June 1995, DSF report 95/32

SPIN-RAISING OPERATORS AND SPIN-$\frac{3}{2}$ POTENTIALS IN QUANTUM COSMOLOGY

Giampiero Esposito[1,2,3] **and Giuseppe Pollifrone**[1]

[1] *Istituto Nazionale di Fisica Nucleare*

Mostra d'Oltremare Padiglione 20, 80125 Napoli, Italy;

[2] *Dipartimento di Scienze Fisiche*

Mostra d'Oltremare Padiglione 19, 80125 Napoli, Italy;

[3] *Dipartimento di Fisica Teorica*

Università degli Studi di Salerno

84081 Baronissi, Italy.

Abstract. Local boundary conditions involving field strengths and the normal to the boundary, originally studied in anti-de Sitter space-time, have been recently considered in one-loop quantum cosmology. This paper derives the conditions under which spin-raising operators preserve these local boundary conditions on a 3-sphere for fields of spin $0, \frac{1}{2}, 1, \frac{3}{2}$ and 2. Moreover, the two-component spinor analysis of the four potentials of the totally symmetric and independent field strengths for spin $\frac{3}{2}$ is applied to the case of a 3-sphere boundary. It is shown that such boundary conditions can only be imposed in a flat Euclidean background, for which the gauge freedom in the choice of the potentials remains.

PACS numbers: 04.20.Cv, 04.60.+n, 98.80.Dr

Recent work in the literature has studied the quantization of gauge theories and supersymmetric field theories in the presence of boundaries, with application to one-loop quantum cosmology [1-9]. In particular, in the work described in [9], two possible sets of local boundary conditions were studied. One of these, first proposed in anti-de Sitter space-time [10-11], involves the normal to the boundary and Dirichlet or Neumann conditions for spin 0, the normal and the field for massless spin-$\frac{1}{2}$ fermions, the normal and totally symmetric field strengths for spins 1, $\frac{3}{2}$ and 2. Although more attention has been paid to alternative local boundary conditions motivated by supersymmetry, as in [2-3,8-9], the analysis of the former boundary conditions remains of mathematical and physical interest by virtue of its links with twistor theory [9]. The aim of this paper is to derive further mathematical properties of the corresponding boundary-value problems which are relevant for quantum cosmology and twistor theory.

In section 5.7 of [9], a flat Euclidean background bounded by a 3-sphere was studied. On the bounding S^3, the following boundary conditions for a spin-s field were required:

$$2^s \ {}_{e}n^{AA'} ... {}_{e}n^{LL'} \ \phi_{A...L} = \pm \widetilde{\phi}^{A'...L'} \quad . \tag{1}$$

With our notation, ${}_{e}n^{AA'}$ is the Euclidean normal to S^3 [3,9], $\phi_{A...L} = \phi_{(A...L)}$ and $\widetilde{\phi}_{A'...L'} = \widetilde{\phi}_{(A'...L')}$ are totally symmetric and independent (i.e. not related by any conjugation) field strengths, which reduce to the massless spin-$\frac{1}{2}$ field for $s = \frac{1}{2}$. Moreover, the complex scalar field ϕ is such that its real part obeys Dirichlet conditions on S^3 and its imaginary part obeys Neumann conditions on S^3, or the other way around, according to the value of the parameter $\epsilon \equiv \pm 1$ occurring in (1), as described in [9].

In flat Euclidean 4-space, we write the solutions of the twistor equations [9,12]

$$D_{A'}^{\ (A}\ \omega^{B)} = 0 \qquad (2)$$

$$D_{A}^{\ (A'}\ \widetilde{\omega}^{B')} = 0 \qquad (3)$$

as [9]

$$\omega^{A} = (\omega^{o})^{A} - i\left({}_{e}x^{AA'}\right)\pi^{o}_{A'} \qquad (4)$$

$$\widetilde{\omega}^{A'} = (\widetilde{\omega}^{o})^{A'} - i\left({}_{e}x^{AA'}\right)\widetilde{\pi}^{o}_{A} \quad . \qquad (5)$$

Note that, since unprimed and primed spin-spaces are no longer isomorphic in the case of Riemannian 4-metrics, Eq. (3) is not obtained by complex conjugation of Eq. (2). Hence the spinor field $\widetilde{\omega}^{B'}$ is independent of ω^{B}. This leads to distinct solutions (4)-(5), where the spinor fields $\omega^{o}_{A}, \widetilde{\omega}^{o}_{A'}, \widetilde{\pi}^{o}_{A}, \pi^{o}_{A'}$ are covariantly constant with respect to the flat connection D, whose corresponding spinor covariant derivative is here denoted by $D_{AB'}$. In section 5.7 of [9] it was shown that the spin-lowering operator [9,12] preserves the local boundary conditions (1) on a 3-sphere of radius r if and only if

$$\omega^{o}_{A} = -\frac{i\epsilon r}{\sqrt{2}}\ \widetilde{\pi}^{o}_{A} \qquad (6)$$

$$\widetilde{\omega}^{o}_{A'} = -\frac{i\epsilon r}{\sqrt{2}}\ \pi^{o}_{A'} \quad . \qquad (7)$$

To derive the corresponding preservation condition for spin-raising operators [12], we begin by studying the relation between spin-$\frac{1}{2}$ and spin-1 fields. In this case, the independent spin-1 field strengths take the form [9,11-12]

$$\psi_{AB} = i\ \widetilde{\omega}^{L'}\left(D_{BL'}\ \chi_{A}\right) - 2\chi_{(A}\ \widetilde{\pi}^{o}_{B)} \qquad (8)$$

$$\widetilde{\psi}_{A'B'} = -i\,\omega^L \Bigr(D_{LB'}\,\widetilde{\chi}_{A'}\Bigr) - 2\widetilde{\chi}_{(A'}\,\pi^o_{B')} \qquad (9)$$

where the independent spinor fields $\Bigr(\chi_A, \widetilde{\chi}_{A'}\Bigr)$ represent a massless spin-$\frac{1}{2}$ field obeying the Weyl equations on flat Euclidean 4-space and subject to the boundary conditions

$$\sqrt{2}\;{}_e n^{AA'}\;\chi_A = \epsilon\,\widetilde{\chi}^{A'} \qquad (10)$$

on a 3-sphere of radius r. Thus, by requiring that (8) and (9) should obey (1) on S^3 with $s = 1$, and bearing in mind (10), one finds

$$2\epsilon\biggr[\sqrt{2}\,\widetilde{\pi}^o_A\,\widetilde{\chi}^{(A'}\;{}_e n^{AB')} - \widetilde{\chi}^{(A'}\;\pi^{o\;B')}\biggr] = i\biggr[2\;{}_e n^{AA'}\;{}_e n^{BB'}\,\widetilde{\omega}^{L'}\;D_{L'(B}\;\chi_{A)}$$
$$+\,\epsilon\,\omega^L\;D_L^{\;(B'}\;\widetilde{\chi}^{A')}\biggr] \qquad (11)$$

on the bounding S^3. It is now clear how to carry out the calculation for higher spins. Denoting by s the spin obtained by spin-raising, and defining $n \equiv 2s$, one finds

$$n\epsilon\biggr[\sqrt{2}\,\widetilde{\pi}^o_A\;{}_e n^{A(A'}\;\widetilde{\chi}^{B'\ldots K')} - \widetilde{\chi}^{(A'\ldots D'}\;\pi^{o\;K')}\biggr] = i\biggr[2^{\frac{n}{2}}\;{}_e n^{AA'}\ldots{}_e n^{KK'}\,\widetilde{\omega}^{L'}\;D_{L'(K}\;\chi_{A\ldots D)}$$
$$+\,\epsilon\,\omega^L\;D_L^{\;(K'}\;\widetilde{\chi}^{A'\ldots D')}\biggr] \qquad (12)$$

on the 3-sphere boundary. In the comparison spin-0 vs spin-$\frac{1}{2}$, the preservation condition is not obviously obtained from (12). The desired result is here found by applying the spin-raising operators [12] to the independent scalar fields ϕ and $\widetilde{\phi}$ (see below) and bearing in mind (4)-(5) and the boundary conditions

$$\phi = \epsilon\,\widetilde{\phi} \quad \text{on} \quad S^3 \qquad (13)$$

$$ {}_{e}n^{AA'} D_{AA'} \phi = -\epsilon \ {}_{e}n^{BB'} D_{BB'} \widetilde{\phi} \quad \text{on} \quad S^3 \quad . \tag{14} $$

This leads to the following condition on S^3 (cf Eq. (5.7.23) of [9]):

$$ 0 = i\phi \left[\frac{\widetilde{\pi}_A^o}{\sqrt{2}} - \pi_{A'}^o \ {}_{e}n_A^{\ A'} \right] - \left[\frac{\widetilde{\omega}^{K'}}{\sqrt{2}} \Bigl(D_{AK'} \phi \Bigr) - \frac{\omega_A}{2} \ {}_{e}n_C^{\ K'} \Bigl(D^C_{\ K'} \phi \Bigr) \right] $$

$$ + \epsilon \ {}_{e}n_{(A}^{\ \ A'} \ \omega^B \ D_{B)A'} \ \widetilde{\phi} \quad . \tag{15} $$

Note that, whilst the preservation conditions (6-7) for spin-lowering operators are purely algebraic, the preservation conditions (12) and (15) for spin-raising operators are more complicated, since they also involve the value at the boundary of four-dimensional covariant derivatives of spinor fields or scalar fields. Two independent scalar fields have been introduced, since the spinor fields obtained by applying the spin-raising operators to ϕ and $\widetilde{\phi}$ respectively are independent as well in our case.

In the second part of this paper, we focus on the totally symmetric field strengths ϕ_{ABC} and $\widetilde{\phi}_{A'B'C'}$ for spin-$\frac{3}{2}$ fields, and we express them in terms of their potentials, rather than using spin-raising (or spin-lowering) operators. The corresponding theory in Minkowski space-time (and curved space-time) is described in [13-16], and adapted here to the case of flat Euclidean 4-space with flat connection D. It turns out that $\widetilde{\phi}_{A'B'C'}$ can then be obtained from two potentials defined as follows. The first potential satisfies the properties [13-16]

$$ \gamma_{A'B'}^C = \gamma_{(A'B')}^C \tag{16} $$

$$ D^{AA'} \ \gamma_{A'B'}^C = 0 \tag{17} $$

$$ \widetilde{\phi}_{A'B'C'} = D_{CC'} \ \gamma_{A'B'}^C \tag{18} $$

with the gauge freedom of replacing it by

$$\widehat{\gamma}_{A'B'}^{C} \equiv \gamma_{A'B'}^{C} + D^{C}{}_{B'} \; \widetilde{\nu}_{A'} \tag{19}$$

where $\widetilde{\nu}_{A'}$ satisfies the positive-helicity Weyl equation

$$D^{AA'} \; \widetilde{\nu}_{A'} = 0 \quad . \tag{20}$$

The second potential is defined by the conditions [13-16]

$$\rho_{A'}^{BC} = \rho_{A'}^{(BC)} \tag{21}$$

$$D^{AA'} \; \rho_{A'}^{BC} = 0 \tag{22}$$

$$\gamma_{A'B'}^{C} = D_{BB'} \; \rho_{A'}^{BC} \tag{23}$$

with the gauge freedom of being replaced by

$$\widehat{\rho}_{A'}^{BC} \equiv \rho_{A'}^{BC} + D^{C}{}_{A'} \; \chi^{B} \tag{24}$$

where χ^{B} satisfies the negative-helicity Weyl equation

$$D_{BB'} \; \chi^{B} = 0 \quad . \tag{25}$$

Moreover, in flat Euclidean 4-space the field strength ϕ_{ABC} is expressed in terms of the potential $\Gamma_{AB}^{C'} = \Gamma_{(AB)}^{C'}$, independent of $\gamma_{A'B'}^{C}$, as

$$\phi_{ABC} = D_{CC'} \; \Gamma_{AB}^{C'} \tag{26}$$

with gauge freedom

$$\widehat{\Gamma}_{AB}^{C'} \equiv \Gamma_{AB}^{C'} + D^{C'}{}_{B} \; \nu_{A} \quad . \tag{27}$$

Thus, if we insert (18) and (26) into the boundary conditions (1) with $s = \frac{3}{2}$, and require that also the gauge-equivalent potentials (19) and (27) should obey such boundary conditions on S^3, we find that

$$2^{\frac{3}{2}} \; {}_e n^{A}{}_{A'} \; {}_e n^{B}{}_{B'} \; {}_e n^{C}{}_{C'} \; D_{CL'} \; D^{L'}{}_{B} \; \nu_A = \epsilon \; D_{LC'} \; D^{L}{}_{B'} \; \widetilde{\nu}_{A'} \tag{28}$$

on the 3-sphere. Note that, from now on (as already done in (12) and (15)), covariant derivatives appearing in boundary conditions are first taken on the background and then evaluated on S^3. In the case of our flat background, (28) is identically satisfied since $D_{CL'} \; D^{L'}{}_{B} \; \nu_A$ and $D_{LC'} \; D^{L}{}_{B'} \; \widetilde{\nu}_{A'}$ vanish by virtue of spinor Ricci identities [17-18]. In a curved background, however, denoting by ∇ the corresponding curved connection, and defining $\Box_{AB} \equiv \nabla_{M'(A} \nabla^{M'}{}_{B)}$, $\Box_{A'B'} \equiv \nabla_{X(A'} \; \nabla^{X}{}_{B')}$, since the spinor Ricci identities we need are [17]

$$\Box_{AB} \; \nu_C = \psi_{ABDC} \; \nu^D - 2\Lambda \; \nu_{(A} \; \epsilon_{B)C} \tag{29}$$

$$\Box_{A'B'} \; \widetilde{\nu}_{C'} = \widetilde{\psi}_{A'B'D'C'} \; \widetilde{\nu}^{D'} - 2\widetilde{\Lambda} \; \widetilde{\nu}_{(A'} \; \epsilon_{B')C'} \tag{30}$$

one finds that the corresponding boundary conditions

$$2^{\frac{3}{2}} \; {}_e n^{A}{}_{A'} \; {}_e n^{B}{}_{B'} \; {}_e n^{C}{}_{C'} \; \nabla_{CL'} \; \nabla^{L'}{}_{B} \; \nu_A = \epsilon \; \nabla_{LC'} \; \nabla^{L}{}_{B'} \; \widetilde{\nu}_{A'} \tag{31}$$

are identically satisfied if and only if one of the following conditions holds: (i) $\nu_A = \widetilde{\nu}_{A'} = 0$; (ii) the Weyl spinors $\psi_{ABCD}, \widetilde{\psi}_{A'B'C'D'}$ and the scalars $\Lambda, \widetilde{\Lambda}$ vanish everywhere. However, since in a curved space-time with vanishing $\Lambda, \widetilde{\Lambda}$, the potentials with the gauge freedoms (19) and (27) only exist provided D is replaced by ∇ and the trace-free part Φ_{ab} of the Ricci tensor vanishes as well [19], the background 4-geometry is actually flat Euclidean

4-space. Note that we require that (31) should be identically satisfied to avoid that, after a gauge transformation, one obtains more boundary conditions than the ones originally imposed. The curvature of the background should not, itself, be subject to a boundary condition.

The same result can be derived by using the potential $\rho_{A'}^{BC}$ and its independent counterpart $\Lambda_A^{B'C'}$. This spinor field yields the $\Gamma_{AB}^{C'}$ potential by means of

$$\Gamma_{AB}^{C'} = D_{BB'} \, \Lambda_A^{B'C'} \tag{32}$$

and has the gauge freedom

$$\widehat{\Lambda}_A^{B'C'} \equiv \Lambda_A^{B'C'} + D^{C'}{}_A \, \widetilde{\chi}^{B'} \tag{33}$$

where $\widetilde{\chi}^{B'}$ satisfies the positive-helicity Weyl equation

$$D_{BF'} \, \widetilde{\chi}^{F'} = 0 \quad . \tag{34}$$

Thus, if also the gauge-equivalent potentials (24) and (33) have to satisfy the boundary conditions (1) on S^3, one finds

$$2^{\frac{3}{2}} \, {}_e n^A{}_{A'} \, {}_e n^B{}_{B'} \, {}_e n^C{}_{C'} \, D_{CL'} \, D_{BF'} \, D^{L'}{}_A \, \widetilde{\chi}^{F'} = \epsilon \, D_{LC'} \, D_{MB'} \, D^L{}_{A'} \, \chi^M \tag{35}$$

on the 3-sphere. In our flat background, covariant derivatives commute, hence (35) is identically satisfied by virtue of (25) and (34). However, in the curved case the boundary conditions (35) are replaced by

$$2^{\frac{3}{2}} \, {}_e n^A{}_{A'} \, {}_e n^B{}_{B'} \, {}_e n^C{}_{C'} \, \nabla_{CL'} \, \nabla_{BF'} \, \nabla^{L'}{}_A \, \widetilde{\chi}^{F'} = \epsilon \, \nabla_{LC'} \, \nabla_{MB'} \, \nabla^L{}_{A'} \, \chi^M \tag{36}$$

on S^3, if the *local* expressions of ϕ_{ABC} and $\widetilde{\phi}_{A'B'C'}$ in terms of potentials still hold [13-16]. By virtue of (29)-(30), where ν_C is replaced by χ_C and $\widetilde{\nu}_{C'}$ is replaced by $\widetilde{\chi}_{C'}$, this means that the Weyl spinors $\psi_{ABCD}, \widetilde{\psi}_{A'B'C'D'}$ and the scalars $\Lambda, \widetilde{\Lambda}$ should vanish, since one should find

$$\nabla^{AA'} \; \widehat{\rho}_{A'}^{BC} = 0 \quad \nabla^{AA'} \; \widehat{\Lambda}_{A}^{B'C'} = 0 \quad . \tag{37}$$

If we assume that $\nabla_{BF'} \; \widetilde{\chi}^{F'} = 0$ and $\nabla_{MB'} \; \chi^M = 0$, we have to show that (36) differs from (35) by terms involving a part of the curvature that is vanishing everywhere. This is proved by using the basic rules of two-spinor calculus and spinor Ricci identities [17-18]. Thus, bearing in mind that [17]

$$\square^{AB} \; \widetilde{\chi}_{B'} = \Phi^{AB}{}_{L'B'} \; \widetilde{\chi}^{L'} \tag{38}$$

$$\square^{A'B'} \; \chi_B = \widetilde{\Phi}^{A'B'}{}_{LB} \; \chi^L \tag{39}$$

one finds

$$\nabla^{BB'} \; \nabla^{CA'} \; \chi_B = \nabla^{(BB'} \; \nabla^{C)A'} \; \chi_B + \nabla^{[BB'} \; \nabla^{C]A'} \; \chi_B$$

$$= -\frac{1}{2} \nabla_B{}^{B'} \; \nabla^{CA'} \; \chi^B + \frac{1}{2} \widetilde{\Phi}^{A'B'LC} \; \chi_L \quad . \tag{40}$$

Thus, if $\widetilde{\Phi}^{A'B'LC}$ vanishes, also the left-hand side of (40) has to vanish since this leads to the equation $\nabla^{BB'} \; \nabla^{CA'} \; \chi_B = \frac{1}{2} \nabla^{BB'} \; \nabla^{CA'} \; \chi_B$. Hence (40) is identically satisfied. Similarly, the left-hand side of (36) can be made to vanish identically provided the additional condition $\Phi^{CDF'M'} = 0$ holds. The conditions

$$\Phi^{CDF'M'} = 0 \qquad \widetilde{\Phi}^{A'B'CL} = 0 \tag{41}$$

when combined with the conditions

$$\psi_{ABCD} = \widetilde{\psi}_{A'B'C'D'} = 0 \quad \Lambda = \widetilde{\Lambda} = 0 \tag{42}$$

arising from (37) for the local existence of $\rho_{A'}^{BC}$ and $\Lambda_{A}^{B'C'}$ potentials, imply that the whole Riemann curvature should vanish. Hence, in the boundary-value problems we are interested in, the only admissible background 4-geometry (of the Einstein type [20]) is flat Euclidean 4-space.

In conclusion, in our paper we have completed the characterization of the conditions under which spin-lowering and spin-raising operators preserve the local boundary conditions studied in [9-11]. Note that, for spin 0, we have introduced a pair of independent scalar fields on the real Riemannian section of a complex space-time, following [21], rather than a single scalar field, as done in [9]. Setting $\phi \equiv \phi_1 + i\phi_2, \widetilde{\phi} \equiv \phi_3 + i\phi_4$, this choice leads to the boundary conditions

$$\phi_1 = \epsilon \, \phi_3 \quad \phi_2 = \epsilon \, \phi_4 \quad \text{on} \quad S^3 \tag{43}$$

$${}_{e}n^{AA'} \, D_{AA'} \, \phi_1 = -\epsilon \, {}_{e}n^{AA'} \, D_{AA'} \, \phi_3 \quad \text{on} \quad S^3 \tag{44}$$

$${}_{e}n^{AA'} \, D_{AA'} \, \phi_2 = -\epsilon \, {}_{e}n^{AA'} \, D_{AA'} \, \phi_4 \quad \text{on} \quad S^3 \tag{45}$$

and it deserves further study.

We have then focused on the potentials for spin-$\frac{3}{2}$ field strengths in flat or curved Riemannian 4-space bounded by a 3-sphere. Remarkably, it turns out that local boundary conditions involving field strengths and normals can only be imposed in a flat Euclidean background, for which the gauge freedom in the choice of the potentials remains. In [16] it was found that ρ potentials exist *locally* only in the self-dual Ricci-flat case, whereas γ potentials may be introduced in the anti-self-dual case. Our result may be interpreted as a further restriction provided by (quantum) cosmology.

A naturally occurring question is whether the potentials studied in this paper can be used to perform one-loop calculations for spin-$\frac{3}{2}$ field strengths subject to (1) on S^3. This problem may provide another example (cf [9]) of the fertile interplay between twistor theory and quantum cosmology, and its solution might shed new light on one-loop quantum cosmology and on the quantization program for gauge theories in the presence of boundaries [1-9].

Acknowledgments

The first author is indebted to Roger Penrose for bringing [13-16] to his attention. The second author is grateful to the Istituto Astronomico dell'Università di Roma for hospitality. Anonymous referees suggested the correct form of some statements originally formulated in a misleading way in the first version of our work.

References

[1] Moss I G and Poletti S 1990 *Nucl. Phys.* B **341** 155

[2] Poletti S 1990 *Phys. Lett.* **249B** 249

[3] D'Eath P D and Esposito G 1991 *Phys. Rev.* D **43** 3234

[4] D'Eath P D and Esposito G 1991 *Phys. Rev.* D **44** 1713

[5] Barvinsky A O, Kamenshchik A Y, Karmazin I P and Mishakov I V 1992 *Class. Quantum Grav.* **9** L27

[6] Kamenshchik A Y and Mishakov I V 1992 *Int. J. Mod. Phys.* A **7** 3713

[7] Barvinsky A O, Kamenshchik A Y and Karmazin I P 1992 *Ann. Phys., N.Y.* **219** 201

[8] Kamenshchik A Y and Mishakov I V 1993 *Phys. Rev.* D **47** 1380

[9] Esposito G 1994 *Quantum Gravity, Quantum Cosmology and Lorentzian Geometries* Lecture Notes in Physics, New Series m: Monographs vol m12 second corrected and enlarged edn (Berlin: Springer)

[10] Breitenlohner P and Freedman D Z 1982 *Ann. Phys., N.Y.* **144** 249

[11] Hawking S W 1983 *Phys. Lett.* **126B** 175

[12] Penrose R and Rindler W 1986 *Spinors and Space-Time, Vol. 2: Spinor and Twistor Methods in Space-Time Geometry* (Cambridge: Cambridge University Press)

[13] Penrose R 1990 *Twistor Newsletter* **31** 6

[14] Penrose R 1991 *Twistor Newsletter* **32** 1

[15] Penrose R 1991 *Twistor Newsletter* **33** 1

[16] Penrose R 1991 Twistors as Spin-$\frac{3}{2}$ Charges *Gravitation and Modern Cosmology* eds A Zichichi, V de Sabbata and N Sánchez (New York: Plenum Press)

[17] Ward R S and Wells R O 1990 *Twistor Geometry and Field Theory* (Cambridge: Cambridge University Press)

[18] Esposito G 1993 *Nuovo Cimento* B **108** 123

[19] Buchdahl H A 1958 *Nuovo Cim.* **10** 96

[20] Besse A L 1987 *Einstein Manifolds* (Berlin: Springer)

[21] Hawking S W 1979 The path integral approach to quantum gravity *General Relativity, an Einstein Centenary Survey* eds S W Hawking and W Israel (Cambridge: Cambridge University Press)