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Ising Model formulation of Large Scale Dynamics: Universality in the Universe.

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Abstract

The partition function of a system of galaxies in gravitational interaction can be cast in an Ising Model form, and this reformulated via a Hubbard–Stratonovich transformation into a three dimensional stochastic and classical scalar field theory, whose critical exponents are calculable and known. This allows one to *compute* the galaxy to galaxy correlation function, whose non–integer exponent is predicted to be between 1.530 and 1.862, to be compared with the phenomenological value of 1.6 to 1.8.

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An intergalactic tourist admiring the Universe at the largest scales would perceive it as something akin to a three dimensional 'salt–and–pepper' pattern which, when projected onto a two dimensional picture, would appear very similar to what *we* see in pictures of Large Scale Surveys, such as the Lick [1] or APM surveys [2]. As he reduced the size of his gauge to smaller and smaller distances he would come to the conclusion that the Universe today is dominated by matter which, at large distances is in gravitational interaction. And that this matter seems to organize itself into bodies which roughly group themselves into solar systems, galaxies, groups of galaxies, and even larger structures. We *believe* we see the same as our tourist, and that we call the Cosmological Principle.

At the larger scales he would probably recognize the Universe as a homogeneous object and therefore describe it with a Friedmann–Robertson–Walker metric. He would reproduce the many successes of cosmology based on this metric. As he went on to smaller scales he would find that (i) *'on all observable scales there are structures seen and significant anisotropies are detected'* [3]. In fact, (ii) if he inferred the galaxy–to–galaxy correlation function ($\xi_{Gal}(r)$) from these surveys, he would discover[1] that $\xi(r) \propto r^{-\gamma}$ where [4] $\gamma \sim O(1.6 - 1.8)$, instead of $\gamma = 1$ which is what one would naively expect[2] for a homogeneous distribution of matter in a three dimensional *space*.

Given these phenomenological facts, one may ask: (A) Is there a *fundamental* explanation for this power law behavior? (B) Can it be understood by using some basic *scheme*?

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[1]The value of γ seems to vary somewhat from survey to survey; for example, it is 1.8 for the Lick survey and about 1.6 for the APM.

[2]This can be understood as follows: as we will see below, the gas of galaxies can be put in a one–to–one correspondence with a 3–dimensional scalar field theory, and the galaxy–to–galaxy correlation function corresponds to the scalar field propagator. In units of length, a scalar field in a d–dimensional space–time has a canonical dimension of $-(d/2-1)$. The above statement about γ follows at once.

This note aims at providing answers to the above questions within the framework of known physics. It rests upon the well–known observation [5] that the existence of a non–integer (anomalous) dimension (signalled by $\gamma \neq 1$) is a tell–tale sign betraying the existence of *both* smaller length scales and fluctuations.

We will apply the techniques of Statistical Mechanics to a system made up by many–galaxies (accounting for the 'smaller length scales') in gravitational interaction, and which are subject to fluctuations emerging from the intrinsic properties of the gravitational interaction of this many body system. The actual nature of the fluctuations needs not be *specified* here, but they could, for example, be related to the 'frictional' processes in gravitational systems long ago considered by Chandrasekhar [6] or, as predicted by well known classical theorems of Poincaré, to chaotic processes [7] related to the many body nature of the gravitational system.

We will obtain the partition function for this system and compute the two–point correlation function and its corrections due to the existence of fluctuations.

Let us consider the continuous mass density, $\rho(r)$, describing the spatial–distribution of galaxies. The deviation from an average density $\bar{\rho}$ is $\delta\rho(r) = \rho(r) - \bar{\rho}$. The galaxy–to–galaxy correlation function is defined [4] as $\xi_{Gal}(r_i - r_j) = \langle \delta\rho(r_i)\delta\rho(r_j) \rangle / \bar{\rho}^2$, where the angular brackets mean that a suitable average has been taken. In the (justifiable) non–relativistic limit, the gravitational interaction energy for this system is then given by (assuming no expansion)

$$H_{int} = \frac{-1}{2} G \int\int d^3r_i d^3r_j \rho(r_i) \frac{1}{|r_i - r_j|} \rho(r_j) \, .$$

As a first approximation, it is reasonable to consider the gas of galaxies as made up of discrete, spatially localized 'points' of (equal) mass m_0, and we can set [4] the 'contrast' $\delta\rho(r_i)/\bar{\rho}$ equal to 1 if there is a galaxy at position r_i and equal to –1 if there is a void. The interaction energy becomes

$$H_{int} = -\frac{1}{2} \sum_{ij} m_i \frac{G}{|r_i - r_j|} m_j \equiv -\frac{1}{2} \sum_{ij} m_i L_{ij} m_j \qquad (1)$$

with the following natural correspondence (since we have assumed that all galaxies have a similar mass) between m_i, a two–valued (± 1) '*spin*' variable s_i, the density and the contrast:

$$2\ m_i/m_0 \leftrightarrow \rho(r_i)/\bar{\rho} \tag{2}$$

$$s_i \leftrightarrow \delta\rho(r_i)/\bar{\rho}. \tag{3}$$

Furthermore, m_i and s_i are related by

$$m_i = m_0\,\frac{1}{2}(s_i + 1) \tag{4}$$

which happens to be analogous to the relationship between a lattice gas and an Ising magnet.

This relationship, via a Hubbard–Stratonovich transformation [8], allows one to map the 'gas' of galaxies into a system described by a stochastic 1–component (scalar) classical field $\phi(r)$ in 3 dimensions, whose partition function $Z[\beta]$ can be readily calculated, and gives

$$Z^{Grav}_{Ising}[\beta] = \sum_{\{m\}} e^{\frac{\beta}{2}\sum_{ij} m_i L_{ij} m_j}$$

$$= C \int [d\phi] \exp\left\{ -\frac{\beta}{2} \int (\phi(r) - H(r))\, L^{-1}(r,r')\, (\phi(r') - H(r'))\, dr dr' + \int dr \log\cosh\left[\beta\phi(r)\right] \right\}$$

$$\equiv \int [d\phi] e^{-\beta\mathcal{H}[\phi,H]}. \tag{5}$$

Here the function $H(r)$ is given by $H(r) = -1/2 \int dr' m_0^2\, L(r,r')$ and C is an inessential factor. The field ϕ is the order parameter for this system. Although all direct reference to the original masses has disappeared, the physics described by the hamiltonian of Eq. (5) is completely equivalent to the original description.

Because of Eq. (1) and (4),

$$L^{-1}(r,r') = -[\delta(r-r')/(2\pi G m_0^2)]\nabla_r^2\,,$$

and

$$\mathcal{H}[\phi, H] = -\frac{1}{2}\int (\phi(r) - H(r))\, \frac{1}{2\pi G m_0^2}\nabla^2\, (\phi(r) - H(r))\, dr$$
$$-\frac{1}{\beta}\int dr\, \log\cosh\left[\beta\phi(r)\right].$$

As is well known [8], the connected, two–point correlation function for the spin system and the field theory are the same. Furthermore, because of fluctuations in the field ϕ, its canonical dimension acquires an *anomalous dimension* and shifts away from its canonical value (cf. Footnote 2), such that when $|r_i - r_j| \to \infty$, the connected, 2–point correlation function for *this* hamiltonian, $\xi(|r_i - r_j|)$, scales as [5] [8]

$$\lim_{|r_i - r_j|\to\infty} \langle s_i\, s_j\rangle = \lim_{|r_i - r_j|\to\infty} \xi(|r_i - r_j|) \sim \frac{1}{|r_i - r_j|^{d-2+\eta}} \tag{6}$$

where the first equality follows from the equivalence between the '$spin$' and '$field$' descriptions of the system, d is the dimensionality of space (=3) and η is the critical exponent for the pair correlation function, whose value (0.0198 – 0.064) (cf. the Table below) differs from zero due to the fluctuations in $\phi(r)$.

Because of Eq. (2) and (3)

$$\xi_{Gal}(|r_i - r_j|) = \langle s_i\, s_j\rangle \tag{7}$$

with the average computed using Eq. (5).

Putting together Eqns. (6) and (7), our calculation *shows* that for large separations, the galaxy–to–galaxy correlation function *must* scale as

$$\xi_{Gal}(|r_i - r_j|) \sim r^{-\gamma}$$

with [10] $\gamma = d - 2 + \eta$ between 1.0198 and 1.064.

Thus far our calculation has been static, but the Universe is expanding, and the effects of expansion can (and do [9]) modify the values of critical exponents. For the correlation function, it is known from computer simulations in condensed matter physics [9] combined with dynamical scaling considerations, that time enters in the correlation function by altering

the argument of the correlation function from $|r_i - r_j|$ to $|r_i - r_j|/L(t)$ where $L(t) \propto t^{\zeta}$ and ζ is determined in computer simulations to be $1/3$ for systems with a conserved order parameter, and $1/2$ for systems with a non–conserved order parameter. Separation and time are related in an expanding Universe where, to a first approximation, in a *matter dominated Universe* the scale factor is proportional to $t^{2/3}$. Putting together the expansion of the Universe and the dynamical critical phenomena effects as contained in ζ, the exponent in the galaxy–to–galaxy correlation function is modified from $\gamma = d - 2 + \eta$ to $(d - 2 + \eta) \times (1 + 3\zeta/2)$. That is, we finally get that the predicted (calculated) value for γ will be[3] between 1.530 and 1.596 $(= \gamma^{C}_{Expanding})$ if we assume that the order parameter is conserved, and between 1.785 and 1.862 $(= \gamma^{NC}_{Expanding})$ if we assume that the order parameter is not conserved.

Method of Calculation	γ_{Static}	$\gamma^{C}_{Expanding}$	$\gamma^{NC}_{Expanding}$
Series Estimates	1.056 ± 0.008	1.584 ± 0.012	1.848 ± 0.014
$O(\epsilon)$	0	1.5	1.75
$O(\epsilon^2)$	1.0198	1.530	1.785
$O(\epsilon^3)$	1.037	1.555	1.815
$O(\epsilon^4)$	1.029	1.543	1.801

These values are to be compared with the values inferred from the existing galaxy catalogs, which range between 1.5 for the APM survey to 1.8 for the Lick survey [4].

Therefore, we see that the questions enumerated at the beginning of this note can find an answer within the framework outlined here. In addition, there is a clear and unambiguous prediction: due to the Universal nature of the gravitational force, reflected in the interaction hamiltonian of Eq. (1), the result we have obtained for the galaxy–to–galaxy

[3]Cf. Table. The static values for η are taken from Reference [10].

correlation function must apply also to *any other many–body–gravitational system*, including the interstellar medium in our galaxy. This means that observations must confirm that the interstellar medium has a distribution whose correlation function scales with the same *generic* power law as galaxies. The only possible difference would be in the numerical value of the anomalous dimension, since for intergalactic gas clouds the size of the system is smaller, and therefore, renormalization group arguments tell us that the value of $\gamma_{Interstellar}$ is smaller than for systems of galaxies, where the coupling constant has had more distance to grow on its way into the IR fixed point[4].

Many questions remain. For example, is the order parameter conserved, or not? Can one use renormalization group techniques to also *calculate* the dynamical effect of the expansion of the Universe on η, instead of appealing to phenomenological (albeit well substantiated) computer estimates to generalize the static values of γ to dynamical values? Does the implied r-dependence of $G(r)$ play a rôle in the physics of large scales? How do initial conditions impact on the correlation functions? How does one actually approach the disordered phase? These questions will be considered in a future paper.

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[4]This follows by noticing that, because of conservation of probability (unitarity in field theory) η is positive; perturbation theory tells us that it is proportional to a power of Gm_0^2. Also, in less than four dimensions, the latter coupling tends in the IR to the equivalent of the Wilson–Fisher fixed point, and away from the Gaussian UV–fixed point.

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