TWISTORS AND SPIN-$\frac{3}{2}$ POTENTIALS IN QUANTUM GRAVITY

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Abstract. Local boundary conditions involving field strengths and the normal to the boundary, originally studied in anti-de Sitter space-time, have been recently considered in one-loop quantum cosmology. This paper derives the conditions under which spin-lowering and spin-raising operators preserve these local boundary conditions on a 3-sphere for fields of spin $0, \frac{1}{2}, 1, \frac{3}{2}$ and 2. Moreover, the two-component spinor analysis of the four potentials of the totally symmetric and independent field strengths for spin $\frac{3}{2}$ is applied to the case of a 3-sphere boundary. It is shown that such boundary conditions can only be imposed in a flat Euclidean background, for which the gauge freedom in the choice of the potentials remains. Alternative boundary conditions for supergravity involving the spinor-valued 1-forms for gravitinos and the normal to the boundary are also studied.

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1. Introduction

Recent work in the literature has studied the quantization of gauge theories and supersymmetric field theories in the presence of boundaries, with application to one-loop quantum cosmology [1-9]. In particular, in the work described in [9], two possible sets of local boundary conditions were studied. One of these, first proposed in anti-de Sitter space-time [10-11], involves the normal to the boundary and Dirichlet or Neumann conditions for spin 0, the normal and the field for massless spin-$\frac{1}{2}$ fermions, and the normal and totally symmetric field strengths for spins 1, $\frac{3}{2}$ and 2. Although more attention has been paid to alternative local boundary conditions motivated by supersymmetry, as in [2-3,8-9], the analysis of the former boundary conditions remains of mathematical and physical interest by virtue of its links with twistor theory [9]. The aim of this paper is to derive the mathematical properties of the corresponding boundary-value problems in both cases, since these are relevant for quantum cosmology and twistor theory.

For this purpose, sections 2-3 derive the conditions under which spin-lowering and spin-raising operators preserve local boundary conditions involving field strengths and normals. Section 4 applies the 2-spinor form of spin-$\frac{3}{2}$ potentials to Riemannian 4-geometries with a 3-sphere boundary. Boundary conditions on spinor-valued 1-forms describing gravitino fields are studied in section 5. Concluding remarks and open problems are presented in section 6.

2. Spin-lowering operators in cosmology

In section 5.7 of [9], a flat Euclidean background bounded by a 3-sphere was studied. On the bounding S^3, the following boundary conditions for a spin-s field were required:

$$2^s \; {}_e n^{AA'} ... {}_e n^{LL'} \; \phi_{A...L} = \epsilon \; \widetilde{\phi}^{A'...L'} \quad . \tag{2.1}$$

With our notation, ${}_e n^{AA'}$ is the Euclidean normal to S^3 [3,9], $\phi_{A...L} = \phi_{(A...L)}$ and $\widetilde{\phi}_{A'...L'} = \widetilde{\phi}_{(A'...L')}$ are totally symmetric and independent (i.e. not related by any conjugation) field strengths, which reduce to the massless spin-$\frac{1}{2}$ field for $s = \frac{1}{2}$. Moreover, the complex scalar field ϕ is such that its real part obeys Dirichlet conditions on S^3 and its imaginary part obeys Neumann conditions on S^3, or the other way around, according to the value of the parameter $\epsilon \equiv \pm 1$ occurring in (2.1), as described in [9].

In flat Euclidean 4-space, we write the solutions of the twistor equations [9,12]

$$D_{A'}^{\;\;(A} \, \omega^{B)} = 0 \quad , \tag{2.2}$$

$$D_{A}^{\;\;(A'} \, \widetilde{\omega}^{B')} = 0 \quad , \tag{2.3}$$

as [9]

$$\omega^A = (\omega^o)^A - i \Bigl({}_e x^{AA'} \Bigr) \pi^o_{A'} \quad , \tag{2.4}$$

$$\widetilde{\omega}^{A'} = (\widetilde{\omega}^o)^{A'} - i \Bigl({}_e x^{AA'} \Bigr) \widetilde{\pi}^o_{A} \quad . \tag{2.5}$$

Note that, since unprimed and primed spin-spaces are no longer isomorphic in the case of Riemannian 4-metrics, Eq. (2.3) is not obtained by complex conjugation of Eq. (2.2).

Hence the spinor field $\widetilde{\omega}^{B'}$ is independent of ω^B. This leads to distinct solutions (2.4)-(2.5), where the spinor fields $\omega_A^o, \widetilde{\omega}_{A'}^o, \widetilde{\pi}_A^o, \pi_{A'}^o$ are covariantly constant with respect to the flat connection D, whose corresponding spinor covariant derivative is here denoted by $D_{AB'}$. The following theorem can be now proved:

Theorem 2.1 Let ω^D be a solution of the twistor equation (2.2) in flat Euclidean space with a 3-sphere boundary, and let $\widetilde{\omega}^{D'}$ be the solution of the independent equation (2.3) in the same 4-geometry with boundary. Then a form exists of the spin-lowering operator which preserves the local boundary conditions on S^3:

$$4 \; {}_e n^{AA'} \; {}_e n^{BB'} \; {}_e n^{CC'} \; {}_e n^{DD'} \; \phi_{ABCD} = \epsilon \; \widetilde{\phi}^{A'B'C'D'} \quad , \tag{2.6}$$

$$2^{\frac{3}{2}} \; {}_e n^{AA'} \; {}_e n^{BB'} \; {}_e n^{CC'} \; \phi_{ABC} = \epsilon \; \widetilde{\phi}^{A'B'C'} \quad . \tag{2.7}$$

Of course, the independent field strengths appearing in (2.6)-(2.7) are assumed to satisfy the corresponding massless free-field equations.

Proof. Multiplying both sides of (2.6) by ${}_e n_{FD'}$ one gets

$$-2 \; {}_e n^{AA'} \; {}_e n^{BB'} \; {}_e n^{CC'} \; \phi_{ABCF} = \epsilon \; \widetilde{\phi}^{A'B'C'D'} \; {}_e n_{FD'} \quad . \tag{2.8}$$

Taking into account the total symmetry of the field strengths, putting $F = D$ and multiplying both sides of (2.8) by $\sqrt{2}\,\omega^D$ one finally gets

$$-2^{\frac{3}{2}} \; {}_e n^{AA'} \; {}_e n^{BB'} \; {}_e n^{CC'} \; \phi_{ABCD} \, \omega^D = \epsilon \, \sqrt{2} \, \widetilde{\phi}^{A'B'C'D'} \; {}_e n_{DD'} \, \omega^D \quad , \tag{2.9}$$

$$2^{\frac{3}{2}} \; {}_e n^{AA'} \; {}_e n^{BB'} \; {}_e n^{CC'} \; \phi_{ABCD} \, \omega^D = \epsilon \; \widetilde{\phi}^{A'B'C'D'} \; \widetilde{\omega}_{D'} \quad , \tag{2.10}$$

where (2.10) is obtained by inserting into (2.7) the definition of the spin-lowering operator. The comparison of (2.9) and (2.10) yields the preservation condition

$$\sqrt{2} \; {}_{e}n_{DA'} \; \omega^{D} = -\widetilde{\omega}_{A'} \quad . \tag{2.11}$$

In the light of (2.4)-(2.5), equation (2.11) is found to imply

$$\sqrt{2} \; {}_{e}n_{DA'} \; (\omega^{o})^{D} - i\sqrt{2} \; {}_{e}n_{DA'} \; {}_{e}x^{DD'} \; = -\widetilde{\omega}^{o}_{A'} - i \; {}_{e}x_{DA'} \; (\widetilde{\pi}^{o})^{D} \quad . \tag{2.12}$$

Requiring that (2.12) should be identically satisfied, and using the identity ${}_{e}n^{AA'} = \frac{1}{r} \; {}_{e}x^{AA'}$ on a 3-sphere of radius r, one finds

$$\widetilde{\omega}^{o}_{A'} = i\sqrt{2} \; r \; {}_{e}n_{DA'} \; {}_{e}n^{DD'} \; \pi^{o}_{D'} = -\frac{ir}{\sqrt{2}} \; \pi^{o}_{A'} \quad , \tag{2.13}$$

$$-\sqrt{2} \; {}_{e}n_{DA'} \; (\omega^{o})^{D} = ir \; {}_{e}n_{DA'} \; (\widetilde{\pi}^{o})^{D} \quad . \tag{2.14}$$

Multiplying both sides of (2.14) by ${}_{e}n^{BA'}$, and then acting with ϵ_{BA} on both sides of the resulting relation, one gets

$$\omega^{o}_{A} = -\frac{ir}{\sqrt{2}} \; \widetilde{\pi}^{o}_{A} \quad . \tag{2.15}$$

The equations (2.11), (2.13) and (2.15) completely solve the problem of finding a spin-lowering operator which preserves the boundary conditions (2.6)-(2.7) on S^3. Q.E.D.

If one requires local boundary conditions on S^3 involving field strengths and normals also for lower spins (i.e. spin $\frac{3}{2}$ vs spin 1, spin 1 vs spin $\frac{1}{2}$, spin $\frac{1}{2}$ vs spin 0), then by using the same technique of the theorem just proved, one finds that the preservation condition obeyed by the spin-lowering operator is still expressed by (2.13) and (2.15).

3. Spin-raising operators in cosmology

To derive the corresponding preservation condition for spin-raising operators [12], we begin by studying the relation between spin-$\frac{1}{2}$ and spin-1 fields. In this case, the independent spin-1 field strengths take the form [9,11-12]

$$\psi_{AB} = i \, \widetilde{\omega}^{L'} \Bigl(D_{BL'} \, \chi_A \Bigr) - 2 \chi_{(A} \, \widetilde{\pi}^o_{B)} \quad , \tag{3.1}$$

$$\widetilde{\psi}_{A'B'} = -i \, \omega^L \Bigl(D_{LB'} \, \widetilde{\chi}_{A'} \Bigr) - 2 \widetilde{\chi}_{(A'} \, \pi^o_{B')} \quad , \tag{3.2}$$

where the independent spinor fields $\Bigl(\chi_A, \widetilde{\chi}_{A'} \Bigr)$ represent a massless spin-$\frac{1}{2}$ field obeying the Weyl equations on flat Euclidean 4-space and subject to the boundary conditions

$$\sqrt{2} \, {}_e n^{AA'} \, \chi_A = \epsilon \, \widetilde{\chi}^{A'} \tag{3.3}$$

on a 3-sphere of radius r. Thus, by requiring that (3.1) and (3.2) should obey (2.1) on S^3 with $s = 1$, and bearing in mind (3.3), one finds

$$2\epsilon \biggl[\sqrt{2} \, \widetilde{\pi}^o_A \, \widetilde{\chi}^{(A'} \, {}_e n^{AB')} - \widetilde{\chi}^{(A'} \, \pi^{o \, B')} \biggr] = i \biggl[2 \, {}_e n^{AA'} \, {}_e n^{BB'} \, \widetilde{\omega}^{L'} \, D_{L'(B} \, \chi_{A)} + \epsilon \, \omega^L \, D_L{}^{(B'} \, \widetilde{\chi}^{A')} \biggr] \tag{3.4}$$

on the bounding S^3. It is now clear how to carry out the calculation for higher spins. Denoting by s the spin obtained by spin-raising, and defining $n \equiv 2s$, one finds

$$n\epsilon \biggl[\sqrt{2} \, \widetilde{\pi}^o_A \, {}_e n^{A(A'} \, \widetilde{\chi}^{B' \ldots K')} - \widetilde{\chi}^{(A' \ldots D'} \, \pi^{o \, K')} \biggr] = i \biggl[2^{\frac{n}{2}} \, {}_e n^{AA'} \ldots {}_e n^{KK'} \, \widetilde{\omega}^{L'} \, D_{L'(K} \, \chi_{A \ldots D)} + \epsilon \, \omega^L \, D_L{}^{(K'} \, \widetilde{\chi}^{A' \ldots D')} \biggr] \tag{3.5}$$

on the 3-sphere boundary. In the comparison spin-0 vs spin-$\frac{1}{2}$, the preservation condition is not obviously obtained from (3.5). The desired result is here found by applying the spin-raising operators [12] to the independent scalar fields ϕ and $\widetilde{\phi}$ (see below) and bearing in mind (2.4)-(2.5) and the boundary conditions

$$\phi = \epsilon \, \widetilde{\phi} \quad \text{on} \quad S^3 \quad , \tag{3.6}$$

$${}_e n^{AA'} D_{AA'} \phi = -\epsilon \; {}_e n^{BB'} D_{BB'} \widetilde{\phi} \quad \text{on} \quad S^3 \quad . \tag{3.7}$$

This leads to the following condition on S^3 (cf. equation (5.7.23) of [9]):

$$\begin{aligned} 0 = i\phi \left[\frac{\widetilde{\pi}^o_A}{\sqrt{2}} - \pi^o_{A'} \; {}_e n_A^{\;\;A'} \right] - \left[\frac{\widetilde{\omega}^{K'}}{\sqrt{2}} \Bigl(D_{AK'} \phi \Bigr) - \frac{\omega_A}{2} \; {}_e n_C^{\;\;K'} \Bigl(D^C_{\;\;K'} \phi \Bigr) \right] \\ + \epsilon \; {}_e n_{(A}^{\;\;\;A'} \, \omega^B \, D_{B)A'} \, \widetilde{\phi} \quad . \end{aligned} \tag{3.8}$$

Note that, while the preservation conditions (2.13) and (2.15) for spin-lowering operators are purely algebraic, the preservation conditions (3.5) and (3.8) for spin-raising operators are more complicated, since they also involve the value at the boundary of four-dimensional covariant derivatives of spinor fields or scalar fields. Two independent scalar fields have been introduced, since the spinor fields obtained by applying the spin-raising operators to ϕ and $\widetilde{\phi}$ respectively are independent as well in our case.

4. Spin-$\frac{3}{2}$ potentials in cosmology

In this section we focus on the totally symmetric field strengths ϕ_{ABC} and $\widetilde{\phi}_{A'B'C'}$ for spin-$\frac{3}{2}$ fields, and we express them in terms of their potentials, rather than using spin-raising (or spin-lowering) operators. The corresponding theory in Minkowski space-time (and curved space-time) is described in [13-16], and adapted here to the case of flat Euclidean 4-space with flat connection D. It turns out that $\widetilde{\phi}_{A'B'C'}$ can then be obtained from two potentials defined as follows. The first potential satisfies the properties [13-16]

$$\gamma_{A'B'}^{C} = \gamma_{(A'B')}^{C} \quad , \tag{4.1}$$

$$D^{AA'} \ \gamma_{A'B'}^{C} = 0 \quad , \tag{4.2}$$

$$\widetilde{\phi}_{A'B'C'} = D_{CC'} \ \gamma_{A'B'}^{C} \quad , \tag{4.3}$$

with the gauge freedom of replacing it by

$$\widehat{\gamma}_{A'B'}^{C} \equiv \gamma_{A'B'}^{C} + D^{C}{}_{B'} \ \widetilde{\nu}_{A'} \quad , \tag{4.4}$$

where $\widetilde{\nu}_{A'}$ satisfies the positive-helicity Weyl equation

$$D^{AA'} \ \widetilde{\nu}_{A'} = 0 \quad . \tag{4.5}$$

The second potential is defined by the conditions [13-16]

$$\rho_{A'}^{BC} = \rho_{A'}^{(BC)} \quad , \tag{4.6}$$

$$D^{AA'} \ \rho_{A'}^{BC} = 0 \quad , \tag{4.7}$$

$$\gamma_{A'B'}^{C} = D_{BB'} \ \rho_{A'}^{BC} \quad , \tag{4.8}$$

with the gauge freedom of being replaced by

$$\widehat{\rho}_{A'}^{BC} \equiv \rho_{A'}^{BC} + D^{C}{}_{A'} \ \chi^{B} \quad , \tag{4.9}$$

where χ^B satisfies the negative-helicity Weyl equation

$$D_{BB'} \chi^B = 0 \quad . \tag{4.10}$$

Moreover, in flat Euclidean 4-space the field strength ϕ_{ABC} is expressed in terms of the potential $\Gamma_{AB}^{C'} = \Gamma_{(AB)}^{C'}$, independent of $\gamma_{A'B'}^{C}$, as

$$\phi_{ABC} = D_{CC'} \, \Gamma_{AB}^{C'} \quad , \tag{4.11}$$

with gauge freedom

$$\widehat{\Gamma}_{AB}^{C'} \equiv \Gamma_{AB}^{C'} + D^{C'}{}_{B} \, \nu_A \quad . \tag{4.12}$$

Thus, if we insert (4.3) and (4.11) into the boundary conditions (2.1) with $s = \frac{3}{2}$, and require that also the gauge-equivalent potentials (4.4) and (4.12) should obey such boundary conditions on S^3, we find that

$$2^{\frac{3}{2}} \; {}_e n^A{}_{A'} \; {}_e n^B{}_{B'} \; {}_e n^C{}_{C'} \; D_{CL'} \; D^{L'}{}_{B} \; \nu_A = \epsilon \; D_{LC'} \; D^{L}{}_{B'} \; \widetilde{\nu}_{A'} \tag{4.13}$$

on the 3-sphere. Note that, from now on (as already done in (3.5) and (3.8)), covariant derivatives appearing in boundary conditions are first taken on the background and then evaluated on S^3. In the case of our flat background, (4.13) is identically satisfied since $D_{CL'} \; D^{L'}{}_{B} \; \nu_A$ and $D_{LC'} \; D^{L}{}_{B'} \; \widetilde{\nu}_{A'}$ vanish by virtue of spinor Ricci identities [17-18]. In a curved background, however, denoting by ∇ the corresponding curved connection, and defining $\square_{AB} \equiv \nabla_{M'(A} \nabla^{M'}{}_{B)}$, $\square_{A'B'} \equiv \nabla_{X(A'} \; \nabla^{X}{}_{B')}$, since the spinor Ricci identities we need are [17]

$$\square_{AB} \; \nu_C = \psi_{ABDC} \; \nu^D - 2\Lambda \; \nu_{(A} \; \epsilon_{B)C} \quad , \tag{4.14}$$

$$\Box_{A'B'} \; \widetilde{\nu}_{C'} = \widetilde{\psi}_{A'B'D'C'} \; \widetilde{\nu}^{D'} - 2\widetilde{\Lambda} \; \widetilde{\nu}_{(A'} \; \epsilon_{B')C'} \quad , \tag{4.15}$$

one finds that the corresponding boundary conditions

$$2^{\frac{3}{2}} \; {}_{e}n^{A}{}_{A'} \; {}_{e}n^{B}{}_{B'} \; {}_{e}n^{C}{}_{C'} \; \nabla_{CL'} \; \nabla^{L'}{}_{B} \; \nu_{A} = \epsilon \; \nabla_{LC'} \; \nabla^{L}{}_{B'} \; \widetilde{\nu}_{A'} \tag{4.16}$$

are identically satisfied if and only if one of the following conditions holds: (i) $\nu_A = \widetilde{\nu}_{A'} = 0$; (ii) the Weyl spinors $\psi_{ABCD}, \widetilde{\psi}_{A'B'C'D'}$ and the scalars $\Lambda, \widetilde{\Lambda}$ vanish everywhere. However, since in a curved space-time with vanishing $\Lambda, \widetilde{\Lambda}$, the potentials with the gauge freedoms (4.4) and (4.12) only exist provided D is replaced by ∇ and the trace-free part Φ_{ab} of the Ricci tensor vanishes as well [19], the background 4-geometry is actually flat Euclidean 4-space. Note that we require that (4.16) should be identically satisfied to avoid, after a gauge transformation, obtaining more boundary conditions than the ones originally imposed. The curvature of the background should not, itself, be subject to a boundary condition.

The same result can be derived by using the potential $\rho_{A'}^{BC}$ and its independent counterpart $\Lambda_{A}^{B'C'}$. This spinor field yields the $\Gamma_{AB}^{C'}$ potential by means of

$$\Gamma_{AB}^{C'} = D_{BB'} \; \Lambda_{A}^{B'C'} \quad , \tag{4.17}$$

and has the gauge freedom

$$\widehat{\Lambda}_{A}^{B'C'} \equiv \Lambda_{A}^{B'C'} + D^{C'}{}_{A} \; \widetilde{\chi}^{B'} \quad , \tag{4.18}$$

where $\widetilde{\chi}^{B'}$ satisfies the positive-helicity Weyl equation

$$D_{BF'} \; \widetilde{\chi}^{F'} = 0 \quad . \tag{4.19}$$

Thus, if also the gauge-equivalent potentials (4.9) and (4.18) have to satisfy the boundary conditions (2.1) on S^3, one finds

$$2^{\frac{3}{2}} \; {}_e n^A{}_{A'} \; {}_e n^B{}_{B'} \; {}_e n^C{}_{C'} \; D_{CL'} \; D_{BF'} \; D^{L'}{}_A \; \widetilde{\chi}^{F'} = \epsilon \; D_{LC'} \; D_{MB'} \; D^L{}_{A'} \; \chi^M \tag{4.20}$$

on the 3-sphere. In our flat background, covariant derivatives commute, hence (4.20) is identically satisfied by virtue of (4.10) and (4.19). However, in the curved case the boundary conditions (4.20) are replaced by

$$2^{\frac{3}{2}} \; {}_e n^A{}_{A'} \; {}_e n^B{}_{B'} \; {}_e n^C{}_{C'} \; \nabla_{CL'} \; \nabla_{BF'} \; \nabla^{L'}{}_A \; \widetilde{\chi}^{F'} = \epsilon \; \nabla_{LC'} \; \nabla_{MB'} \; \nabla^L{}_{A'} \; \chi^M \tag{4.21}$$

on S^3, if the *local* expressions of ϕ_{ABC} and $\widetilde{\phi}_{A'B'C'}$ in terms of potentials still hold [13-16]. By virtue of (4.14)-(4.15), where ν_C is replaced by χ_C and $\widetilde{\nu}_{C'}$ is replaced by $\widetilde{\chi}_{C'}$, this means that the Weyl spinors $\psi_{ABCD}, \widetilde{\psi}_{A'B'C'D'}$ and the scalars $\Lambda, \widetilde{\Lambda}$ should vanish, since one should find

$$\nabla^{AA'} \; \widehat{\rho}_{A'}^{BC} = 0 \quad , \quad \nabla^{AA'} \; \widehat{\Lambda}_A^{B'C'} = 0 \quad . \tag{4.22}$$

If we assume that $\nabla_{BF'} \; \widetilde{\chi}^{F'} = 0$ and $\nabla_{MB'} \; \chi^M = 0$, we have to show that (4.21) differs from (4.20) by terms involving a part of the curvature that is vanishing everywhere. This is proved by using the basic rules of 2-spinor calculus and spinor Ricci identities [17-18]. Thus, bearing in mind that [17]

$$\square^{AB} \; \widetilde{\chi}_{B'} = \Phi^{AB}{}_{L'B'} \; \widetilde{\chi}^{L'} \quad , \tag{4.23}$$

$$\square^{A'B'} \; \chi_B = \widetilde{\Phi}^{A'B'}{}_{LB} \; \chi^L \quad , \tag{4.24}$$

one finds

$$\nabla^{BB'} \nabla^{CA'} \chi_B = \nabla^{(BB'} \nabla^{C)A'} \chi_B + \nabla^{[BB'} \nabla^{C]A'} \chi_B$$

$$= -\frac{1}{2} \nabla_B^{\ B'} \nabla^{CA'} \chi^B + \frac{1}{2} \widetilde{\Phi}^{A'B'LC} \chi_L \quad . \qquad (4.25)$$

Thus, if $\widetilde{\Phi}^{A'B'LC}$ vanishes, also the left-hand side of (4.25) has to vanish since this leads to the equation $\nabla^{BB'} \nabla^{CA'} \chi_B = \frac{1}{2} \nabla^{BB'} \nabla^{CA'} \chi_B$. Hence (4.25) is identically satisfied. Similarly, the left-hand side of (4.21) can be made to vanish identically provided the additional condition $\Phi^{CDF'M'} = 0$ holds. The conditions

$$\Phi^{CDF'M'} = 0 \quad , \quad \widetilde{\Phi}^{A'B'CL} = 0 \quad , \qquad (4.26)$$

when combined with the conditions

$$\psi_{ABCD} = \widetilde{\psi}_{A'B'C'D'} = 0 \quad , \quad \Lambda = \widetilde{\Lambda} = 0 \quad , \qquad (4.27)$$

arising from (4.22) for the local existence of $\rho_{A'}^{BC}$ and $\Lambda_A^{B'C'}$ potentials, imply that the whole Riemann curvature should vanish. Hence, in the boundary-value problems we are interested in, the only admissible background 4-geometry (of the Einstein type [20]) is flat Euclidean 4-space.

5. Boundary conditions in supergravity

The boundary conditions studied in the previous sections are not appropriate if one studies supergravity multiplets and supersymmetry transformations at the boundary [9]. By contrast, it turns out one has to impose another set of locally supersymmetric boundary conditions, first proposed in [21]. These are in general mixed, and involve in particular Dirichlet conditions for the transverse modes of the vector potential of electromagnetism, a mixture of Dirichlet and Neumann conditions for scalar fields, and local boundary conditions for the spin-$\frac{1}{2}$ field and the spin-$\frac{3}{2}$ potential. Using two-component spinor notation for supergravity [9,22], the spin-$\frac{3}{2}$ boundary conditions take the form

$$\sqrt{2}\ {}_e n_A^{\ A'}\ \psi^A_{\ i} = \epsilon\ \widetilde{\psi}^{A'}_{\ i} \quad \text{on} \quad S^3 \quad . \tag{5.1}$$

With our notation, $\epsilon \equiv \pm 1$, ${}_e n_A^{\ A'}$ is the Euclidean normal to S^3, and $\left(\psi^A_{\ i}, \widetilde{\psi}^{A'}_{\ i}\right)$ are the *independent* (i.e. not related by any conjugation) spatial components (hence $i = 1, 2, 3$) of the spinor-valued 1-forms appearing in the action functional of Euclidean supergravity [9,22].

It appears necessary to understand whether the analysis in the previous section and in [23] can be used to derive restrictions on the classical boundary-value problem corresponding to (5.1). For this purpose, we study a Riemannian background 4-geometry, and we use the decompositions of the spinor-valued 1-forms in such a background, i.e. [9]

$$\psi^A_{\ i} = h^{-\frac{1}{4}} \left[\chi^{(AB)B'} + \epsilon^{AB}\ \widetilde{\phi}^{B'} \right] e_{BB'i} \quad , \tag{5.2}$$

$$\widetilde{\psi}^{A'}_{\ i} = h^{-\frac{1}{4}} \left[\widetilde{\chi}^{(A'B')B} + \epsilon^{A'B'}\ \phi^{B} \right] e_{BB'i} \quad , \tag{5.3}$$

where h is the determinant of the 3-metric on S^3, and $e_{BB'i}$ is the spatial component of the tetrad, written in 2-spinor language. If we now reduce the classical theory of simple supergravity to its physical degrees of freedom by imposing the gauge conditions [9]

$$e_{AA'}{}^{i} \; \psi^{A}{}_{i} = 0 \quad , \tag{5.4}$$

$$e_{AA'}{}^{i} \; \widetilde{\psi}^{A'}{}_{i} = 0 \quad , \tag{5.5}$$

we find that the expansions of (5.2)-(5.3) on a family of 3-spheres centred on the origin take the forms [9]

$$\psi^{A}{}_{i} = \frac{h^{-\frac{1}{4}}}{2\pi} \sum_{n=0}^{\infty} \sum_{p,q=1}^{(n+1)(n+4)} \alpha_{n}^{pq} \Bigl[m_{np}^{(\beta)}(\tau) \; \beta^{nqABB'} + \widetilde{r}_{np}^{(\mu)}(\tau) \; \overline{\mu}^{nqABB'} \Bigr] e_{BB'i} \quad , \tag{5.6}$$

$$\widetilde{\psi}^{A'}{}_{i} = \frac{h^{-\frac{1}{4}}}{2\pi} \sum_{n=0}^{\infty} \sum_{p,q=1}^{(n+1)(n+4)} \alpha_{n}^{pq} \Bigl[\widetilde{m}_{np}^{(\beta)}(\tau) \; \overline{\beta}^{nqA'B'B} + r_{np}^{(\mu)}(\tau) \; \mu^{nqA'B'B} \Bigr] e_{BB'i} \quad . \tag{5.7}$$

With our notation, α_{n}^{pq} are block-diagonal matrices with blocks $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, and the β- and μ-harmonics on S^3 are given by [9]

$$\beta^{nq}{}_{ACC'} = \rho^{nq}{}_{(ACD)} \; n^{D}{}_{C'} \quad , \tag{5.8}$$

$$\mu^{nq}{}_{A'B'B} = \sigma^{nq}{}_{(A'B'C')} \; n_{B}{}^{C'} \quad . \tag{5.9}$$

In the light of (5.6)-(5.9), one gets the following physical-degrees-of-freedom form of the spinor-valued 1-forms of supergravity (cf. [9,22,24]):

$$\psi^{A}{}_{i} = h^{-\frac{1}{4}} \; \phi^{(ABC)} \; {}_{e}n_{C}{}^{B'} \; e_{BB'i} \quad , \tag{5.10}$$

$$\widetilde{\psi}^{A'}{}_{i} = h^{-\frac{1}{4}} \; \widetilde{\phi}^{(A'B'C')} \; {}_{e}n^{B}{}_{C'} \; e_{BB'i} \quad , \tag{5.11}$$

where $\phi^{(ABC)}$ and $\widetilde{\phi}^{(A'B'C')}$ are totally symmetric and independent spinor fields.

Within this framework, a *sufficient* condition for the validity of the boundary conditions (5.1) on S^3 is

$$\sqrt{2}\ {}_{e}n_{A}^{\ \ A'}\ {}_{e}n_{C}^{\ \ B'}\ \phi^{(ABC)} = \epsilon\ {}_{e}n^{B}_{\ \ C'}\ \widetilde{\phi}^{(A'B'C')} \quad . \tag{5.12}$$

From now on, one can again try to express *locally* $\phi^{(ABC)}$ and $\widetilde{\phi}^{(A'B'C')}$ in terms of four potentials as in section 4 and in [23], providing they are solutions of massless free-field equations. The alternative possibility is to consider the Rarita-Schwinger form of the field strength, written in 2-spinor language. The corresponding potential is no longer symmetric as in (4.1), and is instead subject to the equations (cf. [13-16,25])

$$\epsilon^{B'C'}\ \nabla_{A(A'}\ \gamma^{A}_{\ \ B')C'} = 0 \quad , \tag{5.13}$$

$$\nabla^{B'(B}\ \gamma^{A)}_{\ \ B'C'} = 0 \quad . \tag{5.14}$$

Moreover, the spinor field $\widetilde{\nu}_{A'}$ appearing in the gauge transformation (4.4) is no longer taken to be a solution of the positive-helicity Weyl equation (4.5). Hence the classical boundary-value problem might have new features with respect to the analysis of section 4 and [23].

Indeed, the investigation appearing in this section is incomplete, and it relies in part on the unfinished work in [26]. Moreover, it should be emphasized that our analysis, although motivated by quantum cosmology, is entirely classical. Hence we have not discussed ghost modes. The theory has been reduced to its physical degrees of freedom to make a comparison with the results in [23], but totally symmetric field strengths do not enable one to recover the full physical content of simple supergravity. Hence the 4-sphere background studied in [2] is not ruled out by our work [26].

6. Results and open problems

Following [9] and [23], we have derived the conditions (2.13), (2.15), (3.5), and (3.8) under which spin-lowering and spin-raising operators preserve the local boundary conditions studied in [9-11]. Note that, for spin 0, we have introduced a pair of independent scalar fields on the real Riemannian section of a complex space-time, following [27], rather than a single scalar field, as done in [9]. Setting $\phi \equiv \phi_1 + i\phi_2, \widetilde{\phi} \equiv \phi_3 + i\phi_4$, this choice leads to the boundary conditions

$$\phi_1 = \epsilon \, \phi_3 \quad \phi_2 = \epsilon \, \phi_4 \quad \text{on} \quad S^3 \quad , \tag{6.1}$$

$${}_e n^{AA'} \, D_{AA'} \, \phi_1 = -\epsilon \, {}_e n^{AA'} \, D_{AA'} \, \phi_3 \quad \text{on} \quad S^3 \quad , \tag{6.2}$$

$${}_e n^{AA'} \, D_{AA'} \, \phi_2 = -\epsilon \, {}_e n^{AA'} \, D_{AA'} \, \phi_4 \quad \text{on} \quad S^3 \quad , \tag{6.3}$$

and it deserves further study.

We have then focused on the potentials for spin-$\frac{3}{2}$ field strengths in flat or curved Riemannian 4-space bounded by a 3-sphere. Remarkably, it turns out that local boundary conditions involving field strengths and normals can only be imposed in a flat Euclidean background, for which the gauge freedom in the choice of the potentials remains. In [16] it was found that ρ potentials exist *locally* only in the self-dual Ricci-flat case, whereas γ potentials may be introduced in the anti-self-dual case. Our result may be interpreted as a further restriction provided by (quantum) cosmology. What happens is that the boundary conditions (2.1) fix at the boundary a spinor field involving *both* the field strength ϕ_{ABC} and the field strength $\widetilde{\phi}_{A'B'C'}$. The local existence of potentials for the field strength ϕ_{ABC}, jointly with the occurrence of a boundary, forces half of the Riemann curvature of the background to vanish. Similarly, the remaining half of such Riemann curvature has to vanish on considering the field strength $\widetilde{\phi}_{A'B'C'}$. Hence the background 4-geometry can only be flat Euclidean space. This is different from the analysis in [13-16], since in that case one is not dealing with boundary conditions forcing us to consider both ϕ_{ABC} and $\widetilde{\phi}_{A'B'C'}$.

A naturally occurring question is whether the potentials studied in this paper can be used to perform one-loop calculations for spin-$\frac{3}{2}$ field strengths subject to (2.1) on S^3. This problem may provide another example (cf. [9]) of the fertile interplay between twistor theory and quantum cosmology [26], and its solution might shed new light on one-loop quantum cosmology and on the quantization program for gauge theories in the presence of boundaries [1-9]. For this purpose, as shown in recent papers by ourselves and other co-authors [28-30], it is necessary to study Riemannian background 4-geometries bounded by two concentric 3-spheres (cf. sections 2-5). Moreover, the consideration of non-physical degrees of freedom of gauge fields, set to zero in our classical analysis, is necessary to achieve a covariant quantization scheme.

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