October 1993, DSF preprint 93/7 (revised version)

GAUGE-AVERAGING FUNCTIONALS FOR EUCLIDEAN MAXWELL THEORY IN THE PRESENCE OF BOUNDARIES

Giampiero Esposito

Istituto Nazionale di Fisica Nucleare

Mostra d'Oltremare Padiglione 20, 80125 Napoli, Italy;

Dipartimento di Scienze Fisiche

Mostra d'Oltremare Padiglione 19, 80125 Napoli, Italy;

Dipartimento di Fisica Teorica

Università degli Studi di Salerno

84081 Baronissi, Italy.

Abstract. This paper studies the one-loop expansion of the amplitudes of electromagnetism about flat Euclidean backgrounds bounded by a 3-sphere, recently considered in perturbative quantum cosmology, by using ζ-function regularization. For a specific choice of gauge-averaging functional, the contributions to the full $\zeta(0)$ value owed to physical degrees of freedom, decoupled gauge mode, coupled gauge modes and Faddeev-Popov ghost field are derived in detail, and alternative choices for such a functional are also studied. This analysis enables one to get a better understanding of different quantization techniques for gauge fields and gravitation in the presence of boundaries.

PACS numbers: 0370, 0460

1. Introduction

The way in which quantum fields respond to the presence of boundaries is responsible for many interesting physical effects (e.g. the Casimir effect), and plays a very important role in quantum gravity and quantum cosmology. In that case, the (formal) quantization of gauge fields and gravitation via Wick-rotated Feynman path integrals is expressed in terms of quantum amplitudes of going from a 3-metric and a field configuration on an initial spacelike surface to a 3-metric and a field configuration on a final spacelike surface. Whilst mathematics enables one to understand which compact boundary geometries do actually exist, the methods of quantum field theory fix the boundary conditions for scalar, fermionic and gauge fields, as well as gravitation and corresponding ghost fields for spins 1, $\frac{3}{2}$ and 2. Although the full theory via path integrals is in general ill-defined, since there is little understanding of the measure for quantum gravity and of the corresponding sum over all Riemannian 4-geometries, the one-loop approximations of these ill-defined functional integrals can be evaluated in terms of well-defined mathematical concepts [1-9]. Since one has then to study determinants of second-order, self-adjoint, elliptic operators, the basic tool used by theoretical physicists is the generalized Riemann ζ-function formed by the eigenvalues of these operators as [9]

$$\zeta(s) \equiv \sum_{n=n_0}^{\infty} \sum_{m=m_0}^{\infty} d_m(n) \lambda_{n,m}{}^{-s} \quad . \tag{1.1}$$

With our notation, n, m are degeneracy labels, and $d_m(n)$ is the degeneracy of the eigenvalues $\lambda_{n,m}$, which is taken to depend only upon the integer n, as happens in many physically relevant applications (including the ones described in this paper). The regularized $\zeta(0)$ value yields both the scaling of the one-loop prefactor and the one-loop divergences of physical theories.

In particular, the problem of the one-loop finiteness of (extended) supergravity theories in the presence of boundaries is still receiving careful consideration in the current literature [1-9]. As emphasized in [9-11], one can perform one-loop calculations paying attention to: (1) S-matrix elements; (2) topological invariants; (3) presence of boundaries. For example, in the case of pure gravity with vanishing cosmological constant, $\Lambda = 0$, it is known that one-loop on-shell S-matrix elements are finite. This property is also shared by $N = 1$ supergravity when $\Lambda = 0$, and in that theory two-loop on-shell finiteness also holds. However, when $\Lambda \neq 0$ both pure gravity and $N = 1$ supergravity are no longer one-loop finite in the sense (1) and (2), because the non-vanishing on-shell one-loop counterterm is given by [10]

$$S_{(1)} = \frac{1}{\widetilde{\epsilon}} \left[A\chi - \frac{2BG\Lambda S}{3\pi} \right] \quad . \tag{1.2}$$

In (1.2) $\widetilde{\epsilon} \equiv n - 4$ is the dimensional-regularization parameter, χ is the Euler number, S is the classical action on-shell, and one finds [9,10]: $A = \frac{106}{45}, B = -\frac{87}{10}$ for pure gravity, and $A = \frac{41}{24}, B = -\frac{77}{12}$ for $N = 1$ supergravity. Thus, $B \neq 0$ is responsible for lack of S-matrix one-loop finiteness, and $A \neq 0$ does not yield topological one-loop finiteness.

In the presence of boundaries, however, a much larger number of counterterms can be obtained even just at one-loop order in perturbation theory, using the extrinsic-curvature tensor and the Ricci tensor of the boundary. Thus, if any theory of quantum gravity can be studied from the perturbative point of view, boundary effects play a key role in understanding whether it has interesting and useful finiteness properties. It is therefore necessary to analyze in detail the structure of the one-loop boundary counterterms for fields of various spins, and the techniques developed so far are described in detail in [2-9]. The corresponding problems are as follows.

(i) *Choice of locally supersymmetric boundary conditions* [1-4,8-9]. They involve the normal to the boundary and the field for spin $\frac{1}{2}$, the normal to the boundary and the spin-$\frac{3}{2}$ potential for gravitinos, Dirichlet conditions for real scalar fields, magnetic or electric field for electromagnetism, mixed boundary conditions for the 4-metric of the gravitational field (and in particular Dirichlet conditions on the perturbed 3-metric).

(ii) *Quantization techniques.* One-loop amplitudes can be evaluated by first reducing the classical theory to the physical degrees of freedom by choice of gauge and then quantizing, or by using the gauge-averaging method of Faddeev and Popov, or by applying the extended-phase-space Hamiltonian path integral of Batalin, Fradkin and Vilkovisky [2-9].

(iii) *Regularization techniques.* The generalized Riemann ζ-function [12] and its regularized $\zeta(0)$ value can be obtained by studying the eigenvalue equations obeyed by perturbative modes, once the corresponding degeneracies are known, or by using geometrical formulae for one-loop counterterms which generalize well-known results for scalar fields, but make no use of mode-by-mode eigenvalue conditions and degeneracies [2-9,13].

Since the various quantization and regularization techniques mentioned so far have been found to give rise to different estimates of the $\zeta(0)$ value for all spins > 0, it is crucial to get a better understanding of the $\zeta(0)$ values obtained by using the manifestly gauge-invariant quantization techniques previously listed. The aim of this paper is to perform this analysis in the simplest (but highly non-trivial) case, i.e. the one-loop amplitudes of vacuum Maxwell theory when a 3-sphere boundary is present. One is thus led to make a $3+1$ split of the 4-vector potential, expanding its components on a family of 3-spheres centred on the origin as [9,14]

$$A_0(x,\tau) = \sum_{n=1}^{\infty} R_n(\tau) Q^{(n)}(x) \tag{1.3a}$$

$$A_k(x,\tau) = A_k^T(x,\tau) + A_k^L(x,\tau) \qquad \text{for all } k = 1,2,3 \tag{1.3b}$$

where $Q^{(n)}(x)$ are scalar harmonics on the 3-sphere, whereas the transverse part A_k^T and the longitudinal part A_k^L are expanded, $\forall k = 1,2,3$, as

$$A_k^T(x,\tau) = \sum_{n=2}^{\infty} f_n(\tau) S_k^{(n)}(x) \tag{1.4}$$

$$A_k^L(x,\tau) = \sum_{n=2}^{\infty} g_n(\tau) P_k^{(n)}(x) \quad . \tag{1.5}$$

Of course, the $S_k^{(n)}(x)$ and $P_k^{(n)}(x)$ are the transverse and longitudinal vector harmonics on S^3 respectively, and their properties are described in detail in the appendix of [14]. Note that, strictly, normal and tangential components of A_μ are only well-defined at the 3-sphere boundary, where $\tau = a$, since a unit normal vector field inside matching the normal to S^3 at the boundary is ill-defined at the origin. As in all mode-by-mode calculations, we are performing a *local* analysis, where one takes that scalar field whose expansion on a family of 3-spheres centred on the origin matches the $A_0(x,a)$ value at the boundary. Moreover, one takes that 3-vector field whose expansion on a family of 3-spheres centred on the origin matches the $A_k(x,a)$ value on S^3. One then has to show that an unique regular solution of the corresponding boundary-value problem exists, although the unit normal vector field inside is ill-defined at the origin. It will be shown in section 3 that this is indeed the case. [We are grateful to Dr. A. Kamenshchik for correspondence about this problem]

Our paper is thus organized as follows. Section 2 derives the contribution of the physical degrees of freedom (i.e. the modes $f_n(\tau)$ appearing in (1.4)) to the $\zeta(0)$ value, following [9,14]. Section 3 studies the coupled set of second-order ordinary differential equations expressing the eigenvalue equations obeyed by the gauge modes g_n and R_n, $\forall n \geq 2$. Section 4 derives the contribution of the R_1-mode of (1.3a), which remains decoupled from $R_n(\tau)$ and $g_n(\tau)$, $\forall n \geq 2$. Section 5 studies the corresponding form of the ghost operator, and various possible choices of the gauge-averaging term in the Faddeev-Popov formula. Open problems and concluding remarks are presented in section 6.

2. Physical degrees of freedom

Within the Faddeev-Popov approach to Euclidean Maxwell theory one deals with gauge-invariant amplitudes of the form [15]

$$Z[g] \equiv \int \widetilde{\mu}_1[A]\delta[\Phi(A)] \det \left[\frac{\delta\Phi(A)}{\delta\epsilon}\right] \exp\left[-\int_M \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\sqrt{\det g}\; d^4x\right] \qquad (2.1)$$

where A_μ is the 4-vector potential, $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ denotes the electromagnetic-field tensor, and g is the background 4-metric. These amplitudes are more conveniently re-expressed as

$$Z[g] = \int \widetilde{\mu}_1[A]\widetilde{\mu}_2[c,c^*] \ \exp\Bigl(- \widetilde{I}_E\Bigr) \tag{2.2}$$

where the total Euclidean action $\widetilde{I}_E \equiv \hat{I}_E + I_{GA} + I_{gh} \equiv I_E + I_{gh}$ is given by

$$\widetilde{I}_E = I_{gh} + \int_M \left[\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \frac{\Bigl[\Phi(A)\Bigr]^2}{2\alpha} \right] \sqrt{\det g} \ d^4x \quad . \tag{2.3}$$

In these formulae $\Phi(A)$ is an arbitrary gauge-averaging functional which depends on the $U(1)$ potential A and its covariant derivatives, and α is a positive dimensionless parameter. I_{gh} is the corresponding ghost-field action. Moreover, $\widetilde{\mu}_1[A]$, $\widetilde{\mu}_2[c,c^*]$, $\det \left[\frac{\delta\Phi(A)}{\delta\epsilon}\right]$ are a suitable measure on the space of connections, a suitable measure for ghosts, and the Faddeev-Popov determinant respectively. The inclusion of gauge-averaging functionals and corresponding ghost fields (cf section 5) is necessary to extract the volume of the gauge group. In other words, on integrating over all field configurations one integrates infinitely many times over the volume of the gauge group, whereas we need to concentrate the measure over a subset of configurations containing a single point for each orbit of the gauge group. This is achieved using (2.1)-(2.3).

In recent years, the case of flat Euclidean backgrounds bounded by a 3-sphere has been studied as the first step of a program aiming to get a better understanding of one-loop properties of supersymmetric field theories in the presence of boundaries [9]. As described in section 1, for this purpose one is then led to make a $3+1$ split of the 4-vector potential as in (1.3)-(1.5). In this section we compute the contribution of the physical degrees of freedom $f_n(\tau)$ of (1.4) to the one-loop amplitudes $Z^{(1)}$ of vacuum Maxwell theory in four-dimensions. If the measure in the path integral is scale-invariant (see [9,14] and references therein) such $Z^{(1)}$ amplitudes take the asymptotic form

$$Z^{(1)}(a) \sim W \; a^{\zeta(0)} \; e^{-I} \tag{2.4}$$

where W is an arbitrary constant and a is the 3-sphere radius.

The physical modes $f_n(\tau)$ are always decoupled from the gauge modes by virtue of the properties of the transverse vector harmonics. The corresponding elliptic operator in the Euclidean action is found to be [9,14]

$$D_n \equiv -\frac{1}{\tau}\frac{d}{d\tau}\left(\tau\frac{d}{d\tau}\right) + \frac{n^2}{\tau^2} \qquad \text{for all } n \geq 2 \tag{2.5}$$

whose eigenfunctions are $f_n(\tau) = A_n J_n(\sqrt{E}\tau)$, A_n being a constant [16]. Since locally supersymmetric boundary conditions require that either the magnetic field or the electric field should vanish on S^3 [9,14], one has to compute the regularized $\zeta(0)$ value for the generalized zeta-function obtained from the eigenvalues of D_n when $f_n(\tau)$ is subject to Dirichlet conditions on S^3 (i.e. magnetic case) or Neumann conditions on S^3 (i.e. electric case). In the magnetic case, following the detailed analysis of [14], one finds

$$\zeta_B^{(PDF)}(0) = -\frac{77}{180} \tag{2.6}$$

where the label (PDF) reminds us that (2.6) is the contribution of the physical degrees of freedom to the full $\zeta(0)$. In the electric case, following section 5.9 of [9], we begin by taking the Laplace transform of the heat equation, where $J_n(\sqrt{E}\tau)$ is replaced by the linear combination $A_n I_n(\sigma\tau) + B_n K_n(\sigma\tau)$. The ratio $\frac{A_n}{B_n}$ is found by requiring that $\frac{d}{d\tau}\Bigr(A_n I_n(\sigma\tau) + B_n K_n(\sigma\tau)\Bigr)(a) = 0$, $\forall n \geq 2$, which takes into account the eigenvalue condition $\dot{J}_n(\sqrt{E}a) = 0$, where a is the 3-sphere radius. Thus, the Laplace transform of the kernel of the heat equation for spin 1 when $\dot{f}_n(a) = 0$, $\forall n \geq 2$, is an infinite sum of products G_n of functions $\widetilde{G}_n$ of the type $\widetilde{G}_n = T\Bigr(A_n I_n(\sigma T) + B_n K_n(\sigma T)\Bigr)$. More precisely, defining $\tau_< \equiv min(\tau, \tau')$, $\tau_> \equiv max(\tau, \tau')$, one finds that $G_n\Bigr(\tau, \tau', \sigma^2\Bigr) = \widetilde{G}_n\Bigr(\tau_<, \sigma^2\Bigr)\widetilde{G}_n\Bigr(\tau_>, \sigma^2\Bigr)$, where

$$\widetilde{G}_n\Bigr(\tau_<, \sigma^2\Bigr) = \tau_< I_n(\sigma\tau_<) \tag{2.7a}$$

$$\widetilde{G}_n\Bigr(\tau_>, \sigma^2\Bigr) = \tau_>\left[K_n(\sigma\tau_>) - \frac{K_n'(\sigma a)}{I_n'(\sigma a)} I_n(\sigma\tau_>)\right] \quad . \tag{2.7b}$$

This implies that the free part of the heat kernel is equal to the one found in [14], and hence does not contribute to $\zeta(0)$. We therefore study the interacting part [9]

$$G^{int}\Bigr(\sigma^2\Bigr) = -\sum_{n=2}^{\infty}\Bigr(n^2 - 1\Bigr)\frac{K_n'(\sigma a)}{K_n(\sigma a)}\frac{I_n(\sigma a)}{I_n'(\sigma a)} f(n; \sigma a) \tag{2.8}$$

where $f(n; \sigma a)$ is the function defined in equation (4.1.18) of [9]. Thus, we have to work out the uniform asymptotic expansions of the various terms on the r.h.s. of (2.8) according to the relations (4.4.13)-(4.4.22) of [9]. Setting $a = 1$ for simplicity, and defining $y \equiv \frac{n}{\sqrt{n^2+\sigma^2}}$, this yields

$$\frac{K_n'(\sigma)}{K_n(\sigma)}\frac{I_n(\sigma)}{I_n'(\sigma)} \sim \left[A_0(y) + \frac{A_1(y)}{n} + \frac{A_2(y)}{n^2} + \frac{A_3(y)}{n^3} + ...\right] \tag{2.9}$$

where [9]

$$A_0(y) = -1 \tag{2.10}$$

$$A_1(y) = -y\Big(1 - y^2\Big) \tag{2.11}$$

$$A_2(y) = -\frac{y^2}{2}\Big(1 - y^2\Big)^2 \tag{2.12}$$

$$A_3(y) = -\frac{y^3}{16}\Big(1 - y^2\Big)\Big(10 - 68y^2 + 74y^4\Big) \quad . \tag{2.13}$$

Thus, since $f(n;\sigma) \sim \frac{\sqrt{n^2+\sigma^2}}{\sigma^2}\Big[\frac{B_1(y)}{n} + \frac{B_2(y)}{n^2} + \frac{B_3(y)}{n^3} + \frac{B_4(y)}{n^4} + ...\Big]$, using the explicit forms of $B_i(y)$ appearing in equation (4.4.12) of [9] one finds

$$\frac{K_n'(\sigma)}{K_n(\sigma)}\frac{I_n(\sigma)}{I_n'(\sigma)}f(n;\sigma) \sim \frac{\sqrt{n^2+\sigma^2}}{\sigma^2}\left[\frac{C_1(y)}{n} + \frac{C_2(y)}{n^2} + \frac{C_3(y)}{n^3} + \frac{C_4(y)}{n^4} + ...\right] \tag{2.14}$$

where [9]

$$C_1(y) = -\frac{y}{2}\Big(1 - y^2\Big) \tag{2.15}$$

$$C_2(y) = \frac{y^2}{2}\Big(1 - y^2\Big)\Big(2y^2 - 1\Big) \tag{2.16}$$

$$C_3(y) = -\frac{y^3}{8}\Big(1 - y^2\Big)\Big(3 - 20y^2 + 21y^4\Big) \tag{2.17}$$

$$C_4(y) = -\frac{y^4}{16}\Big(1 - y^2\Big)\Big(9 - 122y^2 + 301y^4 - 196y^6\Big) \quad . \tag{2.18}$$

Note that additional terms in (2.14) have not been computed since they give a contribution equal to $\mathrm{O}(\sqrt{t})$ (t being a parameter not related to Lorentzian time), and hence do not affect the $\zeta(0)$ value. Thus, using (2.8) and (2.14)-(2.18) and taking the inverse Laplace transform, one finds that the integrated heat kernel has an asymptotic expansion as $t \rightarrow 0^+$ given by [9]

$$G^{int}(t) \sim -\sum_{n=2}^{\infty}\Bigl(n^2-1\Bigr)\sum_{i=1}^{4}\widetilde{f}_i(n,t)+\mathrm{O}(\sqrt{t}) \tag{2.19}$$

where [9]

$$\widetilde{f}_1(n,t)=-\frac{1}{2}e^{-n^2 t} \tag{2.20}$$

$$\widetilde{f}_2(n,t)=\frac{4}{3}\frac{t^{\frac{3}{2}}}{\sqrt{\pi}}n^2 e^{-n^2 t}-\sqrt{\frac{t}{\pi}}e^{-n^2 t} \tag{2.21}$$

$$\widetilde{f}_3(n,t)=-\frac{3}{8}te^{-n^2 t}+\frac{5}{4}t^2 n^2 e^{-n^2 t}-\frac{7}{16}t^3 n^4 e^{-n^2 t} \tag{2.22}$$

$$\widetilde{f}_4(n,t)=-\frac{3}{4}\frac{t^{\frac{3}{2}}}{\sqrt{\pi}}e^{-n^2 t}+\frac{61}{15}\frac{t^{\frac{5}{2}}}{\sqrt{\pi}}n^2 e^{-n^2 t}-\frac{301}{105}\frac{t^{\frac{7}{2}}}{\sqrt{\pi}}n^4 e^{-n^2 t}+\frac{392}{945}\frac{t^{\frac{9}{2}}}{\sqrt{\pi}}n^6 e^{-n^2 t} \quad . \tag{2.23}$$

The interacting part $G^{int}(t)$ is an even function of n, and we can compute its contribution to $\zeta(0)$ using the Watson transform defined in [9]. The poles of the integrand at 0 and at ± 1 are excluded, since the sum over all n in (2.19) only starts from $n=2$. The poles at ± 1 do not contribute, because the integrand of the Watson transform has zeros at ± 1. In our case the constant contribution arising from the poles is given by the constant term appearing in the inverse Laplace transform of $-\frac{1}{2}\widetilde{f}(0,\sigma)$, which is equal to $\frac{1}{4}$. One thus finds

$$G^{int}(t) \sim \left[\frac{1}{4}+\mathrm{O}(\sqrt{t})-\int_0^{\infty}\Bigl(\rho^2-1\Bigr)\sum_{i=1}^{4}\widetilde{f}_i(\rho,t)\,d\rho\right]$$

$$\sim \frac{\sqrt{\pi}}{8}t^{-\frac{3}{2}}-\frac{1}{4}t^{-1}-\frac{55}{256}\sqrt{\pi}t^{-\frac{1}{2}}-\frac{1}{6}-\frac{1}{90}+\frac{1}{4}+\mathrm{O}(\sqrt{t}) \tag{2.24}$$

where the contributions $-\frac{1}{6}$ and $-\frac{1}{90}$ are owed to (2.21) and (2.23) respectively. One thus obtains the PDF value

$$\zeta_E^{(PDF)}(0) = \frac{13}{180} \quad . \tag{2.25}$$

3. Coupled gauge modes

The Euclidean action is obtained multiplying the Lorentzian action by $-i$, setting $t = -i\tau$, and bearing in mind that $(A_0)_L dt = (A_0)_E d\tau$, so that the r_n-modes appearing in the Lorentzian formulae of [14] are related to the Euclidean R_n-modes by $r_n = iR_n$. Moreover, for flat Euclidean backgrounds bounded by a 3-sphere, we choose the gauge-averaging term $\frac{\Phi^2}{2\alpha}$, where Φ is defined as (cf (5.1))

$$\Phi \equiv \frac{\partial A_0}{\partial \tau} + {}^{(3)}\nabla^i A_i = \sum_{n=1}^{\infty} \dot{R}_n(\tau) Q^{(n)}(x) - \tau^{-2} \sum_{n=2}^{\infty} g_n(\tau) Q^{(n)}(x) \quad . \tag{3.1}$$

Note that in (3.1) we have used the property of flat backgrounds ${}^{(4)}\nabla^0 A_0 = \frac{\partial A_0}{\partial \tau}$, and the relation $s^{ij} = \tau^{-2} c^{ij}$ between the contravariant 3-metric s^{ij}, and the contravariant 3-metric c^{ij} on a unit 3-sphere. Covariant differentiation on a unit S^3, denoted by a vertical stroke, yields $P_k^{(n)\ |k}(x) = -Q^{(n)}(x)$ [14]. With this choice of gauge-averaging functional, the corresponding differential operator acting on R_n-modes will turn out to be the one-dimensional Laplace operator for scalars if $\alpha = 1$, as we would expect in the light of the expansion (1.3a). Note also that (3.1) does not represent the Lorentz gauge-averaging functional (various alternative possibilities are studied in section 5). Thus, the part $I_E(g,R)$ of the Euclidean action quadratic in gauge modes is in our case [9]

$$I_E(g,R) - \frac{1}{2\alpha}\int_0^1 \tau^3 (\dot{R}_1)^2 \, d\tau = \sum_{n=2}^{\infty} \int_0^1 \left[\frac{\tau}{2(n^2-1)} \Big(\dot{g}_n - (n^2-1) R_n \Big)^2 \right.$$

$$\left. + \frac{\tau}{2\alpha} \Big(-\frac{g_n}{\tau} + \tau \dot{R}_n \Big)^2 \right] \, d\tau \tag{3.2}$$

where we have inserted the flat-background hypothesis $N=1$, $a(\tau)=\tau$. The physical degrees of freedom and the ghost field decouple from (3.2). Because we are here dealing with all degrees of freedom, we need further boundary conditions on the modes for A_0 and the whole of A_k. For example, we may set to zero on S^3 the whole of A_k: $f_n(1) = g_n(1) = 0, \ \forall n \geq 2$. This, of course, implies the vanishing on S^3 of the magnetic field $\mathbf{B}$, whereas the converse does not hold, because $\mathbf{B}$ only depends on the f_n-modes. As explained in [17], in this case the gauge-averaging term has to vanish as well on S^3 by virtue of Becchi-Rouet-Stora-Tyutin (BRST) invariance. In light of (3.1), this implies that $\dot{R}_n(1) = 0, \ \forall n \geq 1$. The ghost operator is then self-adjoint only if Dirichlet boundary conditions are imposed. Viceversa, if Neumann boundary conditions are chosen for the ghost field, remaining boundary conditions compatible with BRST invariance are $\dot{f}_n(1) = 0$ and $\dot{g}_n(1) = 0, \ \forall n \geq 2$; $R_n(1) = 0, \ \forall n \geq 1$. This case is then called electric.

We now integrate by parts in (3.2) and use the generalized magnetic or electric boundary conditions described above. Thus, defining $\forall n \geq 2$ the second-order differential operators

$$\hat{A}_n(\tau) \equiv -\frac{d^2}{d\tau^2} - \frac{1}{\tau}\frac{d}{d\tau} + \frac{(n^2-1)}{\alpha \tau^2} \tag{3.3}$$

$$\hat{B}_n(\tau) \equiv \frac{1}{\alpha}\left(-\frac{d^2}{d\tau^2} - \frac{3}{\tau}\frac{d}{d\tau}\right) + \frac{(n^2-1)}{\tau^2} \tag{3.4}$$

we find $\forall n \geq 2$ the fundamental result [9]

$$I_E^{(n)}(g,R) = \frac{1}{2}\int_0^1 \frac{\tau g_n}{(n^2-1)}(\hat{A}_n g_n)\, d\tau + \frac{1}{2}\int_0^1 \tau^3 R_n(\hat{B}_n R_n)\, d\tau$$

$$+ \left(1 - \frac{1}{\alpha}\right)\int_0^1 \tau g_n \dot{R}_n\, d\tau + \int_0^1 g_n R_n\, d\tau \quad . \tag{3.5}$$

By virtue of gauge-invariance, we can perform the $\zeta(0)$ calculation setting $\alpha = 1$, so that the contribution of $\tau g_n \dot{R}_n$ vanishes. However, our problem remains a coupled one, also with this choice. The eigenvalues for g_n-modes and R_n-modes can be obtained in principle from the boundary conditions and the variational principle $\delta\Big(I_E - \lambda J_E\Big) = 0$, where $J_E \equiv \frac{1}{2}\int A_\mu A^\mu \sqrt{\det g}\; d^4x$. We here make the analytic continuation to the Euclidean time variable $\tau = it$ in computing I_E (cf (3.2)) and J_E, and we use (1.3)-(1.5) and the well-known properties of longitudinal vector harmonics and scalar harmonics [14]. This leads to the coupled system of two second-order ordinary differential equations for arbitrary α and $\forall n \geq 2$ (the case $n = 1$ only involves the R_1-mode, and should be treated separately, as in section 4)

$$\frac{\tau}{(n^2-1)}\left[-\ddot{g}_n - \frac{\dot{g}_n}{\tau} + \frac{(n^2-1)}{\alpha\tau^2}g_n\right] + \left(1 - \frac{1}{\alpha}\right)\tau\dot{R}_n + R_n = \frac{\lambda_n}{(n^2-1)}\tau g_n \tag{3.6}$$

$$\tau^3\left[\frac{1}{\alpha}\left(-\ddot{R}_n - \frac{3}{\tau}\dot{R}_n\right) + \frac{(n^2-1)}{\tau^2}R_n\right] - \tau\dot{g}_n\left(1 - \frac{1}{\alpha}\right) + \frac{g_n}{\alpha} = \lambda_n\tau^3 R_n \quad . \tag{3.7}$$

Now we still choose $\alpha = 1$, because it enables one to decouple much more easily the system (3.6)-(3.7). The boundary conditions are regularity at the origin

$$g_n(0) = R_n(0) = 0 \qquad \text{for all } n \geq 2 \tag{3.8}$$

and magnetic conditions on S^3

$$g_n(1) = \dot{R}_n(1) = 0 \qquad \text{for all } n \geq 2 \tag{3.9}$$

or electric conditions on S^3

$$\dot{g}_n(1) = R_n(1) = 0 \qquad \text{for all } n \geq 2 \quad . \tag{3.10}$$

In the $\alpha = 1$ gauge we can express R_n from (3.6) as

$$R_n = \frac{\lambda_n}{(n^2-1)} \tau g_n + \frac{\tau}{(n^2-1)} \left[\ddot{g}_n + \frac{\dot{g}_n}{\tau} - \frac{(n^2-1)}{\tau^2} g_n \right] \tag{3.11}$$

and its insertion into the corresponding form of (3.7) yields the fourth-order equation

$$\begin{aligned} 0 = & \left[\left(2 - \frac{3}{(n^2-1)} \right) \lambda_n \tau^2 - \frac{\lambda_n^2}{(n^2-1)} \tau^4 - \left(n^2 - 1 \right) \right] g_n \\ & + 2\tau \left[1 - \frac{3\lambda_n \tau^2}{(n^2-1)} \right] \dot{g}_n + \frac{2\tau^2}{(n^2-1)} \left[n^2 - 4 - \lambda_n \tau^2 \right] \ddot{g}_n \\ & - \frac{6\tau^3}{(n^2-1)} g_n^{\mathrm{III}} - \frac{\tau^4}{(n^2-1)} g_n^{\mathrm{IV}} \quad . \end{aligned} \tag{3.12}$$

Moreover, studying first the magnetic case, the relations (3.9) and (3.11) lead to

$$\lambda_n = \left(n^2 - 1 \right) - 2 \frac{\ddot{g}_n(1)}{\dot{g}_n(1)} - \frac{g_n^{\mathrm{III}}(1)}{\dot{g}_n(1)} \qquad \text{for all } n \geq 2 \quad . \tag{3.13}$$

Of course, as shown by (3.6)-(3.7), the eigenvalues λ_n have dimension (length)$^{-2}$. However, in (3.13) we have set $a = 1$ for simplicity, following (3.9)-(3.10). Hence the physical dimension does not appear explicitly. For the solutions of the equations (3.11)-(3.12) subject to the boundary conditions (3.8)-(3.9), an existence and uniqueness theorem holds. Thus, denoting by k an integer ≥ 0, in the light of the form of (3.12) we write its solution as [9]

$$g_n(\tau) = \tau^{\mu} \sum_{k=0}^{\infty} a_{n,k}(n, k, \lambda_n) \tau^k \quad . \tag{3.14}$$

The insertion of (3.14) into (3.12)-(3.13), and the requirement that $g_n(1) = 0, \forall n \geq 2$, leads to a problem formulated in purely algebraic terms. One then finds that only half of the $a_{n,k}$ coefficients are non-vanishing and obey very involved recurrence relations, whereas the value of μ is obtained by solving a fourth-order algebraic equation.

In fact, defining

$$F(k, n, \mu) \equiv 2(k + \mu)^2 - (n^2 - 1) - \frac{(k + \mu)^2 \Big((k + \mu)^2 - 1 \Big)}{(n^2 - 1)} \tag{3.15}$$

we find

$$F\Big(0, n, \mu\Big) a_{n,0} = F\Big(1, n, \mu\Big) a_{n,1} = 0 \tag{3.16}$$

$$F\Big(m, n, \mu\Big) a_{n,m} + \left[2 - \frac{\Big(2(m + \mu)^2 - 4(m + \mu) + 3 \Big)}{(n^2 - 1)} \right] \lambda_n \; a_{n,m-2} = 0 \tag{3.17}$$

where $m = 2, 3$, whereas, $\forall k \geq 4$, we have

$$0 = F(k,n,\mu)a_{n,k} + \left[2 - \frac{\Bigl(2(k+\mu)^2 - 4(k+\mu) + 3\Bigr)}{(n^2-1)}\right] \lambda_n \; a_{n,k-2}$$

$$- \frac{\lambda_n^2}{(n^2-1)} a_{n,k-4} \quad . \tag{3.18}$$

In (3.16)-(3.18), the value of μ can be obtained from the equation $F(0,n,\mu) = 0$, and bearing in mind (3.8), which implies that only a $\mu > 1$ is an acceptable value, in the light of (3.11). In other words, we study the fourth-order algebraic equation [9]

$$\mu^4 - (2n^2 - 1)\mu^2 + (n^2 - 1)^2 = 0 \quad . \tag{3.19}$$

This equation can be easily solved setting $\mu^2 = x$ and studying the corresponding second-order equation for x. One thus finds the four roots

$$\mu_+^{(1)} = +\sqrt{n^2 - \frac{3}{4}} + \frac{1}{2} \tag{3.20}$$

$$\mu_+^{(2)} = +\sqrt{n^2 - \frac{3}{4}} - \frac{1}{2} \tag{3.21}$$

$$\mu_-^{(1)} = -\mu_+^{(1)} \tag{3.22}$$

$$\mu_-^{(2)} = -\mu_+^{(2)} \quad . \tag{3.23}$$

Interestingly, both $\mu_+^{(1)}$ and $\mu_+^{(2)}$ are > 1, $\forall n \geq 2$. They yield the desired regular solution of the system (3.6)-(3.7).

4. Decoupled gauge mode

In the light of (3.2), if we set $\alpha = 1$ and integrate by parts using (3.8)-(3.10), we find that the decoupled gauge mode $R_1(\tau)$ obeys the eigenvalue equation

$$\ddot{R}_1 + \frac{3}{\tau}\dot{R}_1 + \mu_1 R_1 = 0 \tag{4.1}$$

which is solved by $R_1(\tau) = H_1 \tau^{-1} J_1(\sqrt{\mu_1}\ \tau)$, where H_1 is a constant. Thus, in the magnetic case, we study the eigenvalue condition (setting $a = 1$ for simplicity)

$$J_1(\sqrt{\mu_1}) - \sqrt{\mu_1}\ \dot{J}_1(\sqrt{\mu_1}) = 0 \tag{4.2}$$

whereas in the electric case the corresponding eigenvalue condition is

$$J_1(\sqrt{\mu_1}) = 0 \quad . \tag{4.3}$$

If (4.2) holds, we have to perform a $\zeta(0)$ calculation with just one perturbative mode, subject to a complicated eigenvalue condition involving a linear combination of J_1 and $\dot{J}_1$. Of course, a regularization is still needed because there are infinitely many solutions $\hat{\lambda}_n$ of (4.2). For this purpose, it is convenient to use the technique described and applied in [3,9,18]. The basic idea is as follows.

Given the ζ-function at large x

$$\zeta(s, x^2) \equiv \sum_{n=1}^{\infty} \left(\hat{\lambda}_n + x^2\right)^{-s} \tag{4.4}$$

one has in four-dimensions

$$\Gamma(3)\zeta(3,x^2) = \int_0^\infty t^2 e^{-x^2 t} G(t)\, dt \sim \sum_{q=0}^{\infty} B_q \Gamma\Big(1+\frac{q}{2}\Big) x^{-q-2} \tag{4.5}$$

where we have used the asymptotic expansion of the heat kernel $G(t)$ for $t \rightarrow 0^+$, written as

$$G(t) \sim \sum_{q=0}^{\infty} B_q t^{\frac{q}{2}-2} \quad . \tag{4.6}$$

Such an asymptotic expansion does actually exist in the case of Laplace operators subject to Dirichlet, Neumann or Robin boundary conditions [9,13,19,20]. On the other hand, defining (cf (4.2))

$$F_1(z) \equiv J_1(z) - z\dot{J}_1(z) \tag{4.7}$$

one also has the identity

$$\Gamma(3)\zeta(3,x^2) = -N_1\bigg(-\frac{1}{2x}\frac{d}{dx}\bigg)^3 \log\Big((ix)^{-1}F_1(ix)\Big) \tag{4.8}$$

where $N_1 = 1$ is the degeneracy of the problem. Thus, the comparison of (4.5) and (4.8) can yield the coefficients B_q and in particular $\zeta(0) = B_4$, provided we carefully perform an uniform asymptotic expansion of $F_1(ix)$. Now, making the analytic continuation $x \rightarrow ix$ and then defining $\widetilde{\alpha} \equiv \sqrt{1+x^2}$, one obtains the following asymptotic expansions which are uniformly valid in the order as $\mid x \mid \rightarrow \infty$:

$$J_1(ix) \sim \frac{(ix)}{\sqrt{2\pi}} \widetilde{\alpha}^{-\frac{1}{2}} e^{\widetilde{\alpha}} e^{-\log(1+\widetilde{\alpha})} \Sigma_1(1,\widetilde{\alpha}(x)) \tag{4.9}$$

$$J'_1(ix) \sim \frac{1}{\sqrt{2\pi}} \widetilde{\alpha}^{\frac{1}{2}} e^{\widetilde{\alpha}} e^{-\log(1+\widetilde{\alpha})} \Sigma_2(1, \widetilde{\alpha}(x)) \tag{4.10}$$

where $\Sigma_1(1, \widetilde{\alpha}(x)) \sim \sum_{k=0}^{\infty} u_k(\frac{1}{\widetilde{\alpha}})$, $\Sigma_2(1, \widetilde{\alpha}(x)) \sim \sum_{k=0}^{\infty} v_k(\frac{1}{\widetilde{\alpha}})$ [3,9,18]. Using (4.7) and (4.9)-(4.10), and defining $C \equiv -\log(\sqrt{2\pi})$, one thus finds the uniform asymptotic expansion

$$\log\Bigr((ix)^{-1} F_1(ix)\Bigr) \sim C - \log(1+\widetilde{\alpha}) + \frac{1}{2}\log(\widetilde{\alpha}) + \widetilde{\alpha} + \sum_{l=1}^{\infty} \sum_{r=0}^{l} b_{lr} \widetilde{\alpha}^{-l-2r} \tag{4.11}$$

where the double sum on the right-hand side of (4.11) is obtained by expanding in inverse powers of $\widetilde{\alpha}$ the $\log\Bigr(\frac{\Sigma_1}{\widetilde{\alpha}} - \Sigma_2\Bigr)$. In the light of (4.8) and (4.11), we conclude that $\Gamma(3)\zeta(3, x^2) \sim \Bigr[\sigma_1 + \sigma_2\Bigr]$, where

$$\sigma_1 \sim \left(\frac{1}{2x}\frac{d}{dx}\right)^3 \Bigr[-\log(1+\widetilde{\alpha}) + \frac{1}{2}\log(\widetilde{\alpha}) + \widetilde{\alpha}\Bigr] \tag{4.12}$$

$$\sigma_2 \sim \left(\frac{1}{2x}\frac{d}{dx}\right)^3 \sum_{l=1}^{\infty} \sum_{r=0}^{l} b_{lr} \widetilde{\alpha}^{-l-2r} \quad . \tag{4.13}$$

It can be easily checked that the asymptotic expansion of σ_2 in (4.13) does not contribute to $\zeta(0)$, whereas one finds

$$\sigma_1 \sim \frac{3}{8} x^{-5} - x^{-6} + \frac{x^{-6}}{2} + \sum_{k=7}^{\infty} \omega_k x^{-k} \tag{4.14}$$

which implies (cf (4.5))

$$\zeta_{R_1}(0) = \frac{1}{2}\Bigr(-\frac{1}{2}\Bigr) = -\frac{1}{4} \quad . \tag{4.15}$$

By contrast, when the eigenvalue condition (4.3) holds for the R_1-mode, the square bracket on the right-hand side of (4.12) contains $-\frac{1}{2}\log(\widetilde{\alpha})$ rather than $\frac{1}{2}\log(\widetilde{\alpha})$ (cf (4.9)). This leads to

$$\widetilde{\zeta}_{R_1}(0) = \frac{1}{2}\Bigl(-\frac{3}{2}\Bigr) = -\frac{3}{4} \tag{4.16}$$

since, again, the asymptotic expansion of the corresponding σ_2 does not contribute to $\zeta(0)$. Note also that our value (4.16) agrees with the result found in [21] in the case of one perturbative mode subject to Dirichlet boundary conditions. As explained in [22], the perturbative calculations of [21] are correct, whereas the $\zeta(0)$ values with finitely many perturbative modes of appendix A of [23] are incorrect.

5. Ghost-field contribution

In section 3, we have chosen the gauge-averaging term $\frac{\Phi^2}{2\alpha}$, where the gauge-averaging functional $\Phi(A)$ can be written in the form (cf [24])

$$\Phi(A) \equiv \Phi_1(A) \equiv {}^{(4)}\nabla^{\mu}A_{\mu} - K^{i}_{\ i}A_0 = \frac{\partial A_0}{\partial \tau} + {}^{(3)}\nabla^{i}A_i \tag{5.1}$$

where $K^{i}_{\ i} = \frac{3}{\tau}$ is the trace of the extrinsic-curvature tensor of the boundary. This choice of $\Phi(A)$ leads to the familiar one-dimensional Laplace operator acting on the R_n-modes, which simplifies the $\zeta(0)$ calculation for the coupled gauge modes and for the R_1-mode, as shown in sections 3 and 4. However, since $\Phi_1(A)$ is not the Lorentz gauge-averaging functional, the corresponding ghost action does not involve the familiar Laplace operator. This is proved (cf [15]) by studying the gauge transformation

$$ {}^{\epsilon}A_{\mu} \equiv A_{\mu} + {}^{(4)}\nabla_{\mu}\epsilon = A_{\mu} + \partial_{\mu}\epsilon \tag{5.2} $$

where the scalar ϵ is expanded on a family of 3-spheres centred on the origin as

$$ \epsilon(x,\tau) = \sum_{n=1}^{\infty} \epsilon_n(\tau) Q^{(n)}(x) \quad . \tag{5.3} $$

One thus finds

$$ \delta(\Phi_1(A)) \equiv \Phi_1(A) - \Phi_1({}^{\epsilon}A) = \sum_{n=1}^{\infty} Q^{(n)}(x)\biggr[-\frac{d^2}{d\tau^2} + \frac{(n^2-1)}{\tau^2}\biggr]\epsilon_n(\tau) \quad . \tag{5.4} $$

This implies that the eigenfunctions of the ghost operator are of the kind [16]

$$ \widetilde{\epsilon}_n(\tau) = \sqrt{\tau} J_{\sqrt{n^2-\frac{3}{4}}}(\sqrt{E}\tau) \quad . \tag{5.5} $$

More precisely, since the electromagnetic field is bosonic, the corresponding ghost field is fermionic [15]. Its contribution to the full $\zeta(0)$ is thus obtained changing sign to the scalar-eigenfunctions contribution of (5.5), and then multiplying the resulting number by two, since the ghost field is complex. We now have to perform a $\zeta(0)$ calculation which involves Bessel functions of non-integer order, generalizing the technique described in section 4. Here we show that, although eigenvalues and eigenfunctions are different, the $\zeta(0)$ calculation originating from (5.5) is closely related to a standard $\zeta(0)$ calculation involving Bessel functions of integer order. For this purpose, we study the simplest case, i.e. when the ghost field obeys homogeneous Dirichlet conditions on S^3. This leads to the eigenvalue condition

$$J_{\sqrt{n^2-\frac{3}{4}}}(\sqrt{E}a) = 0 \qquad \text{for all } n \geq 1 \quad . \tag{5.6}$$

Following [9], our section 4 and (5.6), it is now useful to define $\forall n \geq 1$ and at large x

$$\nu \equiv +\sqrt{n^2 - \frac{3}{4}} \tag{5.7}$$

$$\alpha_\nu(x) \equiv \sqrt{\nu^2 + x^2} = \sqrt{n^2 - \frac{3}{4} + x^2} \tag{5.8}$$

$$\alpha_n(x) \equiv \sqrt{n^2 + x^2} \quad . \tag{5.9}$$

Since the generalization of the technique of section 4 to infinitely many perturbative modes for the ghost involves defining $\alpha_\nu(x)$, whereas we are only able to perform exact calculations using $\alpha_n(x)$, it is also useful to evaluate the ratio

$$\frac{\alpha_\nu(x)}{\alpha_n(x)} \sim \rho_n(x) \sim \left[1 - \frac{3}{8}\Bigr(n^2 + x^2\Bigr)^{-1} - \frac{9}{128}\Bigr(n^2 + x^2\Bigr)^{-2} + \mathrm{O}\biggr(\Bigr(n^2 + x^2\Bigr)^{-3}\biggr)\right] \quad . \tag{5.10}$$

The asymptotic expansion (5.10) is very useful in that it is uniform in n, i.e. it holds $\forall n \geq 1$, at large x. A careful study of section 7.3 of [9] shows that, if the eigenvalue condition (5.6) holds (whose eigenvalues are positive $\forall n \geq 1$), (4.8) and (4.11) are generalized as

$$\Gamma(3)\zeta(3, x^2) \sim \Bigr[\sigma_1 + \sigma_2\Bigr] \tag{5.11}$$

where

$$\sigma_1 \sim \sum_{n=0}^{\infty} n^2 \Big[-\nu x^{-6} + \nu^2 x^{-6} \alpha_\nu^{-1} + \frac{\nu^2}{2} x^{-4} \alpha_\nu^{-3} + \frac{3}{8} \nu^2 x^{-2} \alpha_\nu^{-5} - \frac{\alpha_\nu^{-6}}{2} + \frac{3}{8} \alpha_\nu^{-5} \Big] \qquad (5.12)$$

$$\sigma_2 \sim -\sum_{l=1}^{\infty} \sum_{r=0}^{l} a_{lr} \Big(r + \frac{l}{2}\Big) \Big(r + \frac{l}{2} + 1\Big) \Big(r + \frac{l}{2} + 2\Big) \sum_{n=0}^{\infty} n^2 \nu^{2r} {\alpha_\nu}^{-l-2r-6} \quad . \qquad (5.13)$$

In these formulae, obtained using uniform asymptotic expansions of Bessel functions of non-integer order, n^2 is the degeneracy resulting from the scalar harmonics appearing in the expansion (5.3), ν is the order of the Bessel functions defined in (5.7), and α_ν has been defined in (5.8). We now re-express ν^2 as $\Big(n^2 - \frac{3}{4}\Big)$, and $\alpha_\nu(x) \sim \alpha_n(x) \rho_n(x)$ as in (5.10). Moreover, we use the contour formula [3,9,18]

$$\sum_{n=0}^{\infty} n^{2k} \alpha_n^{-2k-m} = \frac{\Gamma\Big(k + \frac{1}{2}\Big) \Gamma\Big(\frac{m}{2} - \frac{1}{2}\Big)}{2\Gamma\Big(k + \frac{m}{2}\Big)} \, x^{1-m} \qquad \text{for all } k = 1, 2, 3, ... \quad . \qquad (5.14)$$

We then point out that the asymptotic expansion (5.12) can be cast in the form

$$\sigma_1 \sim \Big[-x^{-6} I_\infty^{(1)} + x^{-6} I_\infty^{(2)} + x^{-4} I_\infty^{(3)} + x^{-2} I_\infty^{(4)} - I_\infty^{(5)} + I_\infty^{(6)} \Big] \qquad (5.15)$$

where (see appendix A)

$$I_\infty^{(2)} - I_\infty^{(1)} \equiv -\sum_{n=0}^{\infty} n^3 + \sum_{n=0}^{\infty} n^2 \left[\frac{\nu^2}{\alpha_\nu} - \Big(\nu - n\Big) \right] \qquad (5.16)$$

$$I_{\infty}^{(3)} \sim \frac{1}{2}\sum_{n=0}^{\infty} n^4 \alpha_n^{-3} + \frac{9}{16}\sum_{n=0}^{\infty} n^4 \alpha_n^{-5} + \frac{135}{256}\sum_{n=0}^{\infty} n^4 \alpha_n^{-7}$$

$$- \frac{3}{8}\sum_{n=0}^{\infty} n^2 \alpha_n^{-3} - \frac{27}{64}\sum_{n=0}^{\infty} n^2 \alpha_n^{-5} - \frac{405}{1024}\sum_{n=0}^{\infty} n^2 \alpha_n^{-7}$$

$$+ \frac{1}{2}\sum_{n=0}^{\infty} n^4 \alpha_n^{-3} \mathrm{O}\Big(\alpha_n^{-6}\Big) - \frac{3}{8}\sum_{n=0}^{\infty} n^2 \alpha_n^{-3} \mathrm{O}\Big(\alpha_n^{-6}\Big) \tag{5.17}$$

$$I_{\infty}^{(4)} \sim \frac{3}{8}\sum_{n=0}^{\infty} n^4 \alpha_n^{-5} + \frac{45}{64}\sum_{n=0}^{\infty} n^4 \alpha_n^{-7} + \frac{945}{1024}\sum_{n=0}^{\infty} n^4 \alpha_n^{-9}$$

$$- \frac{9}{32}\sum_{n=0}^{\infty} n^2 \alpha_n^{-5} - \frac{135}{256}\sum_{n=0}^{\infty} n^2 \alpha_n^{-7} - \frac{2835}{4096}\sum_{n=0}^{\infty} n^2 \alpha_n^{-9}$$

$$+ \frac{3}{8}\sum_{n=0}^{\infty} n^4 \alpha_n^{-5} \mathrm{O}\Big(\alpha_n^{-6}\Big) - \frac{9}{32}\sum_{n=0}^{\infty} n^2 \alpha_n^{-5} \mathrm{O}\Big(\alpha_n^{-6}\Big) \tag{5.18}$$

$$I_{\infty}^{(5)} \equiv \frac{1}{2}\sum_{n=0}^{\infty} n^2 \alpha_\nu^{-6}$$

$$\sim \frac{1}{2}\sum_{n=0}^{\infty} n^2 \alpha_n^{-6} + \frac{9}{8}\sum_{n=0}^{\infty} n^2 \alpha_n^{-8} + \frac{1}{2}\sum_{n=0}^{\infty} n^2 \alpha_n^{-6} \mathrm{O}\Big(\alpha_n^{-4}\Big) \tag{5.19}$$

$$I_{\infty}^{(6)} \equiv \frac{3}{8}\sum_{n=0}^{\infty} n^2 \alpha_\nu^{-5}$$

$$\sim \frac{3}{8}\sum_{n=0}^{\infty} n^2 \alpha_n^{-5} + \frac{45}{64}\sum_{n=0}^{\infty} n^2 \alpha_n^{-7} + \frac{945}{1024}\sum_{n=0}^{\infty} n^2 \alpha_n^{-9}$$

$$+ \frac{3}{8}\sum_{n=0}^{\infty} n^2 \alpha_n^{-5} \mathrm{O}\Big(\alpha_n^{-6}\Big) \quad . \tag{5.20}$$

It is therefore clear, using (5.14), that the $\zeta(0)$ value resulting from σ_1 and σ_2 is given by $\frac{1}{90} = -2\Big(-\frac{1}{180}\Big)$ (which coincides with the $\zeta(0)$ value corresponding to the Lorentz gauge-averaging functional) plus additional terms owed to the second sum in (5.16), the third and fifth sum in (5.17)-(5.18), denoted by T_1, T_2, T_3, T_4, the third sum in (5.20), denoted by T_5, and finally (5.13). Note that $I_{\infty}^{(5)}$ defined in (5.19) does not contribute to the additional terms. The detailed calculation yields

$$x^{-4}T_1 \equiv \frac{135}{256}x^{-4}\sum_{n=0}^{\infty} n^4 \alpha_n^{-7} = \frac{27}{256}x^{-6} \tag{5.21}$$

$$x^{-4}T_2 \equiv -\frac{27}{64}x^{-4}\sum_{n=0}^{\infty} n^2 \alpha_n^{-5} = -\frac{9}{64}x^{-6} \tag{5.22}$$

$$x^{-2}T_3 \equiv \frac{945}{1024}x^{-2}\sum_{n=0}^{\infty} n^4 \alpha_n^{-9} = \frac{27}{512}x^{-6} \tag{5.23}$$

$$x^{-2}T_4 \equiv -\frac{135}{256}x^{-2}\sum_{n=0}^{\infty} n^2 \alpha_n^{-7} = -\frac{9}{128}x^{-6} \tag{5.24}$$

$$T_5 \equiv \frac{945}{1024}\sum_{n=0}^{\infty} n^2 \alpha_n^{-9} = \frac{9}{128}x^{-6} \quad . \tag{5.25}$$

We now focus on (5.13) and (5.16), and we first study the asymptotic expansion (5.13), since (5.16) gives rise to severe technical difficulties (see below). For this purpose, we remark that, studying for all integer values $l \in [1,\infty[$, $r \in [1,l]$ the function (see appendix

A)

$$I_{lr}(x) \sim \sum_{n=0}^{\infty} n^2 \nu^{2r} \alpha_\nu^{-l-2r-6}$$

$$\sim \sum_{n=0}^{\infty} n^2 \left(n^2 - \frac{3}{4}\right)^r \alpha_n^{-l-2r-6} \left[1 + A_{lr}\alpha_n^{-2} + B_{lr}\alpha_n^{-4} + \mathrm{O}\left(\alpha_n^{-6}\right)\right]$$

$$\sim \left[I_{lr}^{(1)} + I_{lr}^{(2)} + I_{lr}^{(3)} + I_{lr}^{(4)}\right](x) \tag{5.26}$$

one finds

$$I_{lr}^{(1)}(x) \sim \sum_{n=0}^{\infty} n^2 \left(n^2 - \frac{3}{4}\right)^r \alpha_n^{-l-2r-6} \tag{5.27}$$

$$I_{lr}^{(2)}(x) \sim A_{lr} \sum_{n=0}^{\infty} n^2 \left(n^2 - \frac{3}{4}\right)^r \alpha_n^{-l-2r-8} \tag{5.28}$$

$$I_{lr}^{(3)}(x) \sim B_{lr} \sum_{n=0}^{\infty} n^2 \left(n^2 - \frac{3}{4}\right)^r \alpha_n^{-l-2r-10} \tag{5.29}$$

$$I_{lr}^{(4)}(x) \sim \sum_{n=0}^{\infty} n^2 \left(n^2 - \frac{3}{4}\right)^r \alpha_n^{-l-2r-6} \mathrm{O}\left(\alpha_n^{-6}\right) \tag{5.30}$$

where A_{lr} and B_{lr} are coefficients which only depend on l and r. The case $r = 0$ is easier. Using (5.13)-(5.14), $r = 0$ leads to a contribution to $\zeta(0)$ related to

$$T_6 \equiv -a_{10} \frac{15}{8} \frac{21}{8} \frac{\Gamma\left(\frac{3}{2}\right)\Gamma(3)}{2\Gamma\left(\frac{9}{2}\right)} x^{-6} = -\frac{3}{8} a_{10} x^{-6} \tag{5.31}$$

where $\frac{21}{8}$ is the coefficient of α_n^{-2} in the asymptotic expansion of $\rho_n^{-7}(x)$ (see appendix A). If the integer r is ≥ 1, one has to study (5.26)-(5.30), where

$$\left(n^2 - \frac{3}{4}\right)^r = n^{2r} - \frac{3}{4} r n^{2r-2} + \ldots + r n^2 \left(-\frac{3}{4}\right)^{r-1} + \left(-\frac{3}{4}\right)^r \quad . \tag{5.32}$$

Inserting (5.32) into (5.27)-(5.30), and using (5.14), a lengthy calculation yields a contribution to $\zeta(0)$ related to (see appendix A)

$$T_7 \equiv \left[\frac{3}{4}a_{11} - \frac{9}{8}a_{11}\right]x^{-6} = -\frac{3}{8}a_{11}x^{-6} \quad . \tag{5.33}$$

Note that the two terms on the r.h.s. of (5.33) are due to the asymptotic expansions of $I_{lr}^{(1)}(x)$ and $I_{lr}^{(2)}(x)$ respectively, whereas (5.29)-(5.30) do not affect the $\zeta(0)$ value, since they do not involve x^{-6}. It now remains to evaluate the contribution of (5.16). Indeed, defining

$$J_\infty \equiv \sum_{n=0}^{\infty} n^2 \nu \left(\frac{\nu}{\alpha_\nu} - 1\right) \tag{5.34}$$

we point out that multiplying and dividing the round bracket by $\Bigr(\nu + \alpha_\nu\Bigr)$, and then adding and subtracting α_ν in the numerator of the corresponding expression, one finds by virtue of (5.8) the useful identity

$$J_\infty = -x^2 \sum_{n=0}^{\infty} n^2 \left[\frac{1}{\alpha_\nu} - \frac{1}{\Bigr(\nu + \alpha_\nu\Bigr)}\right] = J_\infty^{(1)} + J_\infty^{(2)} \quad . \tag{5.35}$$

Moreover, (5.10) and (5.14) show that the contribution to $\zeta(0)$ owed to $J_\infty^{(1)}$ is related to

$$T_8 \equiv -\frac{27}{128}x^{-4} \sum_{n=0}^{\infty} n^2 \alpha_n^{-5} = -\frac{9}{128}x^{-6} \quad . \tag{5.36}$$

A further contribution is owed to

$$T_9 \equiv \frac{1}{120}x^{-6} \tag{5.37}$$

originating from $\sum_{n=0}^{\infty} n^3$ in (5.16). However, we do not yet know how to deal properly with the divergent sum

$$J_{\infty}^{(2)} \equiv x^2 \sum_{n=0}^{\infty} \frac{n^2}{\Bigr(\nu+\alpha_\nu\Bigr)} \quad . \tag{5.38}$$

We should now add up the numerical coefficients appearing in (5.21)-(5.25), (5.31), (5.33), (5.36)-(5.37), divide them by two, and finally multiply by -2 since the ghost is fermionic and complex. This leads to the following *partial* contribution to the $\zeta(0)$ value for the ghost field:

$$\zeta_{gh}^{(I)}(0) = \frac{1}{90} - \frac{9}{512} + \frac{3}{8}\Bigr(a_{10}+a_{11}\Bigr) + \frac{9}{128} - \frac{1}{120} = \frac{1}{360} + \frac{11}{512} \tag{5.39}$$

where $\frac{1}{90}$ is added for the reasons described following (5.20), and we have used the values $a_{10} = \frac{1}{8}$, $a_{11} = -\frac{5}{24}$ appearing in equation (26) of [18].

By contrast, if the Lorentz gauge-averaging functional is chosen, one finds

$$\Phi(A) \equiv \Phi_2(A) \equiv {}^{(4)}\nabla^{\mu} A_{\mu} = \frac{\partial A_0}{\partial \tau} + {}^{(4)}\nabla^{i} A_i \tag{5.40}$$

which implies

$$\delta(\Phi_2(A)) \equiv \Phi_2(A) - \Phi_2({}^{\epsilon}A) = \sum_{n=1}^{\infty} Q^{(n)}(x)\biggr[-\frac{d^2}{d\tau^2} - \frac{3}{\tau}\frac{d}{d\tau} + \frac{(n^2-1)}{\tau^2}\biggr]\epsilon_n(\tau) \tag{5.41}$$

where we have used the property ${}^{(4)}\nabla_i \epsilon = {}^{(3)}\nabla_i \epsilon = \epsilon_{|i} = \partial_i \epsilon$, $\forall i = 1,2,3$. Thus, as we anticipated, the familiar one-dimensional Laplace operator appears in the ghost action, so that the ghost contributions to the full $\zeta(0)$ value are more easily computed as $-2\left(-\frac{1}{180}\right)$

and $-2\left(\frac{29}{180}\right)$ in the Dirichlet and Neumann cases, respectively. However, if $\Phi_2(A)$ is chosen as gauge-averaging functional, the form of the action quadratic in the gauge modes becomes $\forall n \geq 2$

$$
\begin{aligned}
I_E^{(n)}(g,R) = & \frac{1}{2}\int_0^1 \frac{\tau g_n}{(n^2-1)}\left[-\frac{d^2 g_n}{d\tau^2} - \frac{1}{\tau}\frac{dg_n}{d\tau} + \frac{(n^2-1)}{\alpha\tau^2} g_n\right] d\tau \\
& + \frac{1}{2}\int_0^1 \tau^3 R_n \left[\frac{1}{\alpha}\left(-\frac{d^2 R_n}{d\tau^2} + \frac{3}{\tau}\frac{dR_n}{d\tau}\right) + \left(n^2 - 1 + \frac{9}{\alpha}\right)\frac{R_n}{\tau^2}\right] d\tau \\
& + \left(1-\frac{1}{\alpha}\right)\int_0^1 \tau g_n \dot{R}_n \, d\tau + \left(1-\frac{3}{\alpha}\right)\int_0^1 g_n R_n \, d\tau \\
& - \Big[\tau g_n R_n\Big]_0^1 + \frac{1}{2\alpha}\Big[\tau^3 \dot{R}_n R_n\Big]_0^1 \quad .
\end{aligned}
\tag{5.42}
$$

Thus, the second-order differential operator acting on R_n-modes is no longer the one-dimensional Laplace operator for scalars, and the calculation becomes more involved. For example, if we set $\alpha = 1$, the contribution of $R_1(\tau)$ to $\zeta(0)$ involves a Bessel function of order $\sqrt{13}$. Moreover, a non-vanishing boundary term $I_B^{(n)} \equiv \frac{a^3}{2\alpha}\dot{R}_n(a)R_n(a) = -\frac{3a^2}{2\alpha}R_n^2(a)$ survives in the action, if the whole functional $\Phi_2(A)$ is required to vanish on the boundary in the magnetic case (cf [17]). Thus, one has to add to the action a boundary term equal to $-I_B^{(n)}$, if the whole of $\Phi_2(A)$ is set to zero on S^3.

Of course, since the theory is gauge-invariant, *infinitely* many other choices for $\Phi(A)$ (but not all choices) are still possible. A very relevant class of choices can be cast in the form

$$
\Phi^{(b)}(A) \equiv {}^{(4)}\nabla^\mu A_\mu + bK^i{}_i A_0 \tag{5.43}
$$

where b is a real number. With our parametrization, $b = -1$ leads to $\Phi_1(A)$, and $b = 0$ leads to $\Phi_2(A)$. Note that, even if we set $\alpha = 1$, it does not seem possible to decouple gauge modes using $\frac{\Phi^2}{2\alpha}$ *and* obtain a well-defined ghost action, since the decoupling of g_n and R_n, $\forall n \geq 2$, is then obtained setting

$$\Phi(A) \equiv \Phi_3(A) \equiv \sum_{n=2}^{\infty} \sqrt{\left[\left(-\frac{g_n}{\tau^2}+\dot{R}_n\right)^2+\frac{2}{\tau^2}\frac{d}{d\tau}(g_n R_n)\right]} \; Q^{(n)}(x) \quad . \tag{5.44}$$

However, the ghost action should be derived by functionally differentiating the infinite sum of square roots on the right-hand side of (5.44) as in (5.4) and (5.41), and this does not lead to a linear, second-order differential operator. This is why we believe that the coupling of gauge modes is an intrinsic property of problems with boundaries, as well as the choice of gauge-averaging functionals of the form (5.43), which all reduce to the Lorentz choice in the absence of boundaries. Note that the work in this section supersedes earlier work appearing in section 6.5 of [9], where the ghost-field operator (cf (5.4)) was not derived.

In light of (2.6), (4.15) and (5.39), the full $\zeta(0)$ value for vacuum Euclidean Maxwell theory in the case of magnetic boundary conditions on S^3 takes the form

$$\zeta(0) = \zeta_B^{(PDF)}(0) + \zeta_{R_1}(0) + \zeta_{GM}(0) + \zeta_{gh}(0) = -\frac{243}{360} + \frac{11}{512} + \zeta_{GM}(0) + \zeta_{gh}^{(II)}(0) \tag{5.45}$$

where $\zeta_{GM}(0)$ and $\zeta_{gh}^{(II)}(0)$ are the as yet unknown contributions to $\zeta(0)$ arising from coupled gauge modes (section 3) and from (5.38) respectively. We have been unable to evaluate $\zeta_{GM}(0)$ since we do not know explicitly the uniform asymptotic expansion as $\lambda_n \rightarrow \infty$ of the power series in (3.14), which is not (obviously) related to well-known special functions (see appendix B). Moreover, the regularized contribution $\zeta_{gh}^{(II)}(0)$ of (5.38) to $\zeta(0)$ involves $\nu \equiv +\sqrt{n^2-\frac{3}{4}}$, which is a source of complication. However, it should be emphasized that all divergences are only *fictitious*, since the starting point for the derivation of (5.12) is the identity [9]

$$\left(\frac{1}{2x}\frac{d}{dx}\right)^3 \log\left(\frac{1}{\nu+\alpha_\nu}\right) = \left(\nu+\alpha_\nu\right)^{-3}\left[-\alpha_\nu^{-3} - \frac{9}{8}\nu\alpha_\nu^{-4} - \frac{3}{8}\nu^2\alpha_\nu^{-5}\right] \quad . \tag{5.46}$$

This proves that by summing over all integer values of n from 0 to ∞ one gets a convergent series.

In this section we have not studied the case of Neumann boundary conditions for the ghost field, i.e. the electric case. This complicated calculation may be, by itself, the object of another paper. However, interestingly, in light of (2.6), (2.25) and (4.15)-(4.16) one finds

$$\zeta_B^{(PDF)}(0) + \zeta_{R_1}(0) = \zeta_E^{(PDF)}(0) + \widetilde{\zeta}_{R_1}(0) = -\frac{61}{90} \quad . \tag{5.47}$$

In other words, if the gauge-averaging functional of (5.1) is chosen, physical degrees of freedom and decoupled gauge mode give the same partial contribution to the full $\zeta(0)$, i.e. $-\frac{61}{90}$, both in the magnetic and in the electric case.

6. Concluding remarks and open problems

One-loop quantum cosmology may add further evidence in favour of different approaches to quantizing gauge theories being inequivalent [2-9,25-34]. Studying flat Euclidean backgrounds bounded by a 3-sphere, for vacuum Maxwell theory the PDF method yields $\zeta(0) = -\frac{77}{180}$ and $\zeta(0) = \frac{13}{180}$ in the magnetic and electric cases respectively [6,9,14], whereas the *indirect* method (by this we mean that one-loop amplitudes are expressed using the boundary-counterterms technique and evaluating the various coefficients in a covariant way as in [2,17]) was found to yield $\zeta(0) = -\frac{38}{45}$ in both cases in [2]. For $N = 1$ supergravity, the PDF method yields *partial* cancellations between spin 2 and spin $\frac{3}{2}$ [7-9], whereas the *indirect* method yields a one-loop amplitude which is even more divergent than in the pure-gravity case [2]. Finally, for pure gravity, the PDF method yields $\zeta(0) = -\frac{278}{45}$ in the Dirichlet case, whereas the *indirect* method yields $\zeta(0) = -\frac{803}{45}$ [2,9]. Moreover, within the PDF method, it is possible to set to zero on S^3 the linearized magnetic curvature. This yields a well-defined one-loop calculation, and the corresponding $\zeta(0)$ value is $\frac{112}{45}$ [9]. By contrast, using the Faddeev-Popov formula, magnetic boundary conditions for pure gravity are ruled out [2]. Interestingly, recent work in [35] seems to add evidence in favour of *direct* $\zeta(0)$ calculations being correct. The authors of [35] have shown that different formulae for $\zeta(0)$ previously obtained in [8] for Majorana and Dirac fermions on the part of a de Sitter sphere bounded by a 3-sphere, with local and spectral boundary conditions, have the same limiting value in the case of a full sphere. This value coincides with the covariant one obtained by the method in [2,13,17]. Moreover, all these expressions are found to give the same results in the case of a hemisphere [35]. The authors of [35] have also suggested that $3+1$ splits of the kind considered in many papers, including Eqs. (1.3a)-(1.3b) of our paper, might be the reason of the discrepancies for higher-spin $\zeta(0)$ values found using covariant and non-covariant methods. However, there is not yet a proof of this statement, and in the case of real scalar fields subject to Neumann (or Dirichlet) conditions on a 3-sphere boundary, $\zeta(0)$ values obtained from various methods coincide, although the boundary 3-geometry is the same as in higher-spin calculations.

It is therefore necessary to get a better understanding of the manifestly gauge-invariant formulae for one-loop amplitudes used so far in the literature, by performing a mode-by-mode analysis of the eigenvalue equations, rather than relying on general formulae which contain no explicit information about degeneracies and eigenvalue conditions. This detailed analysis has been attempted here in the simplest case, i.e. vacuum Maxwell theory at one-loop about a flat Euclidean background bounded by a 3-sphere. Our results are here summarized for the sake of clarity:

(1) In the light of (2.6), (2.25) and (4.15)-(4.16) the physical degrees of freedom, and the decoupled gauge mode, give a contribution to the full $\zeta(0)$ equal to $-\frac{61}{90}$ both in the magnetic and in the electric case (this important property had not been realized in section 6.5 of [9]), if the gauge-averaging functional $\Phi_1(A)$ of (5.1) is chosen. Since in [2] it was found that the full $\zeta(0)$ values for spin 1 are equal in the magnetic and electric cases, it appears relevant that also our partial contributions to the full $\zeta(0)$ coincide in these two cases for the spin-1 problem about flat Euclidean backgrounds.

(2) Remaining gauge modes g_n and R_n always obey a coupled system of linear, second-order ordinary differential equations, $\forall n \geq 2$. The solution of such a system corresponding to $\Phi_1(A)$ has been given in section 3 and appendix B.

(3) If $\Phi_1(A)$ is chosen, the ghost eigenfunctions involve Bessel functions of non-integer order. The corresponding contribution to the full $\zeta(0)$ can be obtained using the method of section 5. Such a technical point appears interesting, since to our knowledge no previous

mode-by-mode analysis for the ghost is appearing in the literature in the case of Bessel functions of non-integer order.

It now remains to evaluate the contribution to the full $\zeta(0)$ of the divergent sum in (5.38), and the uniform asymptotic expansion of g_n- and R_n-modes as $\lambda_n \rightarrow \infty$ at the end of section 3. Unfortunately, the generalization of the method described in [5-8] is highly non-trivial. By contrast, a simpler form of the ghost eigenfunctions is obtained using the Lorentz gauge-averaging functional $\Phi_2(A)$. However, this leads to a further complication of the calculations involving gauge modes, since the decoupled mode $R_1(\tau)$ involves a Bessel function of order $\sqrt{13}$ (this implies a contribution to $\zeta(0)$ proportional to $\sqrt{13}$, which we find very puzzling), and coupled gauge modes require the addition to the action, in the magnetic case, of a boundary term equal to $\frac{3a^2}{2\alpha}\sum_{n=2}^{\infty} R_n^2(a)$.

Thus, although some evidence exists that different $\zeta(0)$ values for gauge fields in the presence of boundaries are due to inequivalent quantization techniques [9], the most important check, i.e. the mode-by-mode analysis of eigenvalue equations for gauge modes and ghost fields, remains a very difficult problem. We hope that our paper, through its detailed (although incomplete) analysis, may contribute to shed new light on this long-standing problem in quantum field theory.

Acknowledgments

I am much indebted to Bruce Allen, Andrei Barvinsky, Peter D'Eath, Gary Gibbons, Chris Isham, Alexander Kamenshchik, Jorma Louko, Ian Moss, Stephen Poletti and Peter van Nieuwenhuizen for enlightening conversations or correspondence. Anonymous referees made comments which led to a substantial improvement of the original manuscript. I am also grateful to Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, and to Professor Dennis Sciama for hospitality at the SISSA of Trieste, during the early stages of this work. Last, but not least, the stimulating atmosphere of the July 1992 Les Houches Summer School on Gravitation and Quantizations has played a key role in preparing this paper.

Appendix A

The derivation of (5.17)-(5.20), (5.31), (5.33) and (5.36) has been obtained using the following asymptotic expansions:

$$\rho_n^{-1}(x) \sim 1 + \frac{3}{8}\alpha_n^{-2} + \frac{27}{128}\alpha_n^{-4} + \mathrm{O}\Big(\alpha_n^{-6}\Big) \tag{A.1}$$

$$\rho_n^{-2}(x) \sim 1 + \frac{3}{4}\alpha_n^{-2} + \frac{9}{16}\alpha_n^{-4} + \mathrm{O}\Big(\alpha_n^{-6}\Big) \tag{A.2}$$

$$\rho_n^{-3}(x) \sim 1 + \frac{9}{8}\alpha_n^{-2} + \frac{135}{128}\alpha_n^{-4} + \mathrm{O}\Big(\alpha_n^{-6}\Big) \tag{A.3}$$

$$\rho_n^{-4}(x) \sim 1 + \frac{3}{2}\alpha_n^{-2} + \frac{27}{16}\alpha_n^{-4} + \mathrm{O}\Big(\alpha_n^{-6}\Big) \tag{A.4}$$

$$\rho_n^{-5}(x) \sim 1 + \frac{15}{8}\alpha_n^{-2} + \frac{315}{128}\alpha_n^{-4} + \mathrm{O}\Big(\alpha_n^{-6}\Big) \tag{A.5}$$

$$\rho_n^{-6}(x) \sim 1 + \frac{9}{4}\alpha_n^{-2} + \frac{27}{8}\alpha_n^{-4} + \mathrm{O}\Big(\alpha_n^{-6}\Big) \tag{A.6}$$

$$\rho_n^{-7}(x) \sim 1 + \frac{21}{8}\alpha_n^{-2} + \frac{567}{128}\alpha_n^{-4} + \mathrm{O}\Bigl(\alpha_n^{-6}\Bigr) \tag{A.7}$$

$$\rho_n^{-8}(x) \sim 1 + 3\alpha_n^{-2} + \frac{45}{8}\alpha_n^{-4} + \mathrm{O}\Bigl(\alpha_n^{-6}\Bigr) \tag{A.8}$$

$$\rho_n^{-9}(x) \sim 1 + \frac{27}{8}\alpha_n^{-2} + \frac{891}{128}\alpha_n^{-4} + \mathrm{O}\Bigl(\alpha_n^{-6}\Bigr) \quad . \tag{A.9}$$

Note that these expansions are valid uniformly in the integer n, $\forall n \geq 1$, as $\mid x \mid \rightarrow \infty$. They are obtained using repeatedly (5.10) and the well-known expansion of $(1+Y)^{-1}$ as $Y \rightarrow 0$.

Appendix B

Following section 3, coupled gauge modes can be written as

$$g_n^{(j)}(\tau) = \sum_{k=0}^{\infty} a_{n,k}^{(j)}\Bigl(n, k, \lambda_n^{(j)}\Bigr)\tau^{k+\mu} \tag{B.1}$$

$$R_n^{(j)}(\tau) = \sum_{k=0}^{\infty} b_{n,k}^{(j)}\Bigl(n, k, \lambda_n^{(j)}\Bigr)\tau^{k+\mu-1} \quad . \tag{B.2}$$

The label j is introduced because, for each integer value of $n \geq 2$, there is a whole family $\Bigl\{\lambda_n^{(j)}\Bigr\}$ of eigenvalues labelled by the integer j, say. They are solutions of the equation $g_n(a) = 0$, and their degeneracy $d_j(n) = n^2$, $\forall j \geq 1$ and $\forall n \geq 2$ [14]. Now, defining $\forall k \geq 2$ and $\forall n \geq 2$

$$G(k, n, \mu) \equiv 2 - \frac{\Bigl(2(k+\mu)^2 - 4(k+\mu) + 3\Bigr)}{(n^2 - 1)} \tag{B.3}$$

one finds $\forall j \geq 1$

$$\frac{a_{n,2}^{(j)}}{a_{n,0}^{(j)}} = -\frac{G(2,n,\mu)}{F(2,n,\mu)} \lambda_n^{(j)} \qquad (B.4)$$

$$F(k,n,\mu) a_{n,k}^{(j)} + G(k,n,\mu) \lambda_n^{(j)} a_{n,k-2}^{(j)} - \frac{\left(\lambda_n^{(j)}\right)^2}{(n^2-1)} a_{n,k-4}^{(j)} = 0 \qquad \forall k \geq 4 \qquad (B.5)$$

$$b_{n,0}^{(j)} = \left(\frac{\mu^2}{(n^2-1)} - 1 \right) a_{n,0}^{(j)} \qquad (B.6)$$

$$b_{n,k}^{(j)} = \frac{\lambda_n^{(j)}}{(n^2-1)} a_{n,k-2}^{(j)} + \left(\frac{(k+\mu)^2}{(n^2-1)} - 1 \right) a_{n,k}^{(j)} \qquad \forall k \geq 2 \qquad (B.7)$$

$$a_{n,k}^{(j)} = b_{n,k}^{(j)} = 0 \qquad \forall k = (2m+1) \qquad m = 0,1,2,... \quad . \qquad (B.8)$$

Moreover, setting the 3-sphere radius a to 1 for simplicity, magnetic boundary conditions (i.e. $g_n(1) = \dot{R}_n(1) = 0$) lead to

$$\sum_{k=0}^{\infty} a_{n,k}^{(j)} = 0 \qquad (B.9)$$

$$\lambda_n^{(j)} = \left(n^2 - 1\right) - \mu(3\mu - 2) - \frac{\sum_{k=0}^{\infty} k^3 a_{n,k}^{(j)}}{\sum_{k=0}^{\infty} k a_{n,k}^{(j)}} - (3\mu - 1) \frac{\sum_{k=0}^{\infty} k^2 a_{n,k}^{(j)}}{\sum_{k=0}^{\infty} k a_{n,k}^{(j)}} \quad . \qquad (B.10)$$

Since, $\forall n \geq 2$, there are two values of $\mu > 1$, a further label is necessary to characterize completely the coupled gauge modes as follows:

$$g_{1,n}^{(j)}\left(n, \lambda_{1,n}^{(j)}, \tau\right) \text{ and } R_{1,n}^{(j)}\left(n, \lambda_{1,n}^{(j)}, \tau\right) \text{ if } \mu = \mu_+^{(1)}$$

$$g_{2,n}^{(j)}\left(n, \lambda_{2,n}^{(j)}, \tau\right) \text{ and } R_{2,n}^{(j)}\left(n, \lambda_{2,n}^{(j)}, \tau\right) \text{ if } \mu = \mu_+^{(2)}$$

(see (3.20)-(3.21)).

Note that it is extremely difficult (if not impossible) to find the eigenvalues $\lambda_n^{(j)}$ by analytic or numerical methods, since (B.4)-(B.5) imply that a function H exists such that

$$\frac{a_{n,k}^{(j)}}{a_{n,0}^{(j)}} = H(k,n,\mu)\Bigl(\lambda_n^{(j)}\Bigr)^{\frac{k}{2}} \qquad (B.11)$$

for all even values of $k \geq 2$, and $\forall n \geq 2$. Thus, when (B.11) is inserted into (B.9)-(B.10), it is not clear how to find an explicit solution for $\lambda_n^{(j)}$ and $a_{n,k}^{(j)}$.

References

[1] Luckock H C and Moss I G 1989 *Class. Quantum Grav.* **6** 1993

[2] Poletti S 1990 *Phys. Lett.* **249B** 249

[3] D'Eath P D and Esposito G V M 1991 *Phys. Rev.* D **43** 3234

[4] D'Eath P D and Esposito G V M 1991 *Phys. Rev.* D **44** 1713

[5] Barvinsky A O, Kamenshchik A Y, Karmazin I P and Mishakov I V 1992 *Class. Quantum Grav.* **9** L27

[6] Kamenshchik A Y and Mishakov I V 1992 *Int. J. Mod. Phys.* A **7** 3713

[7] Barvinsky A O, Kamenshchik A Y and Karmazin I P 1992 *Ann. Phys., N.Y.* **219** 201

[8] Kamenshchik A Y and Mishakov I V 1993 *Phys. Rev.* D **47** 1380

[9] Esposito G 1992 *Quantum Gravity, Quantum Cosmology and Lorentzian Geometries* Lecture Notes in Physics, New Series m: Monographs vol m12 (Berlin: Springer)

[10] Duff M J 1982 Ultraviolet divergences in extended supergravity *Supergravity 1981* eds S Ferrara and J G Taylor (Cambridge: Cambridge University Press)

[11] D'Eath P D 1986 *Nucl. Phys.* B **269** 665

[12] Hawking S W 1977 *Commun. Math. Phys.* **55** 133

[13] Branson T P and Gilkey P B 1990 *Commun. Part. Diff. Eqns.* **15** 245

[14] Louko J 1988 *Phys. Rev.* D **38** 478

[15] Itzykson C and Zuber J B 1985 *Quantum Field Theory* (New York: McGraw-Hill)

[16] Gradshteyn I S and Ryzhik I M 1965 *Table of Integrals, Series and Products* (New York: Academic)

[17] Moss I G and Poletti S 1990 *Nucl. Phys.* B **341** 155

[18] Moss I G 1989 *Class. Quantum Grav.* **6** 759

[19] Greiner P 1971 *Archs. Ration. Mech. Analysis* **41** 163

[20] Kennedy G 1978 *J. Phys. A: Math. Gen.* **11** L173

[21] Louko J 1991 *Class. Quantum Grav.* **8** L37

[22] Louko J 1991 *Class. Quantum Grav.* **8** 1947

[23] Louko J 1988 *Ann. Phys., N.Y.* **181** 318

[24] Laenen E and van Nieuwenhuizen P 1991 *Ann. Phys., N.Y.* **207** 77

[25] Gitman D M and Tyutin I V 1990 *Quantization of Fields with Constraints* (Berlin: Springer)

[26] Govaerts J and Troost W 1991 *Class. Quantum Grav.* **8** 1723

[27] Govaerts J 1991 *Hamiltonian Quantization and Constrained Dynamics*, Leuven Notes in Mathematical and Theoretical Physics, Series B, Volume 4 (Leuven: Leuven University Press)

[28] Vassilevich D V 1991 *Nuovo Cimento* A **104** 743

[29] Vassilevich D V 1992 *Nuovo Cimento* A **105** 649

[30] Guven J and Ryan Jr. M P 1992 *Phys. Rev.* D **45** 3559

[31] Kunstatter G 1992 *Class. Quantum Grav.* **9** 1469

[32] Henneaux M and Teitelboim C 1992 *Quantization of Gauge Systems* (Princeton: Princeton University Press)

[33] Vassilevich D V 1993 *Int. J. Mod. Phys.* A **8** 1637

[34] Barvinsky A O 1993 *Phys. Rep.* **230** 237

[35] Kamenshchik A Y and Mishakov I V 1994 *Phys. Rev.* D **49** 816