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NON PERTURBATIVE ASPECTS OF SCREENING PHENOMENA IN ABELIAN AND NON ABELIAN GAUGE THEORIES

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Abstract

When computed to one-loop order in resummed perturbation theory, the non-abelian Debye mass appears to be logarithmically sensitive to the magnetic scale g^2T. More generally, we show that in higher orders power-like infrared divergences forbid the use of perturbation theory to calculate the corrections to Debye screening. A similar infrared problem occurs in the determination of the mass-shell for the scalar propagator in 2+1-dimensional scalar electrodynamics. In this context, we provide a non-perturbative approach which solves the infrared problems and allows for an accurate calculation of the scalar propagator in the vicinity of the mass-shell.

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1 Introduction

Significant progress has been achieved during the last few years toward the understanding of the infrared structure of high temperature QCD[1, 2, 3]. The prominent role of the soft energy scales gT and g^2T has been recognized, and the collective nature of the dominant behaviour at the scale gT has been properly understood. (Here, $g \equiv g(T)$ is the coupling constant at the temperature T, and we assume that $g \ll 1$ in the high-temperature, deconfined phase of QCD.) This led to a systematic description, in classical terms, of a variety of collective phenomena like screening, Landau damping, or color oscillations[3]. In the case of screening, it is known that, to leading order in g, the electrostatic interactions are screened, with a screening mass $m_D \sim gT$, while the magnetostatic interactions are not screened[4, 5]. These properties are shared by abelian and non-abelian plasmas (see Refs. [6, 7, 8, 9] for the abelian case).

Important differences occur between abelian and non-abelian gauge theories when corrections to the leading order Debye screening are considered. In the abelian case, perturbation theory can be used to calculate the corrections to the leading order Debye mass (see [9] and references therein). In QCD, infrared divergences occur in such a calculation, whose origin is the coupling of the chromoelectric field to the unscreened magnetostatic fields (see, e.g., Ref. [10] for a survey of the computations prior to 1993, and also Refs. [11–15] for more recent calculations). For example, at one-loop order, there is a logarithmic singularity, widely discussed in the literature[11, 12, 13]. But we shall see that the difficulty is actually more serious, since power-like infrared divergences occur in the higher orders.

The existence of infrared divergences invalidating the perturbative expansion of thermal QCD is well known for the magnetostatic sector, where power-like infrared divergences are indeed expected in higher order calculations of thermodynamical quantities[16, 17]. It has long been recognized that, because only static modes are involved, these divergences are essentially those of an effective *three-dimensional* theory[18]. The divergences that we shall encounter here, which are also those of an effective three-dimensional theory, are of a slightly different nature. They occur in the perturbative evaluation of the polarisation tensor of the electrostatic gluon *on the tree-level mass-shell.* Similar divergences are encountered in the calculation of the quasiparticle damping rates (see, e.g., [19] and references therein). All such divergences could be removed by introducing an infrared cut-off λ in the magnetostatic sector. However, this is not a satisfactory solution for at least two reasons. In QCD, there is a common belief that such a cut-off is indeed generated dynamically in the form of a magnetic mass $\lambda \sim g^2T$ [17, 20]. But for such a value of λ there are infinitely many terms in the perturbative expansion which contribute to the same order, a situation analogous to the Linde problem[16]. The second reason is that we shall identify similar mass-shell divergences in the evaluation of the static scalar propagator in thermal scalar electrodynamics (SQED). And we know that there is no magnetic mass in abelian gauge theories[6, 9].

Thus, although we expect the picture of Debye screening to hold in higher order calculations, for reasons which will be detailed in the next section, it appears that the corresponding value of the screening mass cannot be computed in perturbation theory beyond the leading order. We are thus led to look for a non perturbative description which allows for the treatment of the large degeneracy of states involving massless magnetostatic fields. We shall propose such a treatment for SQED, and obtain the mass-shell behaviour of the scalar propagator without any infrared regulator.

Our analysis relies on a non-perturbative approximation to the Dyson-Schwinger equations. The method that we use, known as the *gauge technique*, has been developed originally[21, 22] in relation to abelian gauge theories in four dimensions, and has been found to be particularly convenient for the study of the infrared structure of the propagator. It has the advantage to preserve the correct Ward identities, and the expected analytical properties of the propagator. Within this formalism, we shall be able to determine the infrared behaviour of the three dimensional scalar propagator. We shall find that the mass-shell singularity, which is a simple pole in leading order, turns into a branch point, whose location can be shown to be gauge-fixing independent.

The plan for the rest of the paper is as follows. In section 2, we introduce the screening function, and discuss its analytic properties. In section 3, we critically analyze the previous computations of the Debye mass in the resummed one-loop approximation, and show that the loop expansion generates power-like infrared divergences. We identify a similar problem in the charged sector of SQED. In section 4, we present a non-perturbative approach which allows us to study the mass-shell behaviour of the scalar propagator in 2+1-dimensional SQED. The last section summarizes the conclusions.

2 The screening function

In electrodynamics, the potential between two static charges in a medium can be calculated from the electrostatic propagator in the medium, to be referred as the *screening* *function* in what follows,

$$S(x) \equiv \int_{-\infty}^{\infty} \mathrm{d}x_0\, D_{00}(x_0, x) \;=\; \int \frac{\mathrm{d}^3k}{(2\pi)^3}\; \frac{\mathrm{e}^{ik\cdot x}}{k^2 - \Pi_{00}(0,k)}\,, \tag{2.1}$$

where $x = |\mathbf{x}|$ is the distance between the two static charges, $k = |\mathbf{k}| = \sqrt{\mathbf{k}^2}$, $\Pi_{00}(0,k)$ is the static ($k_0 = 0$) electric polarisation tensor, and $D_{00}(0,k)$ the electrostatic propagator: $D_{00}(0,k) = 1/(k^2 - \Pi_{00}(0,k)) \equiv S(k)$. We shall need later to analytically continue $S(k)$ to complex values of k. In most cases to be discussed in this paper, $S(k)$ will be obtained explicitly as an even function of k. When this is not so, we shall regard S as a function of $\sqrt{k^2}$, i.e. $S(k) = S(\sqrt{k^2})$, before doing the analytic continuation.

A similar screening function describes the interaction of two static color charges in a quark-gluon plasma. In this case, however, the polarization tensor is gauge-dependent[24, 25, 11]. A fully gauge invariant treatment of chromoelectric screening should start from a gauge invariant object, such as the Polyakov loop. But, in perturbation theory, the leading long range behaviour of the correlator of two Polyakov loops is determined by the screening function (2.1) (see Refs. [12, 13]). (We are not implying here that perturbation theory correctly describes the long range behaviour of the Polyakov loop correlator, which is presumably dominated by glue ball intermediate states[26].) For this reason, we shall concentrate on this simpler object here. In fact, the long-distance behaviour of $S(x)$ turns out to be gauge-independent. This may be expected from general arguments [27], and will be verified explicitly.

Let us then return to eq. (2.1) which we rewrite as

$$S(x) = \frac{1}{2\pi x} \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi i}\, k\, S(k)\, \mathrm{e}^{ikx}\,. \tag{2.2}$$

In leading order, $S_0(k) = 1/(k^2 + m^2)$ where

$$m^2 \;=\; -\delta\Pi_{00}(0,0) = \frac{g^2NT^2}{3} \tag{2.3}$$

is the leading-order screening mass squared and $\delta\Pi_{\mu\nu}$ is the polarization tensor in the "hard thermal loop" approximation [1-5](we consider here a pure gluonic plasma). The integral (2.2) may be computed by continuing the integrand to complex values of k, and by closing the integration path in the upper half of the complex k-plane. One then picks up the contribution of the pole $k = im$, and gets the familiar screened Coulomb potential, $S_0(x) = \mathrm{e}^{-mx}/4\pi x$.

In abelian plasmas, the higher order corrections do not change significantly the picture. The singularity of $S(k)$ which controls the asymptotic behaviour of $S(x)$ remains a pole on the imaginary axis; recently, this has been verified explicitly up to two orders beyond the hard thermal loop approximation [9]. In such a case, we can define a screening mass m_D as the solution of [11]

$$m_D^2 = -\Pi_{00}(0,k)\,|_{k^2=-m_D^2}\,. \tag{2.4}$$

This self-consistent equation admits also a meaningful perturbative solution[9].

Higher order corrections to $\Pi_{00}(0,k)$ play a more dramatic role in QCD, where they alter the nature of the singularities of $S(k)$. In order to discuss these corrections, it is convenient to remark that, in leading order, they can be considered as loop corrections in the effective three-dimensional theory obtained after integrating the non-static loops with static external lines (see [28, 29] and references therein). At the order of interest, and in Coulomb or covariant gauges, the corresponding Euclidean action reads

$$S_{eff} = \int \mathrm{d}^3x\,\mathrm{Tr}\,\Big(\frac{1}{2}\,F_{ij}^2 + (D_iA_0)^2 + m^2A_0^2 + \frac{1}{2\zeta}(\partial_iA_i)^2 + \partial_i\bar\eta D_i\eta)\Big)\,, \tag{2.5}$$

where $D_i = \partial_i - ig\sqrt{T}A_i$, $F_{ij} = [D_i, D_j]/(ig\sqrt{T})$, m is the leading-order electric mass from eq. (2.3), ζ is the gauge fixing parameter, and η and $\bar\eta$ are the ghost fields. The effective theory describes interacting static and long-wavelength ($k \lesssim gT$) fields. The magnetostatic gauge fields $A_i^a(\mathbf{x})$ and the electrostatic field $A_0^a(\mathbf{x})$ are, up to normalizations, the zero-frequency components of the original gluonic fields.

In the effective theory, A_0^a enters as a massive scalar field, whose propagator is the screening function. We shall write

$$S^{-1}(k) \equiv k^2 + m^2 + \Sigma(k) \tag{2.6}$$

with $\Sigma(k)$ denoting the self-energy corrections in the three-dimensional theory, i.e. we set $\Pi_{00}(0,k) = -m^2 - \Sigma(k)$. It is easy to see that all the Feynman diagrams contributing to $\Sigma(k)$ are analytic in k^2 for small momenta. Indeed, in any such diagram, one can choose the independent loop momenta so that the external momentum $\mathbf{k}$ flows only along the massive propagators. These can be expanded with respect to $\mathbf{k}$, when $|k| \ll m$. In the resulting expression, the external momentum appears then only in the numerator, and rotational symmetry ensures that only the terms with even powers of $\mathbf{k}$ survive the angular integration.

One may regard the effective action (2.5) as the Euclidean version of a Minkovskian action in 2+1 dimensions. From this point of view, one expects $S(k)$ to be analytic in the whole complex k-plane, except on the imaginary axis. We shall assume that this property indeed holds and write

$$S(k) \;=\; \int_0^\infty \mathrm{d}\omega\,\frac{\rho(\omega)}{\omega^2 + k^2}\,. \tag{2.7}$$

The spectral density $\rho(\omega)$ may be calculated from the discontinuity of $S(k)$ accross the imaginary axis:

$$\rho(\omega) \equiv \frac{2\omega}{\pi} \operatorname{Im} S(k = i(\omega + i\epsilon)) \tag{2.8}$$

with ω real. Because S is an even function of k, $\rho(\omega)$ is an even function of ω (this is easily verified by noticing that $S(k)$ is real for k real, and applying the Schwartz reflexion principle: $S^*(k) = S(k^*)$). Thus, only the positive values of ω are needed to represent $S(k)$. At leading order, $S(k) = 1/(k^2 + m^2)$ and $\rho(\omega) = 2m\delta(\omega^2 - m^2)$. We define

$$\tilde{\Sigma}(\omega) \equiv \Sigma(k = i(\omega + i\epsilon)) \tag{2.9}$$

where ω is real. Then

$$\rho(\omega) = -\frac{2\omega}{\pi} \frac{\operatorname{Im} \tilde{\Sigma}(\omega)}{\left[\omega^2 - m^2 - \operatorname{Re}\tilde{\Sigma}(\omega)\right]^2 + \left[\operatorname{Im} \tilde{\Sigma}(\omega)\right]^2} . \tag{2.10}$$

In a Minkovskian theory, one expects $\rho(\omega)$ to be positive in a physical gauge. However, what is meant by a physical gauge is not the same in the present (2+1)-dimensional problem and in the original (3+1)-dimensional one. For the original problem at finite temperature, one can choose, as a physical gauge, the strict Coulomb gauge, that is, the limit $\zeta \to 0$ in eq. (2.5). In the Minkovskian theory in 2+1 dimensions, this does not correspond to a Coulomb gauge, but rather to a Landau gauge which involves unphysical degrees of freedom. Since these latter do not give positive definite contributions to the spectral density, $\rho(\omega)$ is not then necessarily positive for the theory defined by the effective action eq. (2.5).

Since the three-dimensional theory (2.5) is superrenormalisable, $\Sigma(k)/k^2 \to 0$ as $k \to \infty$. It follows then from eq.(2.6) that, as $|k| \to \infty$, $S(k) \simeq 1/k^2$. This property allows us to derive a sum rule for ρ. First we note that, owing to the asymptotic property of S just mentioned, we have

$$\oint \frac{\mathrm{d}k}{2\pi i} \, k \, S(k) = 1 \, , \tag{2.11}$$

where the contour is a circle at infinity in the complex k-plane. Then we replace in this equation $S(k)$ by its expression (2.7) in terms of the spectral function. The contributions of the two poles at $k = \pm i\omega$ give

$$\int_0^\infty \mathrm{d}\omega \, \rho(\omega) = 1. \tag{2.12}$$

We shall see later that, in some approximations, $S(k)$ may have poles away from the imaginary axis, which are not accounted for by the spectral function $\rho(\omega)$. We shall argue later that such poles are unphysical, but it is nevertheless useful to keep track of them by writing

$$S(k)=\int_0^\infty d\omega\frac{\rho(\omega)}{\omega^2+k^2}+\sum_i\left(\frac{a_i}{k^2-k_i^2}+\frac{a_i^*}{k^2-k_i^{*2}}\right). \tag{2.13}$$

The sum rule is then modified into

$$\int_0^\infty \mathrm{d}\,\omega\rho(\omega)+\sum_i(a_i+a_i^*)=1. \tag{2.14}$$

We conclude this section by summarizing the analytic properties that we expect the screening function $S(k)$ to satisfy. We have given arguments suggesting that $S(k)$ is analytic in the complex k-plane, with singularities on the imaginary axis. Furthermore, in most gauges, $S(k)$ is analytic in k^2 for small $|k|$, i.e. $k \ll gT$. We shall verify that these properties are satisfied by the approximate $S(k)$ that we shall obtain. We shall find that $S(k)$ has branch cuts along the imaginary axis, starting at $k=\pm im^*$ (see Fig. 1). The branch point dominates the asymptotic behaviour of $S(x)$. According to eqs. (2.2) and (2.7), we can write

$$S(x)=\frac{1}{4\pi x}\int_{m^*}^\infty \mathrm{d}\omega\,\mathrm{e}^{-\omega x}\,\rho(\omega)\,, \tag{2.15}$$

which expresses the screening function as the Laplace transform of the spectral density. At large x, $S(x)\sim f(x)e^{-m^*x}$, and we shall verify that m^* is gauge-fixing independent.

The previous analyticity arguments, which are sufficient to establish the exponential fall off of the screening function at large distances, may become invalid in some particular gauges. In particular, in the temporal axial gauge, the asymptotic fall-off of $S(x)$ was found to be a power law rather than an exponential[14, 15]. It is likely however that this peculiar behaviour is a gauge artifact (see also Ref. [13]). Indeed, the temporal axial gauge is known to lead to specific difficulties in the imaginary-time formalism[17]. It prevents in particular the power counting arguments leading to the effective three dimensional action (2.5).

3 Corrections to Debye screening are non-perturbative

In perturbation theory, we expect the dominant singularity of $S(k)$ to remain close to the leading-order pole at $k=im$. Thus, the determination of the Debye mass involves the calculation of $\Sigma(k)$ for $k\sim im$. Since $\Sigma(k=im)$ is infrared singular in perturbation theory, this leads to difficulties whose physical origin is analyzed in this section. We shall be led finally to the conclusion that perturbation theory cannot be used to estimate the corrections to the leading-order Debye mass.

3.1 The polarisation tensor at next to leading order

The one-loop graph contributing to Σ is displayed in Fig. 2. It is readily evaluated as

$$\Sigma(k;m)=-g^2NT\int\frac{\mathrm{d}^Dq}{(2\pi)^D}\,\frac{(2k_i+q_i)(2k_j+q_j)}{(\mathbf{q}+\mathbf{k})^2+m^2}\,D^0_{ij}(0,\mathbf{q})\,, \tag{3.1}$$

where ($\hat{q}_i=q_i/q$)

$$D^0_{ij}(0,\mathbf{q})=\frac{\delta_{ij}-\hat{q}_i\hat{q}_j}{\mathbf{q}^2}+\zeta\,\frac{\hat{q}_i\hat{q}_j}{\mathbf{q}^2} \tag{3.2}$$

is the free propagator for magnetic gluons. We are using here, and throughout, dimensional continuation in order to regularise the *ultraviolet* (mass) divergences. After computing the integrals, no UV singularity will actually subsist in the limit $D\to3$. It is important to keep in mind that the limit $D\to3$ will always be taken *before* discussing the infrared structure of the integrals. After a simple rearrangement, eq. (3.1) takes the form

$$\Sigma(k;m)=g^2NT\int\frac{\mathrm{d}^Dq}{(2\pi)^D}\left\{\frac{1}{\mathbf{q}^2+m^2}+\frac{2}{\mathbf{q}^2}\frac{m^2-\mathbf{k}^2}{(\mathbf{q}+\mathbf{k})^2+m^2}\right. \\ \left.+(\zeta-1)\left(\mathbf{k}^2+m^2\right)\,\frac{\mathbf{q}\cdot(\mathbf{q}+\mathbf{2k})}{\mathbf{q}^4\left((\mathbf{q}+\mathbf{k})^2+m^2\right)}\right\}, \tag{3.3}$$

and gives, after an elementary integration,

$$\Sigma(k;m)=\alpha m\left\{\frac{2(m^2-k^2)}{mk}\arctan\frac{k}{m}\;+\;(\zeta-2)\right\}, \tag{3.4}$$

where $\alpha\equiv g^2NT/4\pi$ has the dimension of a mass.

The function (3.4) has logarithmic branch points at $k=\pm im$. The origin of this singularity may be seen on eq. (3.3): as $k^2\to-m^2$, the dominant contribution to the integral comes from the small q region, and when $k=\pm im$ the integral in fact diverges. To understand physically what happens, it is convenient to do a Wick rotation. We have, for real ω (recall eq. (2.9)),

$$\tilde{\Sigma}(\omega;m)\;=\;\alpha m\left\{\frac{(m^2+\omega^2)}{m\omega}\ln\frac{m+\omega+i\epsilon}{m-\omega-i\epsilon}\;+\;(\zeta-2)\right\}, \tag{3.5}$$

from which one gets

$$\mathrm{Im}\,\tilde{\Sigma}(\omega;m)\;=\;\pi\alpha\,\frac{\omega^2+m^2}{\omega}\,\theta(\omega^2-m^2)\,. \tag{3.6}$$

This imaginary part is proportional to $\Phi(\omega)$, the invariant phase-space for the decay of a particle of energy ω into a particle of mass $m < \omega$ and a massless particle. This is easily computed in the rest frame of the decaying particle as

$$\Phi(\omega) = \int \frac{d^2q}{(2\pi)^2} \frac{1}{2\epsilon_q\, 2q} \,\delta(\omega - q - \epsilon_q) = \frac{1}{8\pi\,\omega}\theta(\omega^2 - m^2) \tag{3.7}$$

where $\epsilon_q = \sqrt{q^2 + m^2}$. Note that this 2-dimensional phase space does not vanish when $\omega \to m_+$, in contrast to the 3-dimensional one. This behaviour of the phase-space factor is responsible for the infrared divergences to be discussed further in section 3.3.

Another noteworthy feature of eq. (3.6) is that $\mathrm{Im}\,\tilde{\Sigma}(\omega; m) > 0$ for $\omega > 0$, whereas with our conventions we would expect the opposite sign. The sign of $\mathrm{Im}\,\tilde{\Sigma}$ is related to that of the spectral function according to eq. (2.10), so that, in the one-loop approximation, $\rho(\omega)$ is negative for all momenta $\omega > m$. In particular, from eq. (2.10) and the asymptotic form of the one-loop self-energy (3.5), namely $\tilde{\Sigma}(\omega \to \infty) \simeq i\pi\alpha\omega$, one obtains, for large ω,

$$\rho(\omega) \sim -2\frac{\alpha}{\omega^2}. \tag{3.8}$$

Thus, as alluded to after eq. (2.10), in the present covariant gauge $\rho(\omega)$ cannot be regarded as a physical spectral density.

Because the poles at $k = \pm im$ of the unperturbed propagator coincide with the branch points in the self-energy, which furthermore diverges in these points, the equation (2.4) cannot be used to calculate perturbatively the correction to the Debye mass. In fact the analytic structure of the propagator $S(k) = 1/(k^2 + m^2 + \Sigma(k))$ is very different from that of the unperturbed one. It has branch points at $k = \pm im$ and, besides, a set of four simple poles at $k = \pm a \pm ib$, where the real numbers a and b are gauge-dependent [13]. To leading order in α, the values of a and b are given by $a = \alpha(\pi - \theta)$, $b - m \approx (\alpha/2)[\zeta - 2 + \ln(4m^2/(a^2 + (b-m)^2))]$, with $\theta = \arctan(a/(b-m))$. It is instructive to follow the trajectory of these poles in the complex k-plane, as a function of the dimensionless parameter $\alpha/m \sim g$. In order to do so, we rewrite the inverse propagator as

$$S^{-1}(k) = m^2 \left\{ 1 + x^2 + \frac{\alpha}{m} f(x) \right\} \tag{3.9}$$

where $x \equiv k/m$ and $f(x) \equiv \Sigma/(\alpha m)$. For small coupling, the poles behave as indicated above. When the coupling increases, they follow the trajectories displayed in Fig. 3 (for the gauge $\zeta = 2$). There exists a critical coupling at which the poles become real. Beyond that, one of the pole flows toward m, the other being equal to $\alpha\pi$. We note that the latter regime corresponds to the small mass regime, which is attained here at strong coupling. The pole at $k = \alpha\pi$ is the tachyonic pole already identified in the studies of the massless theory[30, 18].

This analytic structure of the one-loop propagator, which contradicts the expected properties of $S(k)$, leads to unphysical properties. Indeed, the relative magnitudes of b and m, which determine the asymptotic behaviour of $S(x)$, depends on the gauge. If $b > m$, the long-range behaviour of the screening function remains dominated by the logarithmic singularity at $z = im$ so that for $x \to \infty$, $S(x) \approx f(x)\,\mathrm{e}^{-mx}/x$. However, since the spectral function is negative, the pre-exponential factor $f(x)$ is strictly *negative* (see eq. (2.15)), in contrast to the leading order result $f_0 = 1/4\pi$. If now $b < m$, the pole contributions dominate, and the screening function oscillates asymptotically. These changes of regimes for small changes in the parameters are physically not satisfactory.

As we have mentioned after eq. (3.4), the singularities of the integral (3.3) are determined by the small q region. They are therefore very sensitive to the small momentum behaviour of the magnetic gluon propagator. By allowing for a small gluon mass λ, i.e. replacing $1/\mathbf{q}^2 \to 1/(\mathbf{q}^2+\lambda^2)$ in the integral (3.3), one separates the mass-shell of the scalar particle and the threshold for gluon emission, and this removes the infrared divergences. A simple calculation gives then[11] (for $\lambda \ll m$)

$$\Sigma_\lambda(k;m) = \alpha m \left\{ \frac{2(m^2-k^2)}{mk} \arctan \frac{k}{m+\lambda} - 1 + (\zeta - 1)\,\frac{m^2+k^2}{(m+\lambda)^2+k^2} \right\}. \tag{3.10}$$

In eq. (3.10), the branch point has now moved to $m+\lambda$. In perturbation theory, $S(k)$ has a pole at $k = i(m+\delta m)$ where $\delta m \equiv \Sigma_\lambda(k=im;m)/2m \approx \alpha \ln(2m/\lambda)$. This perturbative analysis is consistent as long as the new pole does not move back into the cut, that is, as long as $\alpha \ln(2m/\lambda) < \lambda$. However, if $\lambda \sim g^2T$, which is the order of magnitude expected for the magnetic mass, $\alpha \ln(2m/\lambda) \sim g^2 T \ln(1/g) \gg \lambda$, and one gets an inconsistency.

One way to keep the pole separated from the branch cut is to change m in eq. (3.10) into m_D. That puts the branch point at $m_D+\lambda$, whatever the value of m_D is. The Debye mass m_D is then obtained by solving self-consistently the equation [13]

$$m_D^2 \;=\; m^2 \;+\; \Sigma_\lambda(k = im_D\,; m_D)\,. \tag{3.11}$$

The pole at $k = im_D$ remains below the branch point at $k = i(m_D+\lambda)$, so that it controls the long range behaviour of the screening function. However the sign problem alluded to earlier is not solved. The sign of $S(x)$ at large distances is determined by the residue at the pole. A simple calculation gives

$$S(x) \sim_{x\to\infty} \frac{2\lambda}{2\lambda + (\zeta-3)\alpha}\,\frac{\mathrm{e}^{-m_D x}}{4\pi x}\,, \tag{3.12}$$

the first fraction being the residue just mentioned. Thus, in Feynman's gauge for instance, $S(x)$ becomes negative if $\lambda < \alpha$. Note that, in perturbation theory, one would expect the residue to be close to unity; the above formula shows that this only happens if the infrared cut-off is large enough, i.e. $\lambda \gg \alpha$, which is not to be expected. The above procedure leading to eq. (3.11) is an attempt to go beyond perturbation theory, which ignores, however, all the vertex corrections. The fact that the latter may be important is suggested by the non-perturbative character of the residue. Nevertheless, we shall see in section 4.4 that, in the presence of a magnetic mass, the result (3.11) remains correct even when vertex corrections are taken into account.

In closing this subsection, let us mention that, for nonvanishing λ, the same correction to the Debye mass as obtained above, i.e. $\delta m \approx \alpha \ln(2m/\lambda)$, can be deduced from the long range behaviour of the correlator of two Polyakov loops[12, 13]. Note however that the two calculations are not independent since, in perturbation theory, the asymptotic behaviours of the Polyakov loop correlator and of the screening function involve the same integrals.

3.2 A similar problem in scalar QED

In the high temperature limit, and at leading order in e, the static and long-wavelength ($k \lesssim eT$) correlation functions of scalar QED can be calculated from the effective three-dimensional euclidean action [9]

$$S_{eff} = \int \mathrm{d}^3x \left(\frac{1}{4} F_{ij}^2 + \frac{1}{2\zeta}(\partial_i A_i)^2 + \frac{1}{2}(\partial_i A_0)^2 + \frac{1}{2} m_{el}^2 A_0^2 \right.$$
$$\left. +(D_i\phi)^\dagger(D_i\phi) + m^2\phi^\dagger\phi + +e^2 T A_0^2 \phi^\dagger\phi + \frac{\lambda}{4}(\phi^\dagger\phi)^2 \right) . \tag{3.13}$$

The new notations here are as follows: $\phi(\mathbf{x})$ is the complex scalar field, $m_{el}^2 = e^2T^2/3$ is the leading order electric mass, and $m^2 = e^2T^2/4$ is the charged particle thermal mass. We are only interested here in the mass-shell singularities associated with the interaction between the charged particles and the massless transverse photons. We shall therefore restrict ourselves to the sector of (3.13) which describes these interactions:

$$S_{eff} = \int \mathrm{d}^3x \left(\frac{1}{4} F_{ij}^2 + \frac{1}{2\zeta}(\partial_i A_i)^2 + (D_i\phi)^\dagger(D_i\phi) + m^2\phi^\dagger\phi \right) . \tag{3.14}$$

There is an obvious similarity between this action and the corresponding one for the hot QCD plasma, eq. (2.5): they both describe massive charged particles (A_0^a in QCD and ϕ in SQED) in interaction with massless gauge fields. In particular, to one loop order, the scalar self-energy $\Sigma(k;m)$ is also given by eq. (3.4) [9], and most of the discussion in section 3.1 applies to SQED as well. Of course, essential differences persist between the two theories in the dynamics of the gauge fields themselves. In particular, in SQED, the transverse photons remain massless to all orders in perturbation theory[6, 9], so that one cannot invoke anymore a magnetic mass to regularise the mass-shell singularities, as we did in eq. (3.10) for QCD.

The massless version of the theory (3.14) has been studied extensively[18]. In this case the one-loop scalar self-energy generates a tachyonic pole at $k \sim \alpha$. This can be seen from the expression (3.4): in the limit $m/k \to 0$, $\Sigma(k) = -\pi\alpha k$, and the inverse propagator $S(k) = k^2 - \pi\alpha k$ vanishes at $k = \pi\alpha$. The infrared behaviour is improved by the resummation of the one-loop polarization tensor in the internal *photon* line in Fig. 2. To see that, consider the photon polarization tensor $\Pi_{ij}(0,q) \equiv (\delta_{ij} - \hat{q}_i\hat{q}_j)\Pi_T(0,q)$. At one-loop order in the effective theory we have[34, 9]

$$\Pi_T^{(1)}(0,q) \;=\; \alpha\, m \left\{ \frac{4m^2+q^2}{2qm} \arctan\frac{q}{2m} - 1 \right\}. \tag{3.15}$$

In the massless limit $m/q \to 0$, $\Pi_T^{(1)}(0,q) \;=\; \pi\alpha\, q/4$ is *linear* in q, so that it dominates over the contribution of the bare inverse propagator at small momentum, that is, as $q \to 0$, $D_{ij}(0,q) \propto 1/q$.

Although this softening of the photon propagator does not solve entirely the infrared problem, it is clear that the loop insertions in the internal photon lines do play an important role in the massless case. This is not so in the massive theory. The reason is that, when $m \neq 0$, and to all orders in perturbation theory, the polarization operator is expected to vanish at least as q^2 when $q \to 0$[9] (in particular, $\Pi_T^{(1)}(0,q) \;\sim\; (\alpha/6m)\, q^2$ as $q \to 0$); thus, in the massive theory, the low momentum behaviour of the resummed photon propagator is not different from that of the bare propagator.

3.3 The need for a non perturbative treatment

The infrared divergences that arise in the one-loop calculation signal, in fact, a breakdown of perturbation theory. There exists indeed an infinite number of multi-loop diagrams contributing to $\Sigma(k)$ which become infrared singular as the external momentum approaches the tree-level mass-shell, i.e. when $k^2 \to -m^2$. The one-loop diagram represented in Fig. 2 is logarithmically divergent as $k \to im$. Consider the two loop diagram of Fig. 4a. Its complete infrared behaviour is calculated explicitly in the Appendix, but the leading terms can be obtained by simple power counting. The most divergent contribution is in the integral

$$\int \frac{d^3q\, d^3p}{(q^2+\lambda^2)(p^2+\lambda^2)(\mathbf{k}\cdot\mathbf{q})^2(\mathbf{k}\cdot(\mathbf{p}+\mathbf{q}))} \tag{3.16}$$

where we have added a small mass to the photon in order to facilitate the power counting. The integral over q is linearly divergent as $\lambda \to 0$. The same result holds for the diagram in Fig. 4b, which involves vertex corrections, and it can be verified that the leading divergences of the diagrams 4a and 4b do not mutually cancel. A similar power counting argument can be extended to all Feynman diagrams involving no correction to the magnetic photon line, such as the one displayed in Fig. 5. The result of power counting is that, close to the mass shell, a n-loop graph ($n \geq 2$) diverges like $(\alpha/\lambda)^{n-1}$, up to powers of $\ln(\alpha/\lambda)$.

Physically, the origin of the infrared divergences is the degeneracy between the mass-shell of the charged particle and the threshold for the emission of n ($n \geq 1$) massless transverse photons. Then, the determination of the mass shell requires solving the theory in the subspace of these degenerate states. Naively, one would expect the coupling to two or more photons — which brings in more powers of g — to be less important than the coupling to a single photon. This is what happens in 3+1 dimensional electrodynamics. In the present case, the low dimensionality of the space-time amplifies the effects of the degeneracy, in such a way that the couplings to any number of photons become equally important.

Similar divergences arise in QCD as well. Some of the relevant diagrams are actually the same as in SQED (e.g., Figs. 4a,b and Fig. 5), the scalar line in these diagrams being interpreted as an electrostatic gluon. Besides, there exist new divergent graphs involving the self-interactions of the magnetic gluons (see Fig. 4c for an example). The same power counting as above leads again to the conclusion that n-loop diagram diverge as $(\alpha/\lambda)^{n-1}$. One may argue that the infrared divergences are cured by the dynamical generation of a magnetic mass λ. However, for $\lambda \sim g^2T \sim \alpha$, as commonly expected[16, 17], all the aforementioned diagrams contribute to the same order in g.

To summarize, the analysis of this section suggests that non perturbative methods are necessary in order to determine the correct mass shell behaviour. Such a method will be presented in the next section for the case of SQED.

4 An integral equation for the spectral density

We present now an approximate, but non-perturbative, solution of the Dyson-Schwinger equation of scalar electrodynamics, which provides the behaviour of the scalar propagator near the mass shell. To this aim, we establish a linear integral equation for the spectral density $\rho(\omega)$ using the so-called *gauge technique*[21]. This equation performs a partial resummation of the most infrared singular diagrams in a gauge-invariant way, i.e. by respecting the Ward identities. When applied to four-dimensional abelian gauge theories, it provides the correct mass-shell behaviour for charged particles[22].

4.1 The quenched approximation

The four skeleton diagrams which enter the Dyson-Schwinger equation for the scalar propagator are displayed in Fig. 6. It can be verified by power counting that, at a given order in e, the most singular diagrams are obtained from the perturbative expansion of the first graph, Fig. 6.a, where we keep the photon propagator at the tree-level (see Fig. 7). These are precisely the diagrams discussed in section 3.3. Thus, the Dyson-Schwinger equation that we wish to solve, and to which we refer as the quenched approximation, is

$$\Sigma(k) \;=\; -e^2 T \int \frac{\mathrm{d}^D q}{(2\pi)^D}\,(2k_i+q_i)\,D^0_{ij}(\mathbf{q})\,\Gamma_j(\mathbf{k}+\mathbf{q},\mathbf{k})\,S(\mathbf{k}+\mathbf{q})\,. \tag{4.1}$$

In this equation, S and Γ_i are the full propagator and vertex, related by the Ward identity:

$$q_i\,\Gamma_i(\mathbf{k},\mathbf{k}+\mathbf{q}) \;=\; S^{-1}(\mathbf{k}+\mathbf{q}) - S^{-1}(\mathbf{k})\,. \tag{4.2}$$

The most general vertex function which is consistent with this identity and which is free of kinematical singularities is of the form[35]

$$\Gamma_i(\mathbf{k},\mathbf{k}+\mathbf{q}) \;=\; \frac{2k_i+q_i}{(\mathbf{2k}+\mathbf{q})\cdot\mathbf{q}}\Big(S^{-1}(\mathbf{k}+\mathbf{q}) - S^{-1}(\mathbf{k})\Big) \;+\; A(\delta_{ij}-\hat{q}_i\hat{q}_j)k_j\,, \tag{4.3}$$

where $A \equiv A(k^2, q^2, \mathbf{k}\cdot\mathbf{q})$ is an unknown scalar function. According to the usual terminology in the literature, we shall refer to the two terms in the r.h.s. as the longitudinal and transverse pieces of the vertex function, respectively. The second term, proportional to A, is indeed transverse to the photon momentum $\mathbf{q}$. However, the first term is *not* parallel to $\mathbf{q}$: it involves a non-trivial transverse piece which is completely determined by the Ward identity (see also eq. (4.6) below). At leading order, $S^{-1}(\mathbf{k}) = \mathbf{k}^2 + m^2$, $A = 0$, and $\Gamma_i(\mathbf{k},\mathbf{k}+\mathbf{q})$ given by eq. (4.3) reduces to the tree-level vertex, $2k_i + q_i$.

As $q \to 0$, eq. (4.3) must reproduce the differential form of the Ward identity, namely

$$\Gamma_i(\mathbf{k},\mathbf{k}) \;=\; \frac{\partial S^{-1}}{\partial k_i} \;=\; 2k_i \frac{\partial S^{-1}}{\partial k^2}\,. \tag{4.4}$$

Together with eq. (4.3), this implies that $A \to 0$ as $q \to 0$, for any k. This suggests that only the longitudinal vertex becomes singular as $q \to 0$ and $k^2 \to -m_D^2$; the corresponding singularities are, of course, those of the 2-point function. This conclusion is supported by perturbative calculations[35]. Thus, in order to get the leading mass-shell behaviour, we need only consider the longitudinal vertex. That is, we make the following ansatz for Γ_i :

$$\Gamma_i(\mathbf{k},\mathbf{k}+\mathbf{q}) \;=\; \frac{2k_i + q_i}{(\mathbf{2k}+\mathbf{q})\cdot\mathbf{q}} \Big(S^{-1}(\mathbf{k}+\mathbf{q}) - S^{-1}(\mathbf{k})\Big)\,. \tag{4.5}$$

As anticipated, this has a component transverse to $\mathbf{q}$:

$$(\delta_{ij} - \hat{q}_i\hat{q}_j)\Gamma_j(\mathbf{k},\mathbf{k}+\mathbf{q}) \;=\; \frac{2(\delta_{ij} - \hat{q}_i\hat{q}_j)k_j}{(\mathbf{2k}+\mathbf{q})\cdot\mathbf{q}} \Big(S^{-1}(\mathbf{k}+\mathbf{q}) - S^{-1}(\mathbf{k})\Big)\,. \tag{4.6}$$

It is this component which couples to the physical, transverse piece of the photon propagator (3.2). The longitudinal piece of (4.5) couples only to the gauge degrees of freedom of $D^0_{ij}(\mathbf{q})$.

With the ansatz (4.5) for Γ_i , the Dyson-Schwinger equation (4.1) becomes

$$\Sigma(k) \;=\; -e^2 T \int \frac{\mathrm{d}^D q}{(2\pi)^D}\, D^0_{ij}(\mathbf{q})\, \frac{(2k_i+q_i)(2k_j+q_j)}{(\mathbf{2k}+\mathbf{q})\cdot\mathbf{q}}\, \Big(1 \,-\, S^{-1}(\mathbf{k})S(\mathbf{k}+\mathbf{q})\Big)\,. \tag{4.7}$$

We are eventually interested in the self-energy close to the mass-shell, $\Sigma(k \to im_D)$. By definition, $S^{-1}(k = im_D) = 0$, so that the term proportional to $S^{-1}(k)$ in the r.h.s. of eq. (4.7) does not contribute to $\Sigma(k = im_D)$, unless the integral over q diverges. Since this integral is indeed potentially divergent, we introduce a regulator, in the form of a small photon mass. In doing so, it is important for what follows to keep explicit the distinction between the physical and the unphysical states in the photon propagator. The general structure of the exact magnetostatic propagator in finite-temperature four-dimensional SQED is[6, 9]:

$$D_{ij}(0,\mathbf{q}) = \frac{\delta_{ij} - \hat{q}_i\hat{q}_j}{\mathbf{q}^2 + \Pi_T(0,q)} + \zeta\,\frac{\hat{q}_i\hat{q}_j}{\mathbf{q}^2}, \tag{4.8}$$

with $\Pi_T(0,q) \equiv \Pi_{ii}(0,q)/2$. We have already mentioned, in section 3.2, that $\Pi_T(0,q) \propto q^2$ as $q \to 0$. However, as a convenient IR regularisation, we give the transverse photons a mass and set temporarily $\Pi_T(0,q) = \lambda_M$ ($\lambda_M \ll m$). We also regularise the gauge sector of D_{ij} by replacing $1/\mathbf{q}^2 \rightarrow 1/(\mathbf{q}^2+\lambda^2)$ in the second term of eq. (4.8). Both ζ and λ should disappear in the evaluation of physical quantities. Then we get from eq. (4.7) $\Sigma = \Sigma_L + \zeta\delta\Sigma$ where

$$\Sigma_L(k) \ = \ -e^2T\int\frac{\mathrm{d}^Dq}{(2\pi)^D}\,\frac{1}{q^2+\lambda_M^2}\,\frac{4(\mathbf{k}\cdot\hat{\mathbf{q}})^2}{(\mathbf{2k}+\mathbf{q})\cdot\mathbf{q}}\,\Big(1\ -\ S^{-1}(\mathbf{k})S(\mathbf{k}+\mathbf{q})\Big)\,, \tag{4.9}$$

is the self-energy in the Landau gauge, while

$$\delta\Sigma(k) = -e^2T\int\frac{\mathrm{d}^Dq}{(2\pi)^D}\,\frac{\mathbf{q}\cdot(\mathbf{2k}+\mathbf{q})}{(q^2+\lambda^2)^2}\,\Big(1 - S^{-1}(\mathbf{k})S(\mathbf{k}+\mathbf{q})\Big)\ , \tag{4.10}$$

is the gauge dependent part of Σ. Note that the above equation for $\delta\Sigma$ is actually independent of the ansatz used for the vertex Γ_i, since it follows directly from eq. (4.1) and the Ward identity (4.2).

Consider now eq. (4.9) in the on-shell limit $k \rightarrow im_D$. As long as we keep the infrared regulator $\lambda_M \neq 0$, the integral is convergent and the term proportional to $S^{-1}(k)$ vanishes on the mass-shell. Thus

$$\Sigma(k=im_D) \ = \ -e^2T\int\frac{\mathrm{d}^Dq}{(2\pi)^D}\,\frac{1}{q^2+\lambda_M^2}\,\frac{4(\mathbf{k}\cdot\hat{\mathbf{q}})^2}{(\mathbf{2k}+\mathbf{q})\cdot\mathbf{q}}\bigg|_{k=im_D} \ \simeq\ 2\alpha m_D\ln\frac{2m_D}{\lambda_M}\,. \tag{4.11}$$

On the other hand, in the physical limit $\lambda_M \rightarrow 0$, not only does the estimate (4.11) become logarithmically divergent, but the integral multiplying $S^{-1}(k)$ also diverges on mass-shell, since, in this limit, it is proportional to

$$\int\frac{\mathrm{d}^3q}{q^2(q^2+2\mathbf{k}\cdot\mathbf{q})(q^2+2\mathbf{k}\cdot\mathbf{q}+\Sigma(\mathbf{k}+\mathbf{q})-\Sigma(k))}\,. \tag{4.12}$$

Thus the use of an infrared regulator does not allow us to explore in a simple way the behaviour of the scalar propagator near the mass shell. For this, more powerful technics are needed, such as that developed in the next subsection.

Consider finally the gauge-dependent part of the self-energy, i.e., $\delta\Sigma$, eq. (4.10). Since

$$\int\frac{\mathrm{d}^Dq}{(2\pi)^D}\,\frac{\mathbf{q}\cdot(\mathbf{2k}+\mathbf{q})}{(q^2+\lambda^2)^2} = \int\frac{\mathrm{d}^Dq}{(2\pi)^D}\,\frac{q^2}{(q^2+\lambda^2)^2} = -\frac{3\lambda}{8\pi}\,, \tag{4.13}$$

which vanishes as $\lambda \rightarrow 0$, the r.h.s. of eq. (4.10) reduces to

$$\delta\Sigma(k) = e^2T\,S^{-1}(\mathbf{k})\int\frac{\mathrm{d}^Dq}{(2\pi)^D}\,\frac{\mathbf{q}\cdot(\mathbf{2k}+\mathbf{q})}{(q^2+\lambda^2)^2}\,S(\mathbf{k}+\mathbf{q}). \tag{4.14}$$

The usefulness of λ appears when considering the on-shell limit of this equation. If we set $\lambda = 0$, then the q-integral diverges on the mass-shell, and the limit $\delta\Sigma(k\rightarrow im_D)$

is not obvious. But if we keep $\lambda \neq 0$, we obtain $\delta\Sigma(k = im_D) = 0$. This guarantees the gauge-independence of the mass-shell, since the equation $S^{-1}(k = im_D) = 0$ reduces then to $m_D^2 = m^2 + \Sigma_L(k^2 = -m_D^2)$. This procedure for taking the on-shell limit in the presence of an infrared regulator has been proposed in Ref. [11], at the level of the resummed one-loop approximation. (See also Refs. [31, 32, 33] for a similar problem in the computation of the damping rates.)

4.2 The integral equation

We shall transform now the Dyson-Schwinger equation (4.7) into an integral equation for the spectral density $\rho(\omega)$. In this way, we take automatically into account the expected analytical properties of both the propagator and the vertex. First we put eq. (4.1) in the form

$$1 = (k^2 + m^2)S(\mathbf{k}) - e^2 T \int \frac{\mathrm{d}^D q}{(2\pi)^D} (2k_i + q_i) D^0_{ij}(\mathbf{q})\, S(\mathbf{k}+\mathbf{q})\, \Gamma_j(\mathbf{k}+\mathbf{q}, \mathbf{k})\, S(\mathbf{k}). \quad (4.15)$$

Then we use the spectral representation (2.7) of the propagator to rewrite the ansatz (4.5) for the vertex function as

$$S(\mathbf{k})\, \Gamma_i(\mathbf{k}, \mathbf{k}+\mathbf{q})\, S(\mathbf{k}+\mathbf{q}) = \int_0^\infty \mathrm{d}\omega \, \frac{(2k_i + q_i)\rho(\omega)}{(\omega^2 + \mathbf{k}^2)\,(\omega^2 + (\mathbf{k}+\mathbf{q})^2)}\,. \quad (4.16)$$

When eqs. (2.7) and (4.16) are used in (4.15), the following equation for ρ is obtained[21, 22]:

$$1 = \int_0^\infty \mathrm{d}s \, \frac{\rho(s)}{s^2 + k^2} \left[k^2 + m^2 + \Sigma(k; s) \right], \quad (4.17)$$

where $\Sigma(k; s)$ is the one-loop expression (3.4) in which the mass m is replaced by s. That is,

$$\Sigma(k; s) = \alpha s \left\{ \frac{(s^2 - k^2)}{isk} \ln \frac{s + ik}{s - ik} + \zeta - 2 \right\}, \quad (4.18)$$

where $\alpha = e^2 T / 4\pi$.

As it stands, eq.(4.17) is not easy to solve. To make progress, we do a Wick rotation $k \to i(\omega + i\epsilon)$ to time like momenta. The integral equation becomes then

$$1 = \int_0^\infty \mathrm{d}s \, \frac{\rho(s)}{s^2 - \omega^2 - i\epsilon} \left[-\omega^2 + m^2 + \tilde{\Sigma}(\omega; s) \right]. \quad (4.19)$$

By taking the imaginary part of (4.17) we obtain an equation which is linear in ρ and homogeneous:

$$\frac{\pi}{2\omega}\, \rho(\omega) \left[\omega^2 - m^2 - \mathrm{Re}\, \tilde{\Sigma}(\omega; \omega) \right] = \int_0^\infty \mathrm{d}s \, \frac{\rho(s)}{s^2 - \omega^2}\, \mathrm{Im}\, \tilde{\Sigma}(\omega; s)\,. \quad (4.20)$$

At this point, all what we have done applies to 3+1 dimensional SQED as well. It is interesting to see what happens in this case to enlighten the difference with the 2+1-dimensional case. In 3+1 dimensions,

$$\mathrm{Im}\,\tilde{\Sigma}(\omega;s) = \frac{(3-\zeta)e^2}{16\pi}\,\frac{\omega^2+s^2}{\omega^2}\,(\omega^2-s^2)\,\theta(\omega^2-s^2). \tag{4.21}$$

The real part, $\mathrm{Re}\,\tilde{\Sigma}(\omega;\omega)$, is regular after UV renormalisation and it can be combined with the parameter m^2 in the l.h.s. of eq. (4.20) to define the physical mass (which we continue to denote by the symbol m for simplicity). The equation for the spectral density is then[22]

$$(\omega^2-m^2)\rho(\omega) = \frac{(\zeta-3)e^2\,\omega}{8\pi^2}\int_m^\omega \mathrm{d}s\,\rho(s)\left(1+\frac{s^2}{\omega^2}\right). \tag{4.22}$$

For $\omega \sim m$, this equation can be transformed into a differential equation which is easily solved. To do that, set $F(\omega) \equiv (\omega^2-m^2)\rho(\omega)$, and verify that when $\omega \to m$, $F(\omega)$ satisfies

$$\frac{\mathrm{d}F}{\mathrm{d}\omega} \approx \frac{(\zeta-3)e^2}{8\pi^2}\,\frac{2m\,F(\omega)}{\omega^2-m^2}. \tag{4.23}$$

The solution of this equation gives the behaviour of $\rho(\omega)$ for $\omega \to m$:

$$\rho(\omega) \propto \frac{1}{\omega^2-m^2}\left(\frac{\omega^2-m^2}{4m^2}\right)^{\frac{(\zeta-3)e^2}{8\pi^2}}. \tag{4.24}$$

This is the correct behaviour, as obtained by a variety of other methods (see [23] and Refs. therein).

An essential step in the previous calculation is the cancelation of the singularity at $s^2=\omega^2$ of the integrand in the r.h.s. of eq. (4.20) by the phase space factor $(\omega^2-s^2)/\omega^2$ contained in $\mathrm{Im}\,\tilde{\Sigma}(\omega;s)$, eq. (4.21). This phase space factor is very much dependent upon the dimension. Recall that, in 2+1 dimensions, it is simply $\sim 1/\omega$ (see eq. (3.7)), so that in this case the imaginary part of the one-loop self-energy does not vanish at threshold; we have indeed

$$\mathrm{Im}\,\tilde{\Sigma}(\omega;s) \;=\; \pi\alpha\,\frac{\omega^2+s^2}{\omega}\,\theta(\omega-s)\,. \tag{4.25}$$

It follows that the integrand in the r.h.s. of eq. (4.20) is singular as $s \to \omega$, suggesting that that eq. (4.20) is ill defined as written. However, according to eq. (3.5), $\mathrm{Re}\,\tilde{\Sigma}(\omega;s)$ is also logarithmically divergent when evaluated for $s \to \omega$, and it turns out that this divergence precisely cancels that of the r.h.s. To see this, we write the real part of the one-loop self-energy as

$$\mathrm{Re}\,\tilde{\Sigma}(\omega;m) \,=\, \zeta\,\alpha\,m \,+\, 2\alpha\,\frac{m}{\omega}\int_0^\omega \mathrm{d}s\,\frac{\omega^2+s^2}{m^2-s^2}\,. \tag{4.26}$$

The last integral diverges at its upper limit when $m\to\omega$. When it is added to the integral in the r.h.s. of eq. (4.20), the limit $m\to\omega$ becomes well-defined. The equation for ρ takes then the form

$$\rho(\omega)\left(\omega^2-m^2-\zeta\,\alpha\,\omega\right) \,=\, 2\alpha\int_0^\omega \mathrm{d}s\,\frac{\omega^2+s^2}{\omega^2-s^2}\left(\rho(\omega)-\rho(s)\right). \tag{4.27}$$

This is our main equation. It has a number of properties which are worth emphasizing.

i) The integral over ω in the r.h.s. now extend over a limited range of values of ω. This makes it well suited to study the behaviour of $\rho(\omega)$ near threshold.

ii) It is a linear integral equation for the spectral density. Being also homogeneous, it determines ρ only up to a constant factor, which in principle is fixed by the inhomogeneous equations (4.17) or (4.19). It can be verified in particular that any normalisable solution of the integral equation (4.17) satisfies the sum rule

$$\int_0^\infty \mathrm{d}\omega\,\rho(\omega) \,=\, 1\,. \tag{4.28}$$

iii) In the limit $\alpha\to 0$, eq. (4.27) admits as a solution the free spectral function, $\rho(\omega) = 2m\delta(\omega^2-m^2)$.

iv) The asymptotic form of $\rho(\omega)$ for $\omega\gg m$ has been obtained from perturbation theory (recall eq. (3.8)):

$$\rho(\omega)\sim -2\,\frac{\alpha}{\omega^2}. \tag{4.29}$$

This is consistent with the integral equation (4.27). Indeed, by assuming that the solution $\rho(\omega)$ falls off rapidly enough in order to be normalisable, one can obtain from eq. (4.27) the following relation:

$$\rho(\omega)\sim -2\frac{\alpha}{\omega^2}\int_0^\infty ds\,\rho(s). \tag{4.30}$$

When combined with the sum rule (4.28), this equation yields the asymptotic behaviour (4.29). It also shows that ρ must change sign.

4.3 Solving the integral equation near the mass shell

From the analyticity of the Feynman diagrams (see the discussion after eq. (2.6)), one expects $\rho(\omega)$ to vanish identically in the vicinity of $\omega = 0$. We show now that this is indeed the case for the solution of eqs. (4.17) and (4.27). Note first that eq. (4.17) is valid for any k. Setting $k = 0$, and using $\Sigma(k = 0; \omega) = \zeta\alpha\omega$, we obtain

$$1 = \int_0^\infty \mathrm{d}\omega \, \frac{\rho(\omega)}{\omega^2} \Big(m^2 + \zeta\alpha\omega \Big). \tag{4.31}$$

For this integral to converge, $\rho(\omega)$ must vanish sufficiently rapidly when $\omega \to 0$. Then, assuming $\rho(\omega)$ to be regular, if it is not zero it is either increasing or decreasing for ω small enough. Assume, for example, that ρ is increasing, so that it is positive for small ω. Then, the l.h.s. of the integral equation (4.27), $\approx -m^2\rho(\omega)$, is negative, while the r.h.s. is positive. One runs into a similar contradiction if one assumes instead that ρ is decreasing for small ω. The only acceptable possibility is that there exists $m^* > 0$ such that $\rho(\omega < m^*) = 0$. It may be furthermore verified on eq. (4.27) that $\rho(\omega)$ cannot have an isolated, δ-type singularity at $\omega = m^*$; that is, the spectral density is non-vanishing in any upper vicinity of m^*.

Because $\rho(\omega)$ vanishes when $\omega < m^*$, one can expand the integrand in eq. (4.17) for small k without generating infrared singularities. In this way, one obtains sum rules involving higher and higher moments of $1/\omega$. For example, in the Landau gauge $\zeta = 0$, we have

$$1 = m^2 \int_{m^*}^\infty \mathrm{d}\omega \, \frac{\rho(\omega)}{\omega^2}\,, \tag{4.32}$$

$$1 = m^2 \int_{m^*}^\infty \mathrm{d}\omega \, \frac{\rho(\omega)}{\omega^3} \left(\frac{8}{3}\,\alpha + \frac{m^2}{\omega} \right). \tag{4.33}$$

Such sum rules suggest that ρ is positive when $\omega \to m^*$.

To study the behaviour of $\rho(\omega)$ for $\omega \gtrsim m^*$, we divide the s-integration in eq. (4.27) in two parts: from 0 to m^*, where $\rho(s) = 0$, and from m^* to ω. After a simple calculation, the integral equation is rewritten as

$$\rho(\omega) \left(\omega^2 - m^2 - \zeta\alpha\,\omega + 2\alpha\, m^* - 2\alpha\,\omega \,\ln \frac{\omega + m^*}{\omega - m^*} \right)$$
$$= 2\alpha \int_{m^*}^{\omega} \mathrm{d}s \, \frac{\omega^2 + s^2}{\omega^2 - s^2} \Big(\rho(\omega) - \rho(s) \Big)\,. \tag{4.34}$$

Assume that $\rho(\omega)$ is positive near the threshold, in conformity with the sum rules above. As $\omega \to m^*$, the l.h.s. is dominated by the singularity of the logarithmic term,

$$\text{l.h.s.} \approx -2\alpha\, m^*\, \rho(\omega) \, \ln \frac{2m^*}{\omega - m^*}, \tag{4.35}$$

which is negative (and gauge-independent). The r.h.s. must be negative as well, and this requires $\rho(\omega)$ to be decreasing. Since ρ is positive and decreasing, it cannot vanish at m^*: $\rho(m^*) > 0$. In fact, ρ is *divergent* at threshold, for, if $\rho(m^*)$ were finite, the integral in the r.h.s. of eq. (4.34) would vanish as $\omega \to m^*$, while the l.h.s., eq. (4.35), would be divergent. It follows that, close to the threshold, the integral in the r.h.s. of eq. (4.34) is dominated by the singularity of $\rho(s)$ as $s \to m^*$, so that we may approximate

$$\text{r.h.s.} \approx -2\alpha \frac{m^*}{\omega - m^*} \int_{m^*}^{\omega} \mathrm{d}s\, \rho(s)\,. \tag{4.36}$$

In writing this equation, we have neglected $\rho(\omega)$ (which is finite as long as $\omega > m^*$) next to $\rho(s)$, and we have made the appropriate replacements, e.g., $\omega + m^* \to 2m^*$. From eqs. (4.35) and (4.36), we obtain the following approximate form of the integral equation:

$$x\, \rho(x) \ln \frac{1}{x} = \int_0^x \mathrm{d}y\, \rho(y)\,, \tag{4.37}$$

with the notation $x \equiv (\omega - m^*)/2m^*$ and the obvious identification $\rho(x) \equiv \rho(\omega = m^* + 2m^* x)$. Note that α and all the other parameters (namely, m and ζ) have dropped from this equation. Its solution is easily found as

$$\rho(x) = \frac{Z}{2m^*} \frac{1}{x\,(\ln x)^2}\,, \tag{4.38}$$

where Z is a dimensionless constant. As expected, this is divergent as $x \to 0_+$, but the divergence is integrable, as required for a solution of the integral equation. In the original variable,

$$\rho(\omega) \approx \frac{Z\,\theta(\omega - m^*)}{(\omega - m^*)\left(\ln \frac{\omega - m^*}{2m^*}\right)^2}\,, \qquad \omega \to m^*. \tag{4.39}$$

This is our main result.

Near the mass-shell, the corresponding Minkowski propagator is ($\omega > 0$)

$$\tilde{S}(\omega) \equiv \int_0^\infty \mathrm{d}s\, \frac{\rho(s)}{s^2 - \omega^2 - i\epsilon} \approx \frac{Z}{(\omega^2 - m^{*2}) \ln \frac{m^* - \omega - i\epsilon}{2m^*}}\,. \tag{4.40}$$

This is qualitatively different from the one-loop result discussed in Sect. 3.1. The inverse propagator $\tilde{S}^{-1}(\omega)$ vanishes at the mass shell, but its derivative is divergent there, a situation which is reminiscent of that in four dimensions[23]. Thus, the integral equation (4.15) provides, in (2+1)-dimensional SQED, a mass-shell behaviour for the charged particle propagator which is analogous to that obtained at the one loop level in (3+1)-dimensional SQED. In particular, the self-energy corresponding to the propagator (4.40) behaves near the mass-shell as

$$\tilde{\Sigma}(\omega) \propto (\omega^2 - m^{*2}) \ln \frac{m^* - \omega - i\epsilon}{2m^*}\,. \tag{4.41}$$

It contains a factor $\omega^2 - m^{*2}$ (compare, in this respect, with eqs. (3.6) and (4.21)) which makes the imaginary part of (4.41) vanish and change sign at the mass-shell.

According to eq. (2.15), the long-range behaviour of the screening function is determined by the spectral density near threshold. With $\rho(\omega)$ given by eq. (4.39), we get $S(x) \sim f(x)\,\mathrm{e}^{-m^* x}$, so that m^* plays the role of the screening mass. Note that if the integral equation (4.37) specifies the correct mass-shell behaviour, it leaves m^* arbitrary. In principle, the value of m^* could be obtained by solving the integral equation (4.17) for $\rho(\omega)$ for *all* the values of ω. But the approximations underlying this equation are only valid at small momenta.

Now, if we cannot specify the value of m^* anyfurther, we can verify that it is independent of the choice of the gauge parameter. To this aim, consider the gauge-dependent contribution to the scalar self-energy, as determined by eq. (4.14). This can be written in terms of the spectral density as

$$\delta\Sigma(k) = e^2 T\, S^{-1}(\mathbf{k}) \int \mathrm{d}\omega\, \rho(\omega) \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{\mathbf{q}\cdot(\mathbf{2k}+\mathbf{q})}{(q^2+\lambda^2)^2} \frac{1}{\omega^2 + (\mathbf{k}+\mathbf{q})^2}\,, \tag{4.42}$$

and is readily evaluated, with the result

$$\delta\Sigma(k) = \alpha\, S^{-1}(\mathbf{k}) \int \mathrm{d}\omega\, \frac{\omega\rho(\omega)}{(\omega+\lambda)^2 + k^2}\,. \tag{4.43}$$

If we were to set $\lambda = 0$, we would obtain a non-vanishing contribution on the mass-shell:

$$\delta\Sigma(k) = \alpha\, S^{-1}(\mathbf{k}) \int \mathrm{d}\omega\, \frac{\omega\rho(\omega)}{\omega^2 + k^2} \;\longrightarrow_{k\to im^*}\; \alpha\, m^*\,. \tag{4.44}$$

When taking the limit $k \to im^*$ in the above equation, we use the fact that, close to the mass-shell, the integral is dominated by the singularity of the spectral density at $\omega = m^*$. In order to get rid of the unwanted contribution (4.44), which would make the mass-shell gauge-dependent, we are thus led to keep $\lambda \neq 0$ before taking the on-shell limit (see the discussion after eq. (4.14)). With this procedure, the coefficient Z in eq. (4.39) is gauge dependent, and also λ-dependent, as we explain now. Consider the equation $S^{-1} = S_L^{-1} + \zeta\delta\Sigma$ (with S_L the propagator in the Landau gauge) in the vicinity of the mass-shell, where the approximate form (4.40) holds (with Z_L replacing Z in the case of S_L). By using eq. (4.43) for $\delta\Sigma(k \to im^*)$, one derives

$$\frac{1}{Z} \;=\; \frac{1}{Z_L} + \frac{\zeta\alpha}{Z} \int_{m^*}^{\infty} \mathrm{d}\omega\, \frac{\omega\rho(\omega)}{(\omega+\lambda)^2 - m^{*2}}\,. \tag{4.45}$$

As $\lambda \to 0$, the integral in the r.h.s. is essentially the on-shell propagator, so that it is divergent. Thus, for λ small enough, the integral is dominated by the singularity of $\rho(\omega)$

as $\omega \to m^*$, where the approximation (4.39) can be used. One deduces that, as $\lambda \to 0$, eq. (4.45) takes the form

$$\frac{1}{Z} \approx \frac{1}{Z_L} + \frac{\zeta\alpha}{2\lambda \ln(2m^*/\lambda)}. \tag{4.46}$$

This equation determines the dependence of Z upon ζ and λ, in the limit $\lambda \to 0$. We recall here that Z_L is trivially independent of ζ and λ, since it is determined by eq. (4.17) with $\zeta = 0$. According to sum rules like (4.32)–(4.33), we expect Z_L to be positive.

4.4 The case of a non-vanishing magnetic mass

For the sake of comparaison with previous computations, in particular those presented in section 3.1, it is instructive to consider the integral equation for the spectral density in the presence of an infrared regulator $\lambda \ll m$. Since the question of gauge invariance is not an issue here, we use a single regulator, in contrast to what we did earlier in section 4.1.

With $\lambda \neq 0$, the real and imaginary parts of $\tilde{\Sigma}_\lambda(\omega; s)$ are obtained from eq. (3.10):

$$\begin{aligned} \mathrm{Re}\,\tilde{\Sigma}_\lambda(\omega;\omega) &= 2\alpha\omega \ln\frac{2\omega}{\lambda} - \alpha\,\omega\,, \\ \mathrm{Im}\,\tilde{\Sigma}_\lambda(\omega;s) &= \pi\alpha\,\frac{\omega^2+s^2}{\omega}\,\theta\Big(\omega-(s+\lambda)\Big) - (\zeta-1)\frac{\pi\alpha}{2}\,\frac{s(\omega^2-s^2)}{\omega}\,\delta\Big(\omega-(s+\lambda)\Big)\,. \end{aligned} \tag{4.47}$$

Then, the homogeneous integral equation takes the form

$$\rho(\omega)\left(\omega^2 - m^2 + \alpha\omega - 2\alpha\omega\ln\frac{2\omega}{\lambda}\right) = (\zeta-1)\alpha\omega\rho(\omega-\lambda) - 2\alpha\int_0^{\omega-\lambda} \mathrm{d}s\,\frac{\omega^2+s^2}{\omega^2-s^2}\rho(s)\,. \tag{4.48}$$

Note the following differences with respect to the case $\lambda = 0$: *i)* $\mathrm{Re}\,\tilde{\Sigma}_\lambda(\omega;\omega)$ is finite; *ii)* the s-integration in eq. (4.48) is now restricted to $s \leq \omega - \lambda$, thus avoiding the singularity of the integrand at $s = \omega$; *iii)* the gauge term is proportional to $\rho(\omega - \lambda)$ and reflects the spurious pole at $k = i(m + \lambda)$ arising in the gauge piece of the self-energy (3.10).

From eq. (4.48), it is easy to establish that the mass-shell corresponds to a δ-type singularity, that is, to a simple pole in the corresponding propagator. To see this, consider eq. (4.48) for some ω satisfying $m^* < \omega < m^* + \lambda$, where m^* is the mass-shell position; then, the r.h.s. of the integral equation vanishes since $\rho(s) = 0$ for $s \leq \omega - \lambda < m^*$. The equation becomes, in this momentum range,

$$\rho(\omega)\,\left(\omega^2 - m^2 + \mathrm{Re}\,\tilde{\Sigma}_\lambda(\omega;\omega)\right) \,=\, 0\,, \tag{4.49}$$

and it is solved by $\rho(\omega) \propto \delta(\omega - m^*)$, with m^* determined by

$$m^{*2} \,=\, m^2 \,+\, \mathrm{Re}\,\tilde{\Sigma}_\lambda(m^*; m^*)\,. \tag{4.50}$$

Since $\mathrm{Re}\,\tilde{\Sigma}_\lambda(\omega;\omega)$, eq. (4.47), is gauge independent, so is m^*, the solution of the above equation. Eq. (4.50) is identical to eq. (3.11), obtained after a partial resummation of the scalar propagator which amounts to the replacement of the leading order mass m by the exact mass m_D, that is, by using the propagator

$$S(k) \,\simeq\, \frac{1}{k^2 + m_D^2}\,, \tag{4.51}$$

in the calculation of $\Sigma(k)$. The fact that vertex corrections do not seem to play any role in the determination of the mass shell is, strictly speaking, illusory. In fact, the true propagator in the vicinity of the pole is not (4.51), but rather $z/(k^2 + m_D^2)$. The residue z may differ significantly from unity; in principle, it could be determined if we were able to solve the full integral equation (4.48). However, because of the Ward identity, the residue enters also the vertex correction, in such a way that it cancels against that of the propagator when the self-energy is computed at the mass-shell. It is therefore not needed to determine m^*, but it enters as a preexponential factor in the asymptotic form of the screening function. We see now that the preexponential factor in eq. (3.12) is not consistently determined.

5 Conclusions

We have shown that the corrections to the Debye mass in high temperature non abelian gauge theories can be analyzed as corrections to the mass-shell of particles coupled to massless magnetic modes in a 2+1 dimensional effective theory. If one attempts to calculate perturbatively the Debye mass within this effective theory, one encounters power-like infrared divergences which signal a breakdown of the perturbative expansion. The possible existence of a magnetic mass in QCD does not remove the essentially non perturbative character of the corrections.

We have shown that similar mass-shell singularities occur in the evaluation of the scalar propagator in scalar electrodynamics. For this case, we have presented a non perturbative approach which allows for a complete description of the mass-shell behaviour. The mass-shell singularity, which is a simple pole in leading order, turns into a branch point as a result of the coupling of the scalar particles to an arbitrary number of soft photons. The propagator near the branch point exibits a behaviour which is reminiscent of that of the one-loop propagator in 3+1 dimensional electrodynamics. The location of the branch point, which plays the role of the screening mass in the thermal problem, is shown to be gauge independent. However, its precise value is left undetermined by our present approach.

The calculation that we have performed for SQED suggests that a similar solution may exist for QCD as well. If a magnetic mass exists, the mass-shell remains a pole whose exact location is determined by the integral equation that we have derived, assuming that this equation applies also to QCD; the correction to the Debye mass thus obtained is then identical to that calculated by Rebhan[13]. Finally, we believe that the present analysis should also shed light on another longstanding problem in finite temperature field theory, that is, the infrared singularity of the damping rates for thermal particles (see, e.g., [19] and references therein).

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Appendix

We verify here that, close to the mass-shell $k^2 \to -m^2$, the two-loop rainbow diagram, Fig. 4.a, becomes as important as the one-loop graph of Fig. 2 if the infrared cut-off λ is of the order g^2T or less.

A straightforward application of the Feynman rules for the action (3.14) gives (with $(\mathrm{d}q) \equiv \mathrm{d}^3q/(2\pi)^3$)

$$\Sigma^{(2)}(k) = -(e^2T)^2 \int \frac{(\mathrm{d}q)}{\mathbf{q}^2+\lambda^2} \int \frac{(\mathrm{d}p)}{\mathbf{p}^2+\lambda^2} \frac{(\mathbf{q}+\mathbf{2k})^2}{[(\mathbf{q}+\mathbf{k})^2+m^2]^2} \frac{(\mathbf{p}+\mathbf{2k}+\mathbf{2q})^2}{(\mathbf{p}+\mathbf{q}+\mathbf{k})^2+m^2}\,, \quad \text{(A.1)}$$

where we allow for a photon mass λ as a convenient IR regularization, and we use the Feynman gauge $\zeta = 1$ for simplicity. We are interested here only in the dominant IR singularity of $\Sigma^{(2)}(k)$ as $k \to \pm im$ and $\lambda \to 0$. After working out the scalar products in the numerator, we isolate the terms with the largest number of factors in the denominator. These terms are the most singular in the double limit $k^2 \to -m^2$ and $\lambda \to 0$. (If either $k^2 \neq -m^2$ or $\lambda \neq 0$, $\Sigma^{(2)}$ is finite.) We denote the corresponding contribution to (A.1) by

$\Sigma^{(2)}_{IR}(k)$. We have

$$\Sigma^{(2)}_{IR}(k) = -16m^4\,(e^2T)^2 \int \frac{\mathrm{d}_3 q}{\mathbf{q}^2+\lambda^2} \int \frac{\mathrm{d}_3 p}{\mathbf{p}^2+\lambda^2}\, \frac{1}{[(\mathbf{q}+\mathbf{k})^2+m^2]^2}\, \frac{1}{(\mathbf{p}+\mathbf{q}+\mathbf{k})^2+m^2} \quad \text{(A.2)}$$

The terms which have been neglected in going from eq. (A.1) to eq. (A.2) are, at most, logarithmically divergent in the double limit mentioned before, and do not matter for the power counting developed in Sect. 3. As we shall see, the integral (A.2) is linearly (or, more accurately, linearly $\times$ logarithmically) divergent in the same limit.

To perform the momentum integrals in eq. (A.2), we use the coordinate-space representation of the Coulomb propagators, e.g.

$$\frac{1}{k^2+m^2} = \int \mathrm{d}^3x\, \mathrm{e}^{i\vec{k}\cdot\vec{x}}\, \frac{\mathrm{e}^{-mx}}{4\pi x}, \quad \text{(A.3)}$$

(with $x \equiv |\vec{x}|$) and obtain

$$\Sigma^{(2)}_{IR}(k) = -\frac{2\alpha^2 m^3}{(2\pi)^2} \int \mathrm{d}^3x \int \mathrm{d}^3y\; \mathrm{e}^{i\vec{k}\cdot\vec{x}}\, \mathrm{e}^{-m|\vec{x}-\vec{y}|}\, \frac{\mathrm{e}^{-\lambda x}}{x}\, \frac{\mathrm{e}^{-(m+\lambda)y}}{y^2} \equiv -2\alpha^2 J(k;m,\lambda), \quad \text{(A.4)}$$

where $\alpha = e^2T/4\pi$. Once the angular integrations are done, J is given by

$$J(k;m,\lambda) = \frac{2m}{k}\int_0^\infty \frac{\mathrm{d}x}{x}\int_0^\infty \frac{\mathrm{d}y}{y}\, \sin kx\, \mathrm{e}^{-\lambda x}\, \mathrm{e}^{-(m+\lambda)y}\, F(x,y)\,, \quad \text{(A.5)}$$

where

$$F(x,y) \equiv \Big(1+m|x-y|\Big)\,\mathrm{e}^{-m|x-y|} \;-\; \Big(1+m(x+y)\Big)\,\mathrm{e}^{-m(x+y)} \;=\; F_1 \;+\; F_2\,, \quad \text{(A.6)}$$

with

$$F_1(x,y) \equiv m\left\{|x-y|\,\mathrm{e}^{-m|x-y|} \;-\; (x+y)\,\mathrm{e}^{-m(x+y)}\right\} \quad \text{(A.7)}$$
$$F_2(x,y) \equiv\; \mathrm{e}^{-m|x-y|} \;-\; \mathrm{e}^{-m(x+y)}\,.$$

The IR divergences occuring in the original momentum integrals as $k^2 \to -m^2$ and $\lambda \to 0$ appear now as UV divergences of the integrals over x and y. The most singular terms are generated by $F_1(x,y)$. The contribution of $F_2(x,y)$ is, at most, logarithmically divergent. Let $J_1(k;m,\lambda)$ denote the contribution of F_1 to the integral (A.5). For $\lambda = 0$, but arbitrary k we have a simple expression:

$$J_1(k;m,\lambda=0) = \frac{2m^2}{k^2+m^2}\left(\ln\frac{4m^2}{k^2+m^2} \;-\; 1\right)\,. \quad \text{(A.8)}$$

As $k^2 \to -m^2$, this has a linear $\times$ logarithmic singularity, as announced. For $\lambda \neq 0$, the expression of J_1 is more complicated. We give here only the explicit form of the *most singular* term, valid for arbitrary k and λ:

$$J_{IR}(k;m,\lambda) = \frac{m}{ik}\left\{\frac{m}{m+\lambda-ik}\ln\frac{2m+\lambda}{m+2\lambda-ik} - \frac{m}{m+\lambda+ik}\ln\frac{2m+\lambda}{m+2\lambda+ik}\right\}. \quad \text{(A.9)}$$

After inserting this in eq. (A.4), we obtain the dominant (i.e. the most singular) mass-shell behaviour of the two-loop diagram (A.1) as

$$\Sigma^{(2)}_{IR}(k^2 \to -m^2) \approx -2\alpha^2 \frac{m}{\lambda}\ln\frac{m}{\lambda} \sim \left(\frac{\alpha}{\lambda}\right)\Sigma^{(1)}(k^2 \to -m^2), \quad \text{(A.10)}$$

(recall eq. (3.10)). Thus, for λ of the order $e^2T \sim \alpha$ or smaller, and in the vicinity of the mass-shell, the two-loop contribution $\Sigma^{(2)}_{IR}$ is of the same order in α as the one-loop self-energy. A similar conclusion is reached in section 3.3 using power counting.

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Figure captions

Figure 1. Analytical structure of the screening function $S(k)$ in the complex k-plane, showing the branch point at $k = im^*$. The contour is that used for the evaluation of the integral in eq. (2.2).

Figure 2. The one-loop contribution to the self energy in the three-dimensional effective theory. Full line: scalar (electrostatic) field. Wavy line: transverse gluon.

Figure 3. Approximate representation of the trajectories followed in the complex k-plane by the poles of the one-loop scalar propagator. The arrows indicate the flows of the poles as the coupling increases. With the present choice of gauge ($\zeta = 2$), the poles become real when $\alpha/m \gtrsim 1.45$.

Figure 4. Two-loop contributions to the scalar self-energy which contain linear mass-shell divergences. Diagram (a) and (b) occur both in QCD and in SQED, while diagram (c) exists only in QCD.

Figure 5. An exemple of a multi-loop contribution to the self-energy $\Sigma(k)$, exhibiting power-like mass-shell divergences.

Figure 6. The skeleton diagrams for the self-energy of the scalar propagator in SQED. The dominant infrared singularities are contained in diagram 6.a.

Figure 7. Diagrammatic representation of the Dyson-Schwinger equation of SQED in the quenched approximation. Both the internal scalar propagator (heavy line) and the vertex (heavy blob) are exact quantities, while the photon propagator (wavy line) is the bare propagator.