EUCLIDEAN MAXWELL THEORY IN THE PRESENCE OF BOUNDARIES. II

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Abstract. ζ-function regularization is applied to complete a recent analysis of the quantized electromagnetic field in the presence of boundaries. The quantum theory is studied by setting to zero on the boundary the magnetic field, the gauge-averaging functional and hence the Faddeev-Popov ghost field. Electric boundary conditions are also studied. On considering two gauge functionals which involve covariant derivatives of the 4-vector potential, a series of detailed calculations shows that, in the case of flat Euclidean 4-space bounded by two concentric 3-spheres, one-loop quantum amplitudes are gauge independent and their mode-by-mode evaluation agrees with the covariant formulae for such amplitudes and coincides for magnetic or electric boundary conditions. By contrast, if a single 3-sphere boundary is studied, one finds some inconsistencies, i.e. gauge dependence of the amplitudes.

1. Introduction

Despite the lack of a mathematically consistent theory of quantum gravity, the elliptic boundary-value problems occurring in quantum cosmology have recently shed new light on the whole quantization programme for gauge fields and gravitation in the presence of boundaries [1-4]. Boundary effects play a key role in the path-integral approach to quantum gravity [1], in the problem of boundary conditions for the quantum state of the universe [5], and in comparing different quantization and regularization techniques for gauge fields and gravitation [1-2]. The latter problem provides the main motivation for our paper. In fact many efforts have been produced in the literature to understand the relation between canonical and manifestly gauge-invariant approaches to quantum field theories, as well as to compare mode-by-mode and covariant formulae for the evaluation of quantum amplitudes.

In particular, in some papers dealing with the calculation of the scaling factor of the wave function of the universe in quantum cosmology, discrepancies were found between results obtained by covariant and non-covariant methods. It is well-known that this scaling factor coincides with the Schwinger-De Witt coefficient A_2 in the heat-kernel expansion [6]. Moreover, this factor can be calculated by using the generalized Riemann ζ-function technique [1]. Within this framework, the prefactor is expressed through the $\zeta(0)$ value, while $\zeta'(0)$ yields the full expression for the one-loop effective action. It was noticed that, for fields with non-zero spin, calculations of the Schwinger-De Witt coefficient A_2 [6] by using general covariant formulae for Riemannian 4-manifolds with boundaries [7] give results which differ from those obtained by using ζ-function technique when one restricts the theory to its physical degrees of freedom [1]. An analogous phenomenon was noticed in [8], where $\zeta(0)$ was calculated for gravitons on the full Riemannian de Sitter sphere. In [9] the hypothesis was put forward that the reason of discrepancies lies in the impossibility to perform a 3+1 decomposition on the Riemannian 4-manifolds under consideration. Indeed, this decomposition is necessary for the separation of physical degrees of freedom. Thus, the direct calculation of $\zeta(0)$ in terms of physical degrees of freedom seems to be inconsistent. The $\zeta(0)$ calculation for fermionic fields was then carried out in [9] on the part of flat Euclidean 4-space bounded by two concentric 3-spheres. It was shown that the discrepancy disappears in this case.

However, to understand discrepancies for the electromagnetic field and other gauge theories, it is necessary to take into account ghost modes and non-physical degrees of freedom, whose contributions may survive in a non-trivial background even in the unitary gauges. We here restrict ourselves to a mode-by-mode analysis of the scaling factor in relativistic gauges about flat Euclidean 4-space with one or two 3-sphere boundaries. One then faces the following problems, here described in the case of Euclidean Maxwell theory, which is the object of our investigation.

(i) *Choice of background 4-geometry.* This can be flat Euclidean 4-space, or a curved Riemannian manifold providing the index of the Dirac operator vanishes and no further obstructions to having a unique, smooth solution of the classical, elliptic boundary-value problem can be found. [Knowledge of the index of the Dirac operator ensures one understands what happens for second-order elliptic operators as well]

(ii) *Choice of boundary 3-geometry.* Motivated by quantum cosmology, this is taken to be a 3-sphere, or two concentric 3-spheres. These choices are necessary to have a unique smooth solution of the corresponding classical boundary-value problem for fields of various spins, and to avoid singularities at the origin of the background 4-geometry (see below).

(iii) *Choice of boundary conditions.* They can be magnetic, which implies setting to zero on the boundary the magnetic field, the gauge-averaging functional and hence the Faddeev-Popov ghost field. They can also be electric, hence setting to zero on the boundary the electric field, and leading to Neumann conditions on the ghost [1].

(iv) *Choice of gauge-averaging functional.* Here we focus on the gauge-averaging functional first proposed in [1-2]: $\Phi_E(A) \equiv {}^{(4)}\nabla^{\mu} A_{\mu} - A_0 \ \mathrm{Tr}\ K$ (K being the extrinsic-curvature tensor of the boundary), and on the Lorentz functional $\Phi_L(A) \equiv {}^{(4)}\nabla^{\mu} A_{\mu}$, as a first check of gauge independence of quantum amplitudes in a mode-by-mode analysis of the quantized electromagnetic field.

The mode-by-mode analysis is performed by relying on the familiar expansions of the components of the 4-vector potential on a family of 3-spheres centred on the origin [1-3], i.e.

$$A_0(x,\tau) = \sum_{n=1}^{\infty} R_n(\tau) Q^{(n)}(x) \tag{1.1}$$

$$A_k(x,\tau) = \sum_{n=2}^{\infty} \Big[f_n(\tau) S_k^{(n)}(x) + g_n(\tau) P_k^{(n)}(x) \Big] \quad \text{for all } k = 1,2,3 \tag{1.2}$$

where $Q^{(n)}(x), S_k^{(n)}(x), P_k^{(n)}(x)$ are scalar, transverse and longitudinal vector harmonics on S^3 respectively. Note that, however, normal and tangential components of A_μ are only well-defined at the 3-sphere boundary where $\tau = \tau_+$, since a unit normal vector field inside matching the normal to S^3 at the boundary is ill-defined at the origin of the coordinate system for flat Euclidean 4-space [9]. Hence the geometrical meaning of (1.1)-(1.2) as normal and tangential components of the 4-vector potential inside the 3-sphere boundary remains unclear, unless one studies an elliptic boundary-value problem where flat Euclidean 4-space is bounded by two concentric 3-spheres of radii τ_+ and τ_-, say. The results of our calculations show that a proper study of non-physical degrees of freedom and ghosts (which do not compensate each other), together with the consideration of manifolds possessing a consistent 3+1 decomposition, eliminates the discrepancies between covariant and non-covariant formalisms.

Our paper is organized as follows. Section 2 studies one-loop amplitudes by choosing the gauge-averaging functional $\Phi_E(A)$ (see above), and imposing magnetic (or electric) boundary conditions on 3-spheres. The one-boundary and two-boundary problems are analyzed. Section 3 repeats the investigation of section 2 in the case of the gauge functional $\Phi_L(A)$. Results and open problems are presented in section 4. Details relevant for $\zeta(0)$ calculations are described in the appendix.

2. One-boundary and two-boundary problems in the Esposito gauge

In this section we first evaluate $\zeta(0)$ for the electromagnetic field on the flat 4-dimensional Euclidean background bounded by a 3-sphere. We choose the magnetic boundary conditions described in the introduction and carry out our calculations in the Esposito gauge [1-2]. For this purpose, we begin by studying the coupled eigenvalue equations for normal and longitudinal components of the electromagnetic field obtained in [1-2]. They have the form (hereafter we set to 1 the parameter appearing in the Faddeev-Popov action (2.3) of [2])

$$\widehat{A}_n \; g_n(\tau) + \widehat{B}_n \; R_n(\tau) = 0 \tag{2.1a}$$

$$\widehat{C}_n \; g_n(\tau) + \widehat{D}_n \; R_n(\tau) = 0 \tag{2.1b}$$

where

$$\widehat{A}_n \equiv \frac{d^2}{d\tau^2} + \frac{1}{\tau} \; \frac{d}{d\tau} - \frac{(n^2-1)}{\tau^2} + \lambda_n \tag{2.2}$$

$$\widehat{B}_n \equiv -\frac{(n^2-1)}{\tau} \tag{2.3}$$

$$\widehat{C}_n \equiv -\frac{1}{\tau^3} \tag{2.4}$$

$$\widehat{D}_n \equiv \frac{d^2}{d\tau^2} + \frac{3}{\tau} \; \frac{d}{d\tau} - \frac{(n^2-1)}{\tau^2} + \lambda_n . \tag{2.5}$$

To study non-physical modes it is convenient to diagonalize the operator matrix

$$\begin{pmatrix} \widehat{A}_n & \widehat{B}_n \\ \widehat{C}_n & \widehat{D}_n \end{pmatrix} .$$

Hence we look for a diagonalized matrix in the form

$$O_{ij}^{(n)} \equiv \begin{pmatrix} 1 & V_n(\tau) \\ W_n(\tau) & 1 \end{pmatrix} \times \begin{pmatrix} \widehat{A}_n & \widehat{B}_n \\ \widehat{C}_n & \widehat{D}_n \end{pmatrix} \times \begin{pmatrix} 1 & \alpha_n(\tau) \\ \beta_n(\tau) & 1 \end{pmatrix} . \tag{2.6}$$

The matrix $\begin{pmatrix} 1 & \alpha_n \\ \beta_n & 1 \end{pmatrix}$ creates the linear combinations of functions $R_n(\tau)$ and $g_n(\tau)$ which can be found from decoupled equations, whilst the matrix $\begin{pmatrix} 1 & V_n \\ W_n & 1 \end{pmatrix}$ selects these decoupled equations.

Setting to zero the off-diagonal matrix elements of (2.6), and defining $\nu \equiv +\sqrt{n^2 - \frac{3}{4}}$, one finds equations solved by $V_n = -\alpha_n$, $W_n = -\beta_n$, where

$$\alpha_n(\tau) = \left(-\frac{1}{2} \pm \nu\right) \tau \tag{2.7}$$

$$\beta_n(\tau) = \frac{1}{(\nu + 1/2)(\nu - 1/2)} \left(\frac{1}{2} \pm \nu\right) \frac{1}{\tau}. \tag{2.8}$$

Choosing the pair of solutions with upper sign for α_n and lower sign for β_n [the opposite choice of signs gives the equivalent system of operators whilst the choice of coinciding signs for α_n and β_n implies the degenerate system of equations] one finds general basis functions for $R_n(\tau)$ and $g_n(\tau)$ in the form

$$g_n(\tau) = C_1 \, I_{\nu-1/2}(\sqrt{\lambda}\tau) + C_2 \, \Big(\nu - 1/2\Big) \, I_{\nu+1/2}(\sqrt{\lambda}\tau) \tag{2.9}$$

$$R_n(\tau) = \frac{1}{\tau} \left(C_1 \, \frac{-1}{(\nu + 1/2)} \, I_{\nu-1/2}(\sqrt{\lambda}\tau) + C_2 \, I_{\nu+1/2}(\sqrt{\lambda}\tau)\right) \tag{2.10}$$

where C_1 and C_2 are constants.

In our particular gauge, magnetic boundary conditions imply Dirichlet boundary conditions for $g_n(\tau)$ and Neumann boundary conditions for $R_n(\tau)$ [1-2]. The resulting equations lead to a 2×2 matrix, hereafter denoted by $\mathcal{I}$, whose determinant has to vanish to find non-trivial solutions. Such an eigenvalue condition for normal and longitudinal components of the electromagnetic potential is best studied by using the algorithm of [4]. It is known that $\zeta(0)$ can be expressed as

$$\zeta(0) = I_{\log} + I_{\rm pole}(\infty) - I_{\rm pole}(0) \tag{2.11}$$

where [4]

$$I(M^2, s) \equiv \sum_{n=n_0}^{\infty} d(n)\, n^{-2s}\, \log f_n(M^2) = \frac{I_{\rm pole}(M^2)}{s} + I^R(M^2) + {\rm O}(s). \tag{2.12}$$

With our notation, $d(n)$ is the degeneracy of the eigenvalues parametrized by the integer n, and $f_n(M^2)$ is the corresponding eigenvalue condition. Moreover, on analytic continuation, $I_{\log} = I_{\log}^R$ is the coefficient of $\log M$ from $I(M^2, s)$ as $M \rightarrow \infty$, and $I_{\rm pole}(M^2)$ is the residue at $s = 0$. The condition det $\mathcal{I} = 0$ should be studied after eliminating fake roots $M = 0$. To obtain that it is enough to divide det $\mathcal{I}$ by the minimal power of M occurring in the determinant. It is easy to see by using the series expansion for modified Bessel functions [10] that such a power is $M^{2\nu-1}$.

We begin with the calculation of $I_{\log}$ for normal and longitudinal modes of the electromagnetic field together with ghosts. Using uniform asymptotic expansions for modified Bessel functions [10] we can see that the only terms in the logarithm of det $\mathcal{I}$ divided by $M^{2\nu-1}$ which are proportional to $\log M$ have the form $-2\nu\, \log M$ whilst the ghost eigenvalue condition, divided by M^{ν}, gives analogous terms $-(2\nu+1)\, \log M$ which contribute to $I_{\log}$ with the opposite sign. Hence we can write

$$I_{\log} = \sum_{n=2}^{\infty} \frac{n^2}{2} = -\frac{1}{2} \tag{2.13}$$

where n^2 is the dimension of the irreducible representation for scalar hyperspherical harmonics. Summation in (2.13) is carried out by the method of ζ-function regularization (see for details [4]). The infinite sum starts from $n = 2$ since effects of zero-modes for ghosts and normal photons should be calculated separately whilst the longitudinal ($n = 1$) photon is absent. In [1-2] it was found that the contribution of the decoupled normal mode with magnetic boundary conditions is

$$\zeta(0)_{\text{decoupled mode}} = -\frac{1}{4}. \tag{2.14}$$

It is easy to calculate the contribution to $\zeta(0)$ resulting from ghost zero-modes by substituting into the corresponding expression for $I_{\log}$ the value $n = 1$. One finds

$$\zeta(0)_{\text{ghost zero-modes}} = 1. \tag{2.15}$$

Now we have to calculate $I_{\text{pole}}(\infty)$ and $I_{\text{pole}}(0)$. As shown in the appendix, the structure of the term generating $I_{\text{pole}}(\infty)$ is $\frac{2\,\nu}{(\nu+1/2)}$. Taking the logarithm of this expression and expanding it in inverse powers of n we can pick out the coefficient of the term $1/n$ in the expression $\frac{n^2}{2}\,\log\frac{2\,\nu}{(\nu+1/2)}$ as $n \to \infty$, and find

$$I_{\text{pole}}(\infty) = -\frac{11}{96}. \tag{2.16}$$

Now we can calculate $I_{\text{pole}}(0)$ simply by using the usual series expansion of Bessel functions [10] in the limit $M \to 0$. The calculation relies on the Stirling formula for the Γ-function appearing in the series expansion for Bessel functions [10]. Omitting the details we write the result for $I_{\text{pole}}(0)$ for normal and longitudinal modes of the electromagnetic field together with ghosts (see appendix)

$$I_{\text{pole (nonphys and ghosts)}}(0) = -\frac{5}{12}. \tag{2.17}$$

Now combining the results (2.13)-(2.17), one has

$$\zeta(0)_{\text{normal, longitudinal, ghosts}} = \frac{53}{96}. \tag{2.18}$$

Remarkably, the total contribution of non-physical degrees of freedom and ghosts does not vanish. Adding to (2.18) the contribution $-\frac{77}{180}$ of physical modes obtained in [3] one finds

$$\zeta(0) = \frac{179}{1440}. \tag{2.19}$$

We now evaluate $\zeta(0)$ for the electromagnetic field on the flat 4-dimensional Euclidean background bounded by two concentric 3-spheres. To begin with, let us calculate the contribution to $\zeta(0)$ of physical degrees of freedom. The basis functions for them are now the linear combination $f_n(\tau) = C_1 \; I_n(M\tau) + C_2 \; K_n(M\tau)$ which should vanish at the 3-sphere boundaries of radii τ_+ and τ_- respectively, where $\tau_+ > \tau_-$. This leads to the eigenvalue condition

$$I_n^- \; K_n^+ - I_n^+ \; K_n^- = 0 \tag{2.20}$$

where $I_n^- \equiv I_n(M\tau_-), I_n^+ \equiv I_n(M\tau_+), K_n^- \equiv K_n(M\tau_-), K_n^+ \equiv K_n(M\tau_+)$. Using series expansions for modified Bessel functions one can see that the eigenvalue condition (2.20) has no fake roots. Since in (2.20) the coefficients of products of Bessel functions are independent of n, one finds that $I_{\text{pole}}(\infty) = 0$. Bearing in mind that the dimension of irreducible representations for transverse vector hyperspherical harmonics is $2 \; (n^2 - 1)$ [3], where $n = 2, \ldots$, and using the hypergeometric expansions for I_n and K_n, one can easily show that $I_{\rm pole}(0) = 0$. Using the asymptotic expansions for I_n and K_n one can also calculate $I_{\log}$ as

$$I_{\log} = -\sum_{n=2}^{\infty}(n^2 - 1) = -\frac{1}{2} \tag{2.21}$$

and correspondingly

$$\zeta(0)_{\rm transversal\ photons} = -\frac{1}{2}. \tag{2.22}$$

It is not difficult to consider also the contribution of ghosts. In this case one has an eigenvalue condition coinciding with (2.20) if we use $\nu \equiv +\sqrt{n^2 - 3/4}$ instead of n. Hence all 3 contributions to $\zeta(0)$ vanish, which implies

$$\zeta(0)_{\rm ghosts} = 0. \tag{2.23}$$

The next problem is the calculation of the contribution to $\zeta(0)$ of the decoupled normal mode, which has the form [1-2] $R_1(\tau) = C_1 \, \frac{1}{\tau} \, I_1(M\tau) + C_2 \, \frac{1}{\tau} \, K_1(M\tau)$. The derivative of this function should vanish at the 3-sphere boundaries. The determinant of the corresponding 2×2 matrix should vanish and this equality gives, as in the previous cases, the eigenvalue condition. Such a determinant has no fake roots. Thus, by using the uniform asymptotic expansions of Bessel functions one can see that the $I_{\log}$ value is $-\frac{1}{2}$. However, since our decoupled mode is non-vanishing for $\tau \in [\tau_-, \tau_+]$, one deals with a zero eigenvalue corresponding to a non-zero eigenfunction satisfying boundary conditions. The number $N_D = 1$ of such decoupled normal modes contributes to the full $\zeta(0)$ value. Hence one finds

$$\zeta(0)_{\rm decoupled\ mode} = I_{\log} + N_D = -\frac{1}{2} + 1 = \frac{1}{2}. \tag{2.24}$$

We now evaluate the contribution of the coupled normal and longitudinal modes of the electromagnetic field. Since in the two-boundary case the singularity at the origin is avoided, both I- and K-functions contribute to gauge modes. Hence the general solutions for $g_n(\tau)$ and $R_n(\tau)$ are

$$g_n(\tau) = C_1\, I_{\nu-1/2}(M\tau) + C_2\, (\nu - 1/2)\, I_{\nu+1/2}(M\tau) \\ + C_3\, K_{\nu-1/2}(M\tau) + C_4\, (\nu - 1/2)\, K_{\nu+1/2}(M\tau) \tag{2.25}$$

$$R_n(\tau) = \frac{1}{\tau}\left(C_1\, \frac{-1}{(\nu + 1/2)}\, I_{\nu-1/2}(M\tau) + C_2\, I_{\nu+1/2}(M\tau) \right. \\ \left. + C_3\, \frac{-1}{(\nu + 1/2)}\, K_{\nu-1/2}(M\tau) + C_4\, K_{\nu+1/2}(M\tau)\right). \tag{2.26}$$

After substitution of (2.25)-(2.26) into magnetic boundary conditions at the 3-sphere boundaries one has a system of 4 equations. The determinant of the corresponding 4×4 matrix should vanish. In such a determinant, the smallest power of M is M^{-2}. Thus, to avoid the appearance of fake roots in the eigenvalue condition, it is necessary to multiply it by M^2. Taking into account the formulae for the asymptotic expansions of Bessel functions one can see that the coefficient of $\log M$ in the logarithm of our eigenvalue condition vanishes, hence $I_{\log}$ vanishes as well. Let us now calculate $I_{\text{pole}}(\infty)$. Just as in the previous problem, this value is determined by n-dependent coefficients in the determinant and can be calculated from the expression $\frac{n^2}{2} \log\left(\frac{4\nu^2}{(\nu+1/2)^2}\right)$ along the lines described in the appendix. One has $I_{\text{pole}}(\infty) = -\frac{3}{16} - \frac{1}{24} = -\frac{11}{48}$.

Now using the usual hypergeometric expansions for modified Bessel functions one obtains $I_{\rm pole}(0) = -\frac{11}{48}$, and since the values of $I_{\rm pole}$ at ∞ and at 0 compensate each other, and $I_{\rm log} = 0$, one finds

$$\zeta(0)_{\text{normal and longitudinal}} = 0. \tag{2.27}$$

Combining the results (2.22)-(2.24) and (2.27) one has

$$\zeta(0) = 0 \tag{2.28}$$

which coincides with the covariant result. Of course, the covariant $\zeta(0)$ value is zero, since the volume contribution on the flat background vanishes whilst two-boundary contributions compensate each other (look at the corresponding formulae in [7]).

On imposing electric boundary conditions, which are motivated by supersymmetric quantum field theory [1], the situation for coupled normal and longitudinal modes is just opposite to that in the magnetic case, since normal modes and the normal derivatives of longitudinal modes vanish at the 3-sphere boundaries [1-2]. Defining $a_\nu \equiv \frac{-1}{(\nu+1/2)}$, $b_\nu \equiv (\nu - 1/2)$, the corresponding eigenvalue condition is the vanishing of the determinant of the matrix

$$\begin{pmatrix} a_\nu I^-_{b_\nu} & I^-_{b_\nu+1} & a_\nu K^-_{b_\nu} & K^-_{b_\nu+1} \\ a_\nu I^+_{b_\nu} & I^+_{b_\nu+1} & a_\nu K^+_{b_\nu} & K^+_{b_\nu+1} \\ (I^-_{b_\nu-1} + I^-_{b_\nu+1}) & b_\nu(I^-_{b_\nu} + I^-_{b_\nu+2}) & -(K^-_{b_\nu-1} + K^-_{b_\nu+1}) & -b_\nu(K^-_{b_\nu} + K^-_{b_\nu+2}) \\ (I^+_{b_\nu-1} + I^+_{b_\nu+1}) & b_\nu(I^+_{b_\nu} + I^+_{b_\nu+2}) & -(K^+_{b_\nu-1} + K^+_{b_\nu+1}) & -b_\nu(K^+_{b_\nu} + K^+_{b_\nu+2}) \end{pmatrix}.$$

This yields

$$I_{\rm log} = 0 \quad I_{\rm pole}(\infty) = I_{\rm pole}(0) = -\frac{11}{48} \tag{2.29}$$

which implies

$$\zeta(0)_{\text{normal and longitudinal}} = 0. \tag{2.30}$$

Hence one finds

$$\zeta(0) = 0. \tag{2.31}$$

Such a $\zeta(0)$ value coincides with the covariant result (as in the case of magnetic boundary conditions, the contributions of two 3-sphere boundaries in covariant formalism cancel each other).

It is easy to carry out the corresponding calculations in the Lorentz gauge (section 3). They yield again $\zeta(0) = 0$. Thus, we have shown that in the case of electric boundary conditions (just as in the case of magnetic ones) on the flat Riemannian 4-manifold with two 3-sphere boundaries leading to a well-defined 3+1 split of the 4-vector potential, results of covariant and mode-by-mode formalisms coincide.

3. One-boundary and two-boundary problems in the Lorentz gauge

In the Lorentz gauge, coupled eigenvalue equations for normal and longitudinal modes are again described by a system of the kind (2.1a)-(2.1b). However, the four operators are replaced by other operators $\widehat{E}_n, \widehat{F}_n, \widehat{G}_n, \widehat{H}_n$ respectively, where the operator $\widehat{E}_n$ coincides with $\widehat{A}_n$ written in section 2, whilst $\widehat{F}_n$, $\widehat{G}_n$ and $\widehat{H}_n$ are different and take the form

$$\widehat{F}_n \equiv \frac{2\,(n^2-1)}{\tau} \tag{3.1a}$$

$$\widehat{G}_n \equiv \frac{2}{\tau^3} \tag{3.1b}$$

$$\widehat{H}_n \equiv \frac{d^2}{d\tau^2} + \frac{3}{\tau}\,\frac{d}{d\tau} - \frac{(n^2+2)}{\tau^2} + \lambda_n. \tag{3.1c}$$

The corresponding system of equations can be diagonalized in the same way as the analogous system (2.1a)-(2.1b). Hence one finds a couple of Bessel-type equations, giving the general solutions for $g_n(\tau)$ and $R_n(\tau)$ in the one-boundary problem as

$$g_n(\tau) = C_1\; I_{n+1}(M\tau) + C_2\;(n+1)\; I_{n-1}(M\tau) \tag{3.2}$$

$$R_n(\tau) = \frac{1}{\tau}\Big(- C_1\; \frac{1}{(n-1)}\; I_{n+1}(M\tau) + C_2\; I_{n-1}(M\tau)\Big). \tag{3.3}$$

In the magnetic case we are studying, one sets to zero on the boundary the gauge-averaging functional. If this is the Lorentz functional, one is choosing Dirichlet conditions for g_n modes and Robin conditions for R_n modes, i.e. $g_n(\tau_+) = 0, \dot{R}_n(\tau_+) + \frac{3}{\tau_+}R_n(\tau_+) = 0$ [1-2]. Such a system leads to the eigenvalue condition

$$\det\begin{pmatrix} I^+_{n+1} & (n+1)I^+_{n-1} \\ -\frac{1}{(n-1)}\left(\frac{2I^+_{n+1}}{(M\tau_+/2)} + I^+_n + I^+_{n+2}\right) & \left(\frac{2I^+_{n-1}}{(M\tau_+/2)} + I^+_{n-2} + I^+_n\right)\end{pmatrix} = 0. \tag{3.4}$$

To get rid of fake roots it is necessary to divide the determinant (3.4) by M^{2n-1}, and then we can calculate $I_{\rm log}$ as

$$I_{\rm log} = -\sum_{n=2}^{\infty} n^3 = \frac{119}{120}. \tag{3.5}$$

$I_{\rm pole}(\infty)$ can be obtained by extracting the n-dependent coefficients in the determinant (3.4), which gives $\frac{n^2}{2}\log\left(\frac{2\,n}{(n-1)}\right)$. From this expression one obtains (see appendix)

$$I_{\rm pole}(\infty) = \frac{1}{6}. \tag{3.6}$$

Now we can calculate $I_{\rm pole}(0)$ by taking the logarithm of the determinant (3.4) in the limit $M \rightarrow 0$ and expanding it in inverse powers of n. The result is (see appendix)

$$I_{\rm pole}(0) = \frac{1}{360} + \frac{1}{3} = \frac{121}{360}. \tag{3.7}$$

Combining together the results (3.5)-(3.7) one finds

$$\zeta(0)_{\rm normal\ and\ longitudinal} = \frac{37}{45}. \tag{3.8}$$

The equation for the decoupled normal mode implies $R_1 = \frac{1}{\tau}\, I_2(M\tau)$, and the magnetic boundary condition leads to

$$\zeta(0)_{\rm decoupled} = -\frac{3}{4}. \tag{3.9}$$

The eigenvalue condition for ghosts in the Lorentz gauge coincides with the one for scalar fields and the corresponding $\zeta(0)$ value can be obtained from the well-known result for a scalar field subject to Dirichlet boundary conditions [1]. Taking into account the change of sign due to the fermionic nature of ghosts corresponding to spin-1 fields [1-2] one has

$$\zeta(0)_{\rm ghosts} = \frac{1}{90}. \tag{3.10}$$

Combining the results (3.8)-(3.10) one obtains

$$\zeta(0)_{\rm normal,\, longitudinal,\, ghosts} = \frac{1}{12}. \tag{3.11}$$

Correspondingly, the total $\zeta(0)$ value including the contribution $-\frac{77}{180}$ of physical modes is

$$\zeta(0) = -\frac{31}{90}. \tag{3.12}$$

Remarkably, (3.12) agrees with the corrected $\zeta(0)$ value obtained in [7], which relies on the analysis by Vassilevich [11]. However, (3.12) differs from the mode-by-mode result (2.19). This discrepancy seem to originate from the ill-definiteness of the 3+1 decomposition of the 4-vector potential on the manifold under consideration [9].

In the two-boundary case, the eigenvalue condition for the coupled normal and longitudinal modes in the Lorentz gauge on the flat Riemannian 4-manifold with two concentric 3-sphere boundaries is (on imposing magnetic boundary conditions)

$$\det \begin{pmatrix} I_{n+1}^{-} & (n+1)I_{n-1}^{-} & K_{n+1}^{-} & (n+1)K_{n-1}^{-} \\ I_{n+1}^{+} & (n+1)I_{n-1}^{+} & K_{n+1}^{+} & (n+1)K_{n-1}^{+} \\ -\frac{1}{(n-1)}\Big(\frac{2I_{n+1}^{-}}{M\tau_{-}/2} & \Big(\frac{2I_{n-1}^{-}}{M\tau_{-}/2} & -\frac{1}{(n-1)}\Big(\frac{2K_{n+1}^{-}}{M\tau_{-}/2} & \Big(\frac{2K_{n-1}^{-}}{M\tau_{-}/2} \\ +I_{n}^{-}+I_{n+2}^{-}\Big) & +I_{n-2}^{-}+I_{n}^{-}\Big) & -K_{n}^{-}-K_{n+2}^{-}\Big) & -K_{n-2}^{-}-K_{n}^{-}\Big) \\ -\frac{1}{(n-1)}\Big(\frac{2I_{n+1}^{+}}{M\tau_{+}/2} & \Big(\frac{2I_{n-1}^{+}}{M\tau_{+}/2} & -\frac{1}{(n-1)}\Big(\frac{2K_{n+1}^{+}}{M\tau_{+}/2} & \Big(\frac{2K_{n-1}^{+}}{M\tau_{+}/2} \\ +I_{n}^{+}+I_{n+2}^{+}\Big) & +I_{n-2}^{+}+I_{n}^{+}\Big) & -K_{n}^{+}-K_{n+2}^{+}\Big) & -K_{n-2}^{+}-K_{n}^{+}\Big) \end{pmatrix} = 0. \tag{3.13}$$

The contribution to $I_{\log}$ of the determinant (3.13) is 0. The function determining the behaviour of $I_{\rm pole}(M^2)$ as $M \to \infty$ and $n \to \infty$ has the form $\frac{n^2}{2}\log\left(\frac{4n^2}{(n-1)^2}\right)$ and correspondingly $I_{\rm pole}(\infty) = \frac{1}{3}$. After taking the logarithm of the determinant (3.13) at $M = 0$ and expanding it in inverse powers of n one obtains $I_{\rm pole}(0) = \frac{1}{3}$. Thus, one finds again compensation of $I_{\rm pole}(\infty)$ and $I_{\rm pole}(0)$ and $\zeta(0)_{\text{normal and longitudinal}} = 0$. Finally one obtains the total result

$$\zeta(0) = 0 \tag{3.14}$$

coinciding with that obtained in section 2 and with the covariant one.

4. Concluding remarks

The main results of our investigation are as follows.

First, we have proved that, in the case of flat Euclidean 4-space bounded by two concentric 3-spheres, the mode-by-mode analysis of one-loop quantum amplitudes for Euclidean Maxwell theory agrees with the covariant formulae used in [7]. Moreover, such Faddeev-Popov quantum amplitudes are indeed gauge independent in the cases studied in our paper (see below). Second, we have shown that contributions of gauge modes and ghost fields to the full $\zeta(0)$ value do not cancel each other. Hence the reduction of a gauge theory to its physical degrees of freedom before quantization only yields the contribution of such degrees of freedom to the quantum theory, but is by itself insufficient to describe a gauge-invariant quantum theory. Third, we have provided evidence that, when the boundary 3-geometry does not lead to a well-defined 3+1 split of the 4-vector potential, some inconsistencies occur, i.e. one-loop quantum amplitudes are gauge dependent.

Interestingly, this seems to complement the Hartle-Hawking programme [5], which relies on a Wick-rotated path integral with just one boundary 3-geometry. More precisely, on the one hand we know that the use of unitary gauge, 3+1 decomposition, Hamiltonian formalism and extraction of physical degrees of freedom is necessary to recover the physical content of the theory. Moreover, the natural way of implementing the Hartle-Hawking programme [5] involves the consideration of the wave function of the universe in the Lorentzian region in terms of physical degrees of freedom (with the subsequent analytic continuation to the Euclidean-time region, whenever this is possible). On the other hand, the Euclidean germ from which our universe might originate [5] is the Riemannian 4-manifold which does not possess a well-defined 3+1 split [9].

Does this mean that it is impossible to carry out the Hartle-Hawking programme in a consistent way ? Indeed, it should be emphasized that our present understanding of quantum field theory and quantum gravity does not enable one to make a conclusive statement. Since quantum amplitudes involve differential operators and their eigenvalues, the mode-by-mode analysis based on ζ-function regularization remains of primary importance. Moreover, the analytic continuation back to real, Lorentzian time may be impossible [1]. Hence there are cases where we may have to limit ourselves to the elliptic boundary-value problems of Riemannian geometry, where one cannot define the notion of time-evolution.

Other interesting problems remain unsolved as well. In fact, one still has to obtain a general proof of gauge invariance in the two-boundary case. Our paper has only focused on two *particular* gauge-averaging functionals. Moreover, one has to repeat our mode-by-mode analysis of one-loop amplitudes for spin-$\frac{3}{2}$ fields and gravitation, as well as deal with curved background 4-geometries. In the latter case, it is impossible to decouple gauge modes without studying fourth-order ordinary differential equations, and one faces the technical problem of working out the uniform asymptotic expansions of their solutions.

The new tools developed in our paper and in the recent literature [1-4,7,9,11] make us feel that a complete mode-by-mode analysis of quantized gauge fields and gravitation in the presence of boundaries is in sight. Although this would be far from having a good theory of quantum gravity, it seems to add evidence in favour of quantum cosmology being at the very heart of modern quantum field theory.

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Appendix

To clarify our $\zeta(0)$ calculations, some of them are here presented in more detail. We first evaluate the contribution to $\zeta(0)$ of non-physical modes in the Lorentz gauge for the manifold with one boundary (section 3). For this purpose, let us write the determinant (3.4) giving the eigenvalue condition in the explicit form

$$I_{n+1}^{+}\left(\frac{2I_{n-1}^{+}}{(M\tau_{+}/2)}+I_{n-2}^{+}+I_{n}^{+}\right)+\frac{(n+1)}{(n-1)}\,I_{n-1}^{+}\left(\frac{2I_{n+1}^{+}}{(M\tau_{+}/2)}+I_{n}^{+}+I_{n+2}^{+}\right)=0. \qquad (A1)$$

The minimal power of M in (A1) is $2n-1$. Thus, to get rid of the fake roots $M=0$ we should divide our eigenvalue condition by M^{2n-1}.

The leading behaviour of $I_n(nM\tau)$ is determined by the exponent $e^{nM\tau}$ multiplied by $\frac{1}{\sqrt{M\tau}}$. Hence, after taking the logarithm of (A1) divided by M^{2n-1} one finds that the only term proportional to $\log M$ is $(-2n)\log M$. Having this expression we can get, by direct summation, the result (3.5) for $I_{\rm log}$. For the calculation of $I_{\rm pole}(\infty)$ it is more convenient to re-write (A.1) through derivatives of Bessel functions instead of using the recurrence formulae. Thus we have, instead of (A1), the equation

$$I_{n+1}^{+}\left(\frac{I_{n-1}^{+}}{(M\tau_{+}/2)}+I_{n-1}^{\prime +}\right)+\frac{(n+1)}{(n-1)}\,I_{n-1}^{+}\left(\frac{I_{n+1}^{+}}{(M\tau_{+}/2)}+I_{n+1}^{\prime +}\right)=0. \qquad (A2)$$

All terms in (A2) have the same leading behaviour determined by exponential functions. It is also known that non-exponential terms, after taking logarithms and after expanding these in inverse powers of n, are proportional to inverse powers of M and hence vanish in the limit $M\to\infty$ [4]. Then equation (A.2) has the form: exp. terms $\times\left(1+\frac{(n+1)}{(n-1)}\right)=0$. Moreover, it can be shown that also exponential terms do not contribute to $I_{\rm pole}(\infty)$ [4]. Thus, one should only take the logarithm of $\left(1+\frac{(n+1)}{(n-1)}\right)$ and pick out the terms which, multiplied by $\frac{n^2}{2}$, yield terms proportional to $1/n$. This leads to the result (3.6) for $I_{\rm pole}(\infty)$.

To calculate $I_{\rm pole}(0)$ it is more convenient to use again the expression (A1) for the eigenvalue condition. Taking the limit $M\to 0$, one finds the limiting form of (A1) as

$$\frac{1}{\Gamma(n+2)}\left(\frac{1}{\Gamma(n-1)}+\frac{2}{\Gamma(n)}\right)+\frac{(n+1)}{(n-1)}\frac{1}{\Gamma(n)}\left(\frac{1}{\Gamma(n+1)}+\frac{2}{\Gamma(n+2)}\right)=0. \qquad (A3)$$

Thus, inserting the asymptotic Stirling formula for Γ-functions in the logarithm of the left-hand side of equation (A3) and expanding it in inverse powers of n, one finds the result

(3.7) for $I_{\rm pole}(0)$. For this purpose, we also use the expansion of $\log(1+\omega)$ as $\omega \rightarrow 0$, i.e. $\log(1+\omega)=\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\omega^k}{k}$ [1].

When one evaluates $I_{\rm pole}(\infty)$ and $I_{\rm pole}(0)$ in the Esposito gauge, it is also necessary to use the asymptotic expansion of ν as $n \rightarrow \infty$ (cf (5.10) of [2])

$$\nu \sim n\left(1-\frac{3}{8}\frac{1}{n^2}-\frac{9}{128}\frac{1}{n^4}+{\rm O}(n^{-6})\right). \tag{A4}$$

The contribution of gauge modes to $I_{\rm pole}(0)$ in (2.17) is then obtained by taking the coefficient of $\frac{1}{n}$ in the asymptotic expansion as $n \rightarrow \infty$ of $\frac{n^2}{2}\log\left[\frac{(\nu-1/2)/(\nu+1/2)}{\Gamma(\nu+1/2)}\right]^2$. This yields the contribution

$$I_A=-\frac{9}{128}-\frac{1}{32}+\frac{1}{360}-\frac{3}{8}-\frac{1}{12}-\frac{1}{48}+\frac{1}{16}. \tag{A5}$$

Moreover, the contribution of ghost modes to $I_{\rm pole}(0)$ in (2.17) is obtained by taking the coefficient of $\frac{1}{n}$ in the asymptotic expansion as $n \rightarrow \infty$ of $\frac{n^2}{2}\log\left(\frac{1}{\Gamma(\nu+1)}\right)$ and then multiplying by -2, since the ghost field is fermionic and complex. Hence one finds the contribution

$$I_B=\frac{9}{128}+\frac{1}{32}-\frac{1}{360}. \tag{A6}$$

Thus, the full $I_{\rm pole}(0)$ in (2.17) is given by

$$I_{\rm pole}(0)=I_A+I_B=-\frac{5}{12}. \tag{A7}$$

When one evaluates $I_{\rm pole}(\infty)$ and $I_{\rm pole}(0)$ for coupled gauge modes, one finds that only K functions at $\tau=\tau_-$ and I functions at $\tau=\tau_+$ contribute. The corresponding $I_{\rm pole}(0)$

values are obtained as the coefficient of $\frac{1}{n}$ when $n \rightarrow \infty$ and $M \rightarrow 0$ in the asymptotic expansions of the terms $\frac{n^2}{2} \log \frac{(\nu-1/2)}{(\nu+1/2)}$, $\frac{n^2}{2} \log \frac{(n+1)}{(n-1)}$, $\frac{n^2}{2} \log \frac{(\nu-1/2)}{(\nu+1/2)}$ respectively.

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