Interaction Hierarchy.
Gonihedric String and Quantum Gravity

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Abstract

We have found that the Regge gravity [1, 2], can be represented as a *superposition* of less complicated theory of random surfaces with *Euler character* as an action. This extends to Regge gravity our previous result [6], which allows to represent the gonihedric string [7] as a superposition of less complicated theory of random paths with *curvature* action. We propose also an alternative linear action $A(M_4)$ for the four and high dimensional quantum gravity, which allows the same representation. From these representations it follows that the corresponding partition functions are equal to the product of Feynman path integrals evaluated on time slices with curvature and length action for the gonihedric string and with Euler character and gonihedric action for the Regge gravity. In both cases the interaction is proportional to the overlapping sizes of the paths or surfaces on the neighboring time slices. On the lattice we constructed spin system with local interaction, which have the same partition function as the quantum gravity. The scaling limit is discussed.

1. In our previous article [6] , devoted to the hierarchical structure of the geometrical interactions, we observed that the physical theories can be considered as a *superposition* of less complicated, primary interactions.

The example on which we have been based is the regularized string theory with linear action $A(M_2)$ [7]. This string can be considered as a superposition of less complicated theory of random paths $\{M_1\}$ with an amplitude which is proportional to the total curvature of the path $k(M_1)$ [9].

In the present article we will extend this result and will show that the regularized quantum gravity [1, 2, 3, 4, 5] can be represented as a superposition of the random surfaces $\{M_2\}$ with an action which is equal to the Euler character of the surfaces $\chi(M_2)$. We propose an alternative linear action $A(M_4)$ for the four and high dimensional quantum gravity, which allows the same representation.

In the next section we present the main ideas and results for the regularized string with linear action $A(M_2)$ and in the subsequent sections we will extend this result to the quantum gravity.

1.1 The regularized string with linear-gonihedric action $A(M_2)$ can be derived from natural physical requirements such as [7] : α) coincidence of the string transition amplitude with the usual Feynman path amplitude for long space-time strips and from β) the continuity principle for the transition amplitudes.

The partition function of the regularized linear string has the form [7]

$$Z_{Gonihedric}(\beta) = \sum_{\{M_2\}} \exp\{-\beta A(M_2)\}, \qquad A(M_2) = \sum_{<i,j>} \lambda_{i,j} \cdot \Theta(\alpha_{ij}), \quad (1)$$

where $\Theta(\alpha_{ij}) = |\pi - \alpha_{i,j}|$ and the summation is extended over all triangulated surfaces $\{M_2\}$ with the linear action $A(M_2)$, and $\alpha_{i,j}$ is the dihedral angle between two neighbor faces of M_2 having a common edge $< i,j >$ of the length $\lambda_{i,j}$. The regularized string (1) is well defined in any dimensions and for an arbitrary topology of the surface M_2.

The linear string (1) can be viewed as a superposition of less complicated, primary theory of random paths $\{M_1\}$ with the curvature action $k(M_1)$. This structure of the linear string (1) comes from the representation of the linear action $A(M_2)$ in terms of the total curvature $k(M_1^E)$ of the paths $\{M_1^E\}$ which appear in the intersection of the plane E with the given surface M_2 [9]

$$A(M_2) = \sum_{\{E\}} k(M_1^E). \qquad (2)$$

In the last formula the paths in the intersection are denoted by $\{M_1^E\}$

$$M_1^E = M_2 \cap E \qquad (3)$$

and the absolute total curvature $k(M_1^E)$ of the path M_1^E is equal to

$$k(M_1^E) = \sum_{<i,j>} |\pi - \alpha_{ij}^E|, \qquad (4)$$

where α_{ij}^E is the dihedral angle in the intersection of the plane E with the edge $< i,j >$ and $\alpha_{ij}^E = \pi$ for the edges of M_2 which are not intersected by the given plane E.

With (2) the partition function of the system (1) can be written in the form

$$Z_{Gonihedric}(\beta) = \sum_{\{M_2\}} \prod_{\{E\}} exp\{-\beta \; k(M_1^E) \; \}. \tag{5}$$

When the continuous Euclidean space is replaced by the Euclidean lattice, where the paths and the surfaces are associated with the collection of the links and plaquettes, then in the last formula the product over all intersecting planes $\{E\}$ can be evaluated to a product over planes $\{E^\tau\}$ which are perpendicular to a given time direction τ [6]

$$Z_{Gonihedric}(\beta) = \sum_{\{..M_1^\tau, M_1^{\tau+1}..\}} \prod_\tau K(M_1^\tau, M_1^{\tau+1}), \tag{6}$$

where

$$K(M_1^\tau, M_1^{\tau+1}) = exp - \beta\{\frac{1}{2}k(M_1^\tau) + l(M_1^\tau) + \frac{1}{2}k(M_1^{\tau+1}) + l(M_1^{\tau+1}) - 2l(M_1^\tau \cap M_1^{\tau+1}) \; \}$$
$$\tag{7}$$

and the *independent* summation is extended over all paths $\{..M_1^\tau, M_1^{\tau+1}..\}$ on diferent time slices. This result is valid if the self-intersection coupling constant k [7] is equal to infinity [13], that is for $self - avoiding$ surfaces. In three dimensions it is valid also for the case when $k = 0$ [6].

We have the propagation of the path M_1^τ in the time direction τ with an amplitude which is proportional to the sum of the curvature $k(M_1^\tau)$ and of the length of the path $l(M_1^\tau)$, and the interaction which is proportional to the overlapping length of the paths on the neighboring time slices $l(M_1^\tau \cap M_1^{\tau+1})$. The advantage of this formula is that one can consider this interaction as a perturbation, or as the *hopping* term in the lattice language, because it describes the hopping from one time slice to another [6]. In that case the tree approximation is associated with the length and curvature terms in (7) and describes the $free \; fermion$ on a given time slice. The separate path integral which describes the fermion on a given time slice

$$\sum_{\{M_1^\tau\}} exp\{-\beta \; \{\frac{1}{2} \; k(M_1^\tau) + l(M_1^\tau)\} \; \} = < exp\{-H_{Gonihedric}^{tree}\} > \tag{8}$$

has been already computed in [6] by Kac-Ward method [16, 17].

From the last formulas one can conclude that the string partition function is equal to the limit of the infinite product of the Dirac-Feynman path integrals which are evaluated on the time slices $\{E^\tau\}$ $\tau = a, 1, ..., N, b$. One can say that the gonihedric string (1) is a superposition of weakly interacting Dirac particles which "jumps" from one time slice to another thanks to the hopping interaction [6]. This result demonstrates how the infinite sandwich of primary theories can generate the physical theory.

The equivalent statistical system for which (6),(7) is an exact expression for the partition function has been constructed in [12, 13, 15]. This equivalence allows to simulate the linear string (1) on the lattice and, as we will see later on, the quantum gravity as well.

2.Let us consider three dimensional manifold $\{M_3\}$ which is constructed by gluing together three-dimensional tetrahedra through their triangular faces. This manifold can be associated with the three-dimensional gravity [1, 2] or with the motion of the two-dimensional membrane [10].

To construct the three dimensional quantum gravity with the linear action $A(M_3)$ and to generate the next species in the hierarchy of the geometrical interactions we should apply the principles $\alpha)$ and $\beta)$. In accordance with $\alpha)$ the quantum mechanical amplitude should be proportional to the linear size of the manifold and thus it must be proportional to the linear combination of the lengths of all edges of tetrahedrated manifold M_3

$$A(M_3) = \sum_{<i,j>} \lambda_{ij} \cdot \Theta_{ij} \tag{9}$$

where λ_{ij} is the length of the edge between two vertices $< i >$ and $< j >$, summation is over all edges $< i, j >$ and Θ_{ij} is unknown factor, which can be defined by use of the continuity principle $\beta)$. Indeed, if we impose a new vertex $< m >$ inside a given $flat$ tetrahedron $< ijkl >$, then for that new manifold we will get an extra contribution $\lambda_{im}\Theta_{im} + \lambda_{jm}\Theta_{jm} + \lambda_{km}\Theta_{km} + \lambda_{lm}\Theta_{lm}$ to the action and we will get more extra terms imposing a new vertices, despite the fact that the manifold does not actually changed. To exclude such tape of contributions we should choose unknown factor Θ_{ij} such that it will vanish in $flat$ cases. This can be done by use of the dihedral angles, therefore

$$A(M_3) = \sum_{<i,j>} \lambda_{ij} \cdot \Theta(\alpha_{ij} + \beta_{ij} + ...), \tag{10}$$

where

$$\Theta(2\pi) = 0, \tag{11}$$

and $\alpha_{ij}, \beta_{ij}...$ are the dihedral angles between triangular faces of tetrahedra which have the common edge $< i, j >$. These angles appear in the normal section of the edge $< i, j >$ with the $d - 1$ dimensional plane E. In analogy with the gonihedric string [7] one can also define more specific theories with the property that

$$\Theta(4\pi - \omega) = \Theta(\omega), \qquad \Theta(\omega) \geq 0 \tag{12}$$

As we will see bellow this generalization will allow to extend the string results to quantum gravity.

The essential difference with the theory of random surfaces with the linear action $A(M_2)$ is that now the linear functional $A(M_3)$ is an $intrinsic$ quantity, because dihedral angles $\alpha_{ij}, \beta_{ij}...$ are well defined without any embedding.

Generally speaking, a suitable selection of the factor Θ can be done if we require a convenient scaling behavior of the theory [7]. We will use the following parametrization of the $\Theta(\omega)$

$$\Theta(\omega) = (2\pi - \omega)^\varsigma, \tag{13}$$

which for the case $\varsigma = 1$ coincides with the Regge action [1] and is the discrete version of the following continuous Hilbert-Einstein action

$$A(M_3) = \int_{M_3} R \, dv_3. \tag{14}$$

Our aim is now to represent the three dimensional gravity as a *superposition* of less complicated geometrical theory of random surfaces with Euler character as an action

$$\chi(M_2) = \sum_{<i,j>} \left(2\pi - \alpha_{ij} - \beta_{ij} - ...\right). \tag{15}$$

In the last particular case $\varsigma = 1$ and $\Theta \equiv \chi$.

Let us consider the intersection of the tetrahedrated manifold M_3 by the $d-1$ dimensional plane E. The intersection is the two-dimensional surface M_2^E

$$M_2^E = M_3 \cap E \tag{16}$$

and as we will see in a moment the linear action $A(M_3)$ is the sum of Euler characters of all surfaces which appear in the intersection $\{M_2^E\}$

$$A(M_3) = \sum_{\{E\}} \chi(M_2^E), \tag{17}$$

where

$$\chi(M_2^E) = \sum_{<i,j>} \left(2\pi - \alpha_{ij}^E - \beta_{ij}^E - ...\right), \tag{18}$$

and α_{ij}^E, β_{ij}^E, ... are the angles in the intersection of the plane E with the edge $<i,j>$ and $\omega_{ij}^E = \alpha_{ij}^E + \beta_{ij}^E + ... = 2\pi$ for the edges of M_3 which are not intersected by the given plane E.

The formula (17) follows from the fact that the average of the angle α_{ij}^E over all intersecting planes $\{E\}$ is equal to dihedral angle α_{ij} [9]

$$< \alpha_{ij}^E > \equiv \int \alpha_{ij}^E \, dE = \alpha_{ij} \cdot \lambda_{ij} \tag{19}$$

and that the number of planes which intersect the given edge $<i,j>$ is proportional to it's length λ_{ij} .

Therefore in the same way as for the linear string $A(M_2)$ we have the following representation of the partition function of the three dimensional quantum gravity

$$Z_{Gravity}(\beta) = \sum_{\{M_3\}} \prod_{\{E\}} exp\{-\beta \, \chi(M_2^E) \}. \tag{20}$$

When the continuous Euclidean space is replaced by the Euclidean lattice, where the surfaces and the manifolds are associated with the collection of the plaquettes and cubes, then in the last formula the product over all intersecting planes $\{E\}$ can be evaluated to a product over planes $\{E^\tau\}$ which are perpendicular to a given time direction τ

$$Z_{Gravity}(\beta) = \sum_{\{..M_2^\tau, M_2^{\tau+1}..\}} \prod_\tau K(M_2^\tau, M_2^{\tau+1}), \tag{21}$$

where

$$K(M_2^\tau, M_2^{\tau+1}) = exp - \beta \{ \frac{1}{2}\Theta(M_2^\tau) + A(M_2^\tau) + \frac{1}{2}\Theta(M_2^{\tau+1}) + A(M_2^{\tau+1}) - 2A(M_2^\tau \cap M_2^{\tau+1}) \}$$
(22)

and the *independent* summation is extended over all surfaces $\{..M_2^\tau, M_2^{\tau+1}..\}$ on different time slices. This formula is valid when

$$\Theta(\omega) = |2\pi - \omega|$$
(23)

and self-intersection coupling constant k is equal to infinity [13]. For three-dimensional universe M_3 self-intersection happens when more than two 3d cubes have been glued face-to-face. To prove this formula one should consider case by case all seven vertex configuration $\theta_1, ..., \theta_7$ which remain in the limit $k \to \infty$ (see bellow).

We have the propagation of the surface M_2^τ in the time direction τ with an amplitude which is proportional to the sum of the generalized Euler character $\Theta(M_2^\tau)$ (23) and of the linear size of the surface $A(M_2^\tau)$ (1), and the interaction which is proportional to the length of the right angle edges of the overlapping surface $A(M_2^\tau \cap M_2^{\tau+1})$.

From the last formulas we conclude that the partition function for the quantum gravity is equal to the limit of the infinite product of the linear string path integrals (21),(22) which are evaluated on every time slices $\tau = a, 1, ..., N, b$.

2.1 Let us consider now four dimensional manifold M_4. Using the same principles we can define the linear action $A(M_4)$ for the four dimensinal quantum gravity as

$$A(M_4) = \sum_{<i,j>} \lambda_{ij} \cdot \Theta \{ \sum_q (2\pi - \alpha_{ij}^q - \beta_{ij}^q - ...) \}$$
(24)

where the first summation is extended over all edges of M_4 and the second summation is over all $d-2$ dimensional normal sections of the given edge $< i, j >$, and α_{ij}^q, β_{ij}^q, ... are dihedral angles on the $q'th$ section. Geometricaly the last factor is equal to the total area of the polyhedron on S^3 which corresponds to the spherical image of the edge $< i, j >$. The same arguments as in the previous section allow to find out that

$$A(M_4) = \sum_{\{E\}} \chi(M_2^E),$$
(25)

where the summation is extended over all $d-2$ dimensiona planes $\{E\}$. With this result we have the same representation (20) for the partition function of the four dimensional Quantum Gravity. This linear theory is again *intrinsic* and have better chances to describe quantum gravity than the standart action [1], which is proportional to the area of the four dimensional universe M_4. In the next section we will construct the spin systems which have the equivalent partition functions.

3. The correspondence between the spin configurations and the geometry of interface allows to define different theories of random manifolds on a lattice [11, 12, 16]. In the recent articles [12, 13, 15] the authors have introduced a spin statistical system on the lattice, the *low* and *high* temperature expansion of which generates random walks $\{M_1\}$, with the amplitude which is proportional to a total curvature

$k(M_1)$ of the paths on the two-dimensional lattice and to the linear size of the surface $A(M_2)$ (1) on three-dimensional lattice. This two spin systems are an example of the primary theory and of the physical theory which is an infinite supperposition of the first one, and allows therefore the reach phase structure [6]. In three dimensions the corresponding Hamiltonian is [12, 13, 15]

$$H^{3d}_{gonihedric}(k) = k \cdot H^{3d}_{self-intersections} + H^{3d}_{gonihedric}(0)$$

$$= -2k \sum_{\vec{r},\vec{\alpha}} \sigma_{\vec{r}}\sigma_{\vec{r}+\vec{\alpha}} + \frac{k}{2} \sum_{\vec{r},\vec{\alpha},\vec{\beta}} \sigma_{\vec{r}}\sigma_{\vec{r}+\vec{\alpha}+\vec{\beta}} - \frac{1-k}{2} \sum_{\vec{r},\vec{\alpha},\vec{\beta}} \sigma_{\vec{r}}\sigma_{\vec{r}+\vec{\alpha}}\sigma_{\vec{r}+\vec{\alpha}+\vec{\beta}}\sigma_{\vec{r}+\vec{\beta}}, \qquad (26)$$

and the low temperature expansion of the partition function is equal to:

$$Z(\beta) = \sum_{\{\sigma\}} exp(-\beta H^{3d}_{gonihedric}) \qquad = \sum_{\{M_2\}} exp(-2\beta A(M_2)) \qquad (27)$$

In this lattice implementation of the linear string $A(M_2)$ the corresponding Hamiltonian (26) depends on the self-intersection coupling constant k and the system simplifies in the supersymmetric point where the self-intersection coupling constant is equal to zero $k = 0$ [13, 15]

$$H^{3d}_{gonihedric}(0) = -\frac{1}{2} \sum_{\vec{r},\vec{\alpha},\vec{\beta}} \sigma_{\vec{r}}\sigma_{\vec{r}+\vec{\alpha}}\sigma_{\vec{r}+\vec{\alpha}+\vec{\beta}}\sigma_{\vec{r}+\vec{\beta}}. \qquad (28)$$

The system (28) is highly symmetric, because one can independently flip spins on any combination of planes (spin layers) of the lattice N^d. The degeneracy of the vacuum state is equal to 2^{dN} and allows to construct the dual Hamiltonian [6, 14, 15]. String path integral (6),(7) which we discussed in the previous section is an exact expression for the spin system with the Hamiltonian (28) and allows to predict the second order phase transition in 3d which should be of the same nature as it is in the case of the 2d Ising ferromagnet [6]. This system has well separated vacuum states with nonzero generalized magnetization at low temperature and symmetric state at high temperature.

3.1 In this section our aim is to construct the lattice spin system with the low temperature partition function which is equal to the partition function of the random surfaces with Euler character. This will allow to construct the spin system which simulates quantum gravity with the linear action $A(M_3)$ and $A(M_4)$ on the lattice.

The Hamiltonian which contains eight different types of spin interactions inside the 3d cube can be written in the form

$$H^{3d} = a \cdot \sum_{\vec{r},\vec{\alpha},\vec{\beta},\vec{\gamma}} \sigma_{\vec{r}}\sigma_{\vec{r}+\vec{\alpha}}\sigma_{\vec{r}+\vec{\alpha}+\vec{\beta}}\sigma_{\vec{r}+\vec{\beta}}\sigma_{\vec{r}+\vec{\gamma}}\sigma_{\vec{r}+\vec{\gamma}+\vec{\alpha}}\sigma_{\vec{r}+\vec{\gamma}+\vec{\alpha}+\vec{\beta}}\sigma_{\vec{r}+\vec{\gamma}+\vec{\beta}}$$

$$+g \cdot \sum_{\vec{r},\vec{\alpha},\vec{\beta},\vec{\gamma}} \sigma_{\vec{r}}\sigma_{\vec{r}+\vec{\gamma}}\sigma_{\vec{r}+\vec{\gamma}+\vec{\alpha}}\sigma_{\vec{r}+\vec{\gamma}+\vec{\alpha}+\vec{\beta}}\sigma_{\vec{r}+\vec{\alpha}+\vec{\beta}}\sigma_{\vec{r}+\vec{\beta}}$$

$$+b \cdot \sum_{\vec{r},\vec{\alpha},\vec{\beta},\vec{\gamma}} \sigma_{\vec{r}}\sigma_{\vec{r}+\vec{\gamma}}\sigma_{\vec{r}+\vec{\alpha}+\vec{\beta}}\sigma_{\vec{r}+\vec{\gamma}+\vec{\alpha}+\vec{\beta}} + 2c \cdot \sum_{\vec{r},\vec{\alpha},\vec{\beta}} \sigma_{\vec{r}}\sigma_{\vec{r}+\vec{\alpha}}\sigma_{\vec{r}+\vec{\alpha}+\vec{\beta}}\sigma_{\vec{r}+\vec{\beta}} + e \cdot \sum_{\vec{r},\vec{\alpha},\vec{\beta},\vec{\gamma}} \sigma_{\vec{r}}\sigma_{\vec{r}+\vec{\alpha}}\sigma_{\vec{r}+\vec{\beta}}\sigma_{\vec{r}+\vec{\gamma}}$$

$$2d \cdot \sum_{\vec{r},\vec{\alpha},\vec{\beta}} \sigma_{\vec{r}}\sigma_{\vec{r}+\vec{\alpha}+\vec{\beta}} + h \cdot \sum_{\vec{r},\vec{\alpha},\vec{\beta},\vec{\gamma}} \sigma_{\vec{r}}\sigma_{\vec{r}+\vec{\alpha}+\vec{\beta}+\vec{\gamma}} + 4f \cdot \sum_{\vec{r},\vec{\alpha}} \sigma_{\vec{r}}\sigma_{\vec{r}+\vec{\alpha}} \qquad (29)$$

where the coupling constants a, g, b, c, e, d, h and f describe eight spin, six spin (without main diagonal), four diagonal spin, four spin (plaquette), four spin (around cube vertex), two diagonal spin, two spin (main diagonal), and usual direct two spin interactions terms. It is convenient to consider the part of the Hamiltonian which belongs to a given 3d cube

$$H^{3d}_{cube} = a \cdot \sigma\sigma\sigma\sigma\sigma\sigma\sigma\sigma + g \cdot \sigma\sigma\sigma\sigma\sigma\sigma (four\ zigzag\ terms)$$

$$+ b \cdot \sigma\sigma\sigma\sigma (six\ diagonal) + c \cdot \sigma\sigma\sigma\sigma (six\ square) + e \cdot \sigma\sigma\sigma\sigma (eight\ terms\ around\ vertex)$$

$$+ d \cdot \sigma\sigma (twelve\ diagonal) + h \cdot \sigma\sigma (four\ main\ diagonal\ terms) + f \cdot \sigma\sigma (twelve\ direct). \tag{30}$$

There are thirteen different configurations of the interface in 3d cube first seven of which are shown on Fig.1. The corresponding seven basic vertex curvature are equal to

$$\theta_1 = -a + 6d + 6f + 2h - 2g, \quad \theta_2 = a - 2b + 2c + 4f, \quad \theta_3 = -a - 2d + 2f - 2h + 2g,$$

$$\theta_4 = a + 6b + 6c - 4d + 4f - 8e - 4h - 4g, \quad \theta_5 = a + 6b - 6c + 8e - 4h - 4g, \quad \theta_6 = a - 2b - 2c - 4d,$$

$$\theta_7 = a + 6b - 6c - 8e + 4h + 4g, \tag{31}$$

and six vertices with self-intersections are equal to $\theta_8 = a - 2b - 2c + 4d$, $\theta_9 = a - 2b + 2c - 4f$, $\theta_{10} = -a - 2d - 2f + 2h - 2g$, $\theta_{11} = -a + 6d - 6f - 2h + 2g$, $\theta_{12} = a + 6b + 6c - 4d - 4f + 8e + 4h + 4g$, $\theta_{13} = a + 6b + 6c + 12d - 12f - 8e - 4h - 4g$. To implement generalized Euler action, the number of constraints should be imposed on these weights.

The first requirement is that $\theta_2 = \theta_4 = 0$, because this vertices are $flat$, and one can always normalize $\theta_1 = 1$ and to parametrize θ_3 by ρ, therefore $\theta_1 = 1$, $\theta_2 = 0$, $\theta_3 = \rho$, $\theta_4 = 0$. From this equations we can find the coupling constants a, b, c and d in terms of free parameters ρ and coupling constants f, g, h, e. The solution is: $a = (72f + 24g - 24h - 18\rho - 6)/24$, $b = (24f + 16g + 16e - 4\rho)/24$, $c = (-60f + 4g + 16e + 12h + 5\rho + 3)/24$, $d = (-12f + 12g - 12h - 3\rho + 3)/24$, therefore the basic vertex curvature become equal to :

$$\theta_1 = 1, \quad \theta_2 = 0, \quad \theta_3 = \rho, \quad \theta_4 = 0,$$

$$\theta_5 = -1 - 3\rho + 24f + 8e - 8h, \quad \theta_6 = -1 - \frac{1}{3}\rho + 8f - \frac{8}{3}g - \frac{8}{3}e, \quad \theta_7 = -1 - 3\rho + 24f + 8g - 8e,$$

and the vertices with self-intersections are equal to: $\theta_8 = -\frac{4}{3}\rho + 4f + \frac{4}{3}g - 4h - \frac{8}{3}e$, $\theta_9 = -8f$, $\theta_{10} = \rho - 4f - 4g + 4h$, $\theta_{11} = -12f + 4g - 4h + 1$, $\theta_{12} = -8f + 8g + 16e + 8h$, $\theta_{13} = 2 - 2\rho - 24f + 8g - 8h$. The second requirement which should be imposed on the vertex curvature is that $\theta_5 = \theta_6$ from which it follows that $6f = \rho + 3h - g - 4e$. The coupling constants are parametrized now in terms of ρ and g, h, e and are equal to $a = (-1 - \rho + 2g + 2h - 8e)/4$, $b = (g + h)/2$, $c = (3 - 5\rho + 14g - 18h + 56e)/24$, $d = (3 - 5\rho + 14g - 18h + 8e)/24$, $f = (\rho - g + 3h - 4e)/6$. Therefore $\theta_1 = 1$, $\theta_2 = 0$, $\theta_3 = \rho$, $\theta_4 = 0$, $\theta_5 = \theta_6 = -1 + \rho + 4h - 4g - 8e$, $\theta_7 = -1 + \rho + 12h + 4g - 24e$ and $\theta_8 = (-2\rho + 2g - 6h - 16e)/3$, $\theta_9 = (-4\rho + 4g - 12h + 16e)/3$, $\theta_{10} = (\rho - 10g + 6h + 8e)/3$, $\theta_{11} = (-2\rho + 6g - 10h + 8e + 1)/3$, $\theta_{12} =$

$(-4\rho + 28g + 12h + 64e)/3$, $\theta_{13} = 2 - 6\rho + 12g - 20h + 16e$. From the last condition $\theta_5 = \theta_6 = \theta_7$ it follows that $2e = g + h$ and

$$a = \frac{-1 - \rho - 2g - 2h}{4}, \quad b = \frac{g + h}{2}, \quad c = \frac{3 - 5\rho + 42g + 10h}{24},$$

$$d = \frac{3 - 5\rho + 18g - 14h}{24}, \quad f = \frac{\rho - 3g + h}{6} \quad e = \frac{g + h}{2} \tag{32}$$

and finally

$$\theta_1 = 1, \quad \theta_2 = 0, \quad \theta_3 = \rho, \quad \theta_4 = 0, \quad \theta_5 = \theta_6 = \theta_7 = -1 + \rho - 8g \tag{33}$$

together with self-intersection vertices $\theta_8 = (-2\rho - 6g - 14h)/3$, $\theta_9 = (-4\rho + 12g - 4h)/3$, $\theta_{10} = (\rho - 6g + 10h)/3$, $\theta_{11} = (-2\rho + 10g - 6h + 1)/3$, $\theta_{12} = (-4\rho + 60g + 44h)/3$, $\theta_{13} = 2 - 6\rho + 20g - 12h$. The (32) and (33) completely solve the problem in terms of parameter ρ and coupling constants g and h. Bellow we will consider two different cases of the prime theories.

3.2 Canonical weights for vertex curvature. In this case we should take $\rho = -1$ and $g = h = 0$ in (33), then first six vertices have canonical Euler value

$$\chi_1 = 1, \quad \chi_2 = 0, \quad \chi_3 = -1, \quad \chi_4 = 0, \quad \chi_5 = \chi_6 = \chi_7 = -2 \tag{34}$$

and from (32), (29) the Haliltonian is equal to

$$H_{Euler}^{3d} = -\sum_{\vec{r},\vec{\alpha}} \sigma_{\vec{r}}\sigma_{\vec{r}+\vec{\alpha}} + \sum_{\vec{r},\vec{\alpha},\vec{\beta}} \sigma_{\vec{r}}\sigma_{\vec{r}+\vec{\alpha}+\vec{\beta}} + \sum_{\vec{r},\vec{\alpha},\vec{\beta}} \sigma_{\vec{r}}\sigma_{\vec{r}+\vec{\alpha}}\sigma_{\vec{r}+\vec{\alpha}+\vec{\beta}}\sigma_{\vec{r}+\vec{\beta}}, \tag{35}$$

The remaining surface vertices $\chi_8, .., \chi_{13}$ have the lines of self-intersections and are equal to $\chi_8 = \frac{2}{3}$, $\chi_9 = \frac{4}{3}$, $\chi_{10} = -\frac{1}{3}$, $\chi_{11} = 3$, $\chi_{12} = \frac{4}{3}$, $\chi_{13} = 8$. Most of these vertices are positive and therefore have less statistical weight compared with other basic vertices $\chi_1, .., \chi_7$. To exclude them completely from the partition function one can ascribe to them a large or infinite curvature. For that we can use the fact that they all have the lines of self-intersections. The Hamiltonian which counts the number of self-intersection lines with the weight equal to k has been already constructed [13] and is equal to $H_{self-intersection}^{3d}$ (26). The total Hamiltonian is equal therefore to

$$H_{Euler}^{3d} = -(12k+4)\cdot\sum_{\vec{r},\vec{\alpha}} \sigma_{\vec{r}}\sigma_{\vec{r}+\vec{\alpha}} + (3k+4)\cdot\sum_{\vec{r},\vec{\alpha},\vec{\beta}} \sigma_{\vec{r}}\sigma_{\vec{r}+\vec{\alpha}+\vec{\beta}} + (3k+4)\cdot\sum_{\vec{r},\vec{\alpha},\vec{\beta}} \sigma_{\vec{r}}\sigma_{\vec{r}+\vec{\alpha}}\sigma_{\vec{r}+\vec{\alpha}+\vec{\beta}}\sigma_{\vec{r}+\vec{\beta}}, \tag{36}$$

and the corresponding vertices have the form $\chi_8 = (2 + 2k)/3$, $\chi_9 = (4 + 4k)/3$, $\chi_{10} = (-1 + 2k)/3$, $\chi_{11} = (9 + 4k)/3$, $\chi_{12} = (4 + 4k)/3$, $\chi_{13} = 8 + 4k$ and χ_{1-7} are the same. The limit $k \to \infty$ completely excludes the vertices with self-intersections from the partition function.

3.3 Absolute value of the weights for vertex curvature. The model with the absolute value of the Euler character (23)

$$\Theta(M_2) = \sum_{<i,j>} |2\pi - \alpha_{ij} - \beta_{ij} - ...| \tag{37}$$

can be constructed if we take in (32) and (33) $\rho = 1$, and $g = -h = -1/4$ then

$$a = -\frac{1}{2}, \quad b = 0, \quad c = d = -\frac{5}{12}, \quad f = \frac{1}{3}, \quad e = 0, \quad g = -h = -\frac{1}{4} \qquad (38)$$

with the corresponding weights

$$\theta_1 = 1, \quad \theta_2 = 0, \quad \theta_3 = 1, \quad \theta_4 = 0, \quad \theta_5 = \theta_6 = \theta_7 = 2,$$

and self-intersection vertices are $\theta_8 = (-4 + 2k)/3$, $\theta_9 = (-8 + 4k)/3$, $\theta_{10} = (5 + 2k)/3$, $\theta_{11} = (-15 + 4k)/3$, $\theta_{12} = (-8 + 4k)/3$, $\theta_{13} = -12 + 4k$.

3.4 Increasing the dimension of the lattice by one and leaving the Hamiltonian (35),(36) and (29) (38) without changes one can see that this prime system "moves" to the the next, physical, member of the hierarchy and describes now the quantum gravity or more exactly the system of fluctuating three dimensional manifolds with linear-gravity action $A(M_3)$ (10) on four dimensional lattice,

$$Z_{Gravity}(\beta) = \sum_{\{\sigma\}} exp(-\beta H_{Euler}^{3d}) \quad = \quad \sum_{\{M_3\}} exp(-2\beta A(M_3)) \qquad (39)$$

where the Hamiltonian H_{Euler}^{3d} exactly coincide with (35),(36) and (29), (38), but the summation over $\vec{r},\vec{\alpha}$....is extended now over four dimensional lattice.

As we already have seen in (20), (21) and (22), the partition function (39) of the three-dimensional quantum gravity can be represented as a superposition of the random surfaces with Euler action (20) or on the lattice as the superposition of linear string (21) and (22).

We can conclude from this that the spin system (39), which simulates three-dimensional gravity undergoes the second order phase transition in four dimensions and that this phase transition should be of the same nature as in 2d Ising ferromagnet or in 3d gonihedric system [6]. This happens because they are the next to primary systems in the geometrical hierarchy, that is they are next to 1d Ising, 2d gonimetric and 3d Euler systems correspondingly. Increasing the dimension of the lattice by one more unit we will describe finely four-dimensional Gravity with linear action $A(M_4)$ which is embedded into five dimensional lattice.

4. In this paper we clarify the point that the gonihedric string is the *superposition* of the weakly interacting fermions. We extend this result to quantum gravity, which now appears as a superposition of weakly interacting gonihedric strings. We have proposed also an alternative linear action $A(M_4)$ for the four and high dimensional quantum gravity, which allows the same representation.

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