$ in simple supergravity makes the one-loop amplitude even more divergent, when perturbative modes for the three-metric are set to zero on S^3. By contrast, within the PDF approach, the gravitino contribution to $\zeta(0)$ in $N = 1$ supergravity partially cancels the contribution of the gravitational field in such a case.

Our result (3.54) may not only add evidence in favour of different quantization techniques for gauge fields being inequivalent, but remains of some value if a mode-by-mode gauge-invariant $\zeta(0)$ calculation is performed. In that case, the physical degrees of freedom decouple from gauge and ghost modes, so that their contribution to $\zeta(0)$ is again given by equation (3.54) if the boundary conditions (2.1) are required. Unfortunately, already in the simpler case of Euclidean Maxwell theory in four dimensions, gauge modes are then found to obey a very complicated set of coupled eigenvalue equations, and it is not yet clear how to evaluate their contribution to the full $\zeta(0)$ value in a mode-by-mode analysis.[9] If this last technical problem could be solved, one would then obtain a very relevant check of $\zeta(0)$ values for gauge fields in the presence of boundaries previously found in the literature. Of course, supergravity multiplets cannot be studied at one-loop about flat Euclidean four-space, since the existence of a cosmological constant is incompatible with a flat background geometry.[9] However, we hope that the calculations in our paper (see also Ref. 10) can be used as a first step towards a mode-by-mode perturbative analysis in the presence of curved backgrounds, at least in the limit of small boundary three-geometry.[9,17]

A further interesting question, arising from the work in Refs. 9,18-19, is whether local boundary conditions involving *field strengths* rather than potentials can be used for spin

$\frac{3}{2}$. It is not yet clear whether, and eventually how, the corresponding one-loop calculation might be performed.

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