NBI-HE-95-19
hep-th/9506180
June 1995

HAMILTONIAN REDUCTION OF $SL(2)$-THEORIES AT THE LEVEL OF CORRELATORS

Jens Lyng Petersen[1], Jørgen Rasmussen [2] and Ming Yu[3]

The Niels Bohr Institute, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark

Abstract

Since the work of Bershadsky and Ooguri it is well known that correlators of $SL(2)$ current algebra for admissible representations should reduce to correlators for conformal minimal models. A precise proposal for this relation has been given at the level of correlators: When $SL(2)$ primary fields are expressed as $\phi_j(z_n, x_n)$ with x_n being a variable to keep track of the $SL(2)$ representation multiplet (possibly infinitely dimensional for admissible representations), then the minimal model correlator is supposed to be obtained simply by putting all $x_n = z_n$. Although strong support for this has been presented, to the best of our understanding a direct, simple proof seems to be missing so in this paper we present one based on the free field Wakimoto construction and our previous study of that in the present context. We further verify that the explicit $SL(2)$ correlators we have published in a recent preprint reduce in the above way, up to a constant which we also calculate. We further discuss the relation to more standard formulations of hamiltonian reduction.

[1] e-mail address: jenslyng@nbivax.nbi.dk
[2] e-mail address: jrasmussen@nbivax.nbi.dk
[3] e-mail address: yuming@nbivax.nbi.dk. Address after 1st March 1996: Inst. of Theor. Phys., Academia Sinica, Beijing, Peoples Republic of China

1 Introduction

The relation between the $SL(2)$ current algebra and the Virasoro algebra via hamiltonian reduction is well known [1]. In particular Bershadsky and Ooguri [2] used the powerful BRST formalism for the reduction to establish equivalence between on the one hand $\widehat{SL}(2)_k$ WZNW theory after reduction, and on the other hand conformal minimal theory labelled by (p,q), provided $k+2=p/q$, and admissible representations were used in the WZNW case. This equivalence was discussed in those references at the level of the algebra and of the BRST cohomology of physical states. A particularly simple and remarkable realization of these ideas has been discussed by Furlan, Ganchev, Paunov and Petkova [3] at the level of N-point conformal blocks on the sphere. The formulation is in terms of primary fields of the affine algebra in a formalism where they depend on two points [4, 5, 3, 6]. We use the notation of [7]. Thus the algebra is given by

$$\begin{aligned} J^+(z)J^-(w) &= \frac{2}{z-w}J^3(w) + \frac{k}{(z-w)^2} \\ J^3(z)J^\pm(w) &= \pm\frac{1}{z-w}J^\pm(w) \\ J^3(z)J^3(w) &= \frac{k/2}{(z-w)^2} \end{aligned} \tag{1}$$

A primary field, $\phi_j(w,x)$, belonging to a representation labelled by spin j, satisfies the following OPE's

$$J^a(z)\phi_j(w,x) = \frac{1}{z-w}[J_0^a, \phi_j(w,x)] \tag{2}$$

where the $SL(2)$ representation is provided by the differential operators

$$\begin{aligned} [J_n^a, \phi_j(z,x)] &= z^n D_x^a \phi_j(z,x) \\ D_x^+ &= -x^2\partial_x + 2xj \\ D_x^3 &= -x\partial_x + j \\ D_x^- &= \partial_x \end{aligned} \tag{3}$$

The statement in ref.[3] and under investigation in the present paper is that correlators of such primary fields should reduce to corresponding ones for a particular minimal model in the limit where all x's are put equal to the corresponding z values. We shall come back to a more precise statement which may also be formulated for conformal blocks. The authors of ref.[3] construct a solution of the Knizhnik-Zamolodchikov equations [8] such that the x_i dependence is described as a power series in $(x_i - z_i)$, and by construction that solution is selected for which the boundary condition is, that $x_i - z_i = 0$ reproduces the corresponding minimal model correlator. The expansion coefficients are given in terms of recurrence relations[1]. To make sure that this solution of the KZ equations really generates the WZNW correlator (up to normalization) a study is performed in [3] of the null vector decoupling that follows from that solution, and whether that is as expected for a WZNW correlator. Although this was checked in many examples no explicit general proof was

[1] In fact it appears that the sums may be explicitly performed, V.B. Petkova, private communication.

provided. The relation between null vector decoupling in WZNW correlators and minimal model Virasoro ones was discussed for example in ref.[9]. Thus, while there is very strong evidence for the validity of the assertion, at least a simple and straightforward explicit proof is not available. This is what we intend to supply in the present paper.

In ref.[7] we have recently shown how to evaluate affine model conformal blocks within the framework of the free field Wakimoto construction [10]. In particular we developed a technique based on fractional calculus for how to deal with the "second" screening charge written down in ref.[2] which involves fractional powers of free fields and which is crucial for being able to deal with the case of admissible representations. Very similar techniques (without the present applications) have also been worked out by Andreev [6].

In sect. 2 we provide a simple direct proof of the reduction between WZNW correlators and minimal model ones as described above. However, we find that the affine model conformal blocks differ from the minimal model ones by a normalization factor which we evaluate and sometimes find to be zero, making the procedure of ref.[3] singular in those cases. Also we find that the result obtains in general just by putting all x_i's *proportional* to all z_i's independent of the factor of proportionality. This is reasonable because such a proportionality constant would depend on normalizations of the currents. In sect. 3 we compare with standard forms of hamiltonian reduction, and in sect. 4 we explicitly verify that the correlator we wrote down in ref.[7] satisfies the theorem proven in this paper.

2 Proof of the reduction at the correlator level

The free field Wakimoto realization [10] is obtained in terms of free bosonic fields of dimensions $(1,0)$ and of a scalar field which we take to have the following contractions

$$\varphi(z)\varphi(w) = \log(z-w), \quad \beta(z)\gamma(w) = \frac{1}{z-w} \tag{4}$$

We only consider one chirality of the fields. The $\widehat{SL}(2)_k$ affine currents may then be represented as

$$\begin{aligned} J^+(z) &= \beta(z) \\ J^3(z) &= -:\gamma\beta:(z) - \sqrt{t/2}\,\partial\varphi(z) \\ J^-(z) &= -:\gamma^2\beta:(z) + k\partial\gamma(z) - \sqrt{2t}\gamma\partial\varphi(z) \\ t &\equiv k+2 \neq 0 \end{aligned} \tag{5}$$

The Sugawara energy momentum tensor is obtained as

$$T(z) =: \beta\partial\gamma:(z) + \frac{1}{2}:\partial\varphi\partial\varphi:(z) + \frac{1}{\sqrt{2t}}\partial^2\varphi(z) \tag{6}$$

with central charge

$$c = \frac{3k}{k+2}$$

The primary field is [7]

$$\begin{aligned}\phi_j(z,x) &= (1+\gamma(z)x)^{2j}\phi_j^{\varphi}(z)\\ \phi_j^{\varphi}(z) &= :e^{-j\sqrt{2/t}\varphi(z)}:\end{aligned} \tag{7}$$

where in general one should asymptotically expand $(1+\gamma(z)x)^{2j}$ as

$$(1+\gamma(z)x)^{2j}_{(\alpha)} = \sum_{n\in\mathbb{Z}} \begin{pmatrix} 2j \\ n+\alpha \end{pmatrix} (\gamma(z)x)^{n+\alpha} \tag{8}$$

Here the choice of the parameter α depends on the monodromy conditions of the primary field $\phi_j(z,x)$ around contours in x-space, and those in turn depend on the other fields present in the correlator, as is further discussed in [7].

The primary field defined in eq.(7) may also be written as

$$\begin{aligned}\phi_j(z,x) &= e^{x\partial_y}\phi_j(z,y)|_{y=0}\\ &= e^{xD_y^-}\phi_j(z,y)|_{y=0}\\ &= e^{xJ_0^-}\phi_j(z,0)e^{-xJ_0^-}\\ &= e^{xJ_0^-}:e^{-j\sqrt{2/t}\varphi(z)}:e^{-xJ_0^-}\end{aligned} \tag{9}$$

Here there is a subtlety in that the way the exponential function $e^{xJ_0^-}$ should be expanded [7], must also respect the monodromy conditions in the x variables. We shall come back to this subtlety presently. The two screening charges are

$$\begin{aligned}S_{k_\pm}(w) &= \beta^{-k_\pm}(w)S_\pm^{\varphi}(w)\\ S_\pm^{\varphi}(w) &= :e^{-k_\pm\sqrt{2/t}\varphi(w)}:\\ k_+ &= -1\\ k_- &= t\end{aligned} \tag{10}$$

A general conformal block (on the sphere) in the affine theory is then given by

$$W_N = \langle j_N|R\phi_{j_{N-1}}(z_{N-1},x_{N-1})...\phi_{j_n}(z_n,x_n)...\phi_{j_2}(z_2,x_2)\prod_i\oint\frac{dw_i}{2\pi i}S_{k_i}(w_i)|j_1\rangle \tag{11}$$

where R stands for radial ordering of the fields. Different choices of integration contours for the screening charges define different intertwining chiral vertex operators [11, 12], and different conformal blocks.

The relation to minimal models is obtained by writing [2]

$$\begin{aligned}2j_{r,s}+1 &= r-st\\ t &= k+2 = p/q\\ \alpha_+ &= \sqrt{\frac{2}{t}} = -2/\alpha_-\\ \alpha_{r,s+1} &= -j_{r,s}\sqrt{\frac{2}{t}} = \frac{1}{2}((1-r)\alpha_+ - s\alpha_-)\end{aligned}$$

$$
\begin{aligned}
2\alpha_0 &= \alpha_+ + \alpha_- \\
h_{r,s+1} &= \frac{j_{r,s}(j_{r,s}+1)}{t} - j_{r,s} = \frac{1}{2}\alpha_{r,s+1}(\alpha_{r,s+1} - 2\alpha_0) \\
\phi_{r,s+1}(z) &= :e^{\alpha_{r,s+1}\varphi(z)}: = \phi_{j_{r,s}}(z) \\
V_{\alpha_\pm}(w) &= :e^{\alpha_\pm\varphi(w)}: = S^{\varphi}_{\pm}(w)
\end{aligned}
\tag{12}
$$

It is now clear that if one truncates the β-dependence of the screening currents and the γ-dependent factor in the primary fields, then the minimal model correlators are obtained [3]. This is true despite the fact that the two theories, the WZNW-model and the minimal model, have different background charges for the φ-field: namely $-\alpha_+ = -\sqrt{\frac{2}{t}}$ for the WZNW model and $-2\alpha_0 = -\sqrt{\frac{2}{t}} + \sqrt{2t}$ for the minimal models. However, this difference is of no consequence in the practical evaluation of the free field correlators since in both cases suitable dual bra-states are used to absorb that background charge [7]. Using eq.(9) we may further write

$$
\begin{aligned}
\phi_j(z, xz) &= e^{zxJ_0^-} :e^{-j\sqrt{2/t}\varphi(z)}: e^{-zxJ_0^-} \\
&= e^{xzD_y^-}\phi_j(z,y)|_{y=0} \\
&= e^{xJ_1^-} :e^{-j\sqrt{2/t}\varphi(z)}: e^{-xJ_1^-}
\end{aligned}
\tag{13}
$$

Consider the following conformal blocks

$$
W_N = \langle j_N|R\phi_{j_{N-1}}(z_{N-1}, xz_{N-1})...\phi_{j_n}(z_n, xz_n)...\phi_{j_2}(z_2, xz_2)\prod_i \oint \frac{dw_i}{2\pi i} S_{k_i}(w_i)|j_1\rangle \tag{14}
$$

Substituting eq.(13) and barring the subtleties to which we shall come back

$$
\begin{aligned}
W_N &= \langle j_N|e^{xJ_1^-} R\phi_{j_{N-1}}(z_{N-1},0)...\phi_{j_n}(z_n,0)...\phi_{j_2}(z_2,0)\prod_i \oint \frac{dw_i}{2\pi i} S_{k_i}(w_i)|j_1\rangle \\
&= \langle j_N|(1-x\gamma_1)^{k-2j_N} R\phi_{j_{N-1}}(z_{N-1},0)...\phi_{j_n}(z_n,0)...\phi_{j_2}(z_2,0)\prod_i \oint \frac{dw_i}{2\pi i} S_{k_i}(w_i)|j_1\rangle \\
&= C\langle j_N|R\phi_{j_{N-1}}(z_{N-1},0)...\phi_{j_n}(z_n,0)...\phi_{j_2}(z_2,0)\prod_i \oint \frac{dw_i}{2\pi i} S^{\varphi}_{k_i}(w_i)|j_1\rangle \\
&= CW_N^{\varphi}
\end{aligned}
\tag{15}
$$

where C is the normalization constant, and W_N^{φ} is exactly the free field expression for the minimal model correlator. The point is that since

$$
\langle j_N|e^{xJ_1^-} = \langle j_N|(1-x\gamma_1)^{k-2j_N}
$$

(see ref.[7] for details on the bra-vacuum, see also eq.(39)) now contains the only γ-dependence of the correlator, all β-dependence is effectively removed from the screening charges since γ_1 contracts only with β_{-1} which is the constant (w_i-independent) mode. Using techniques of [7] the constant is easily worked out to be

$$
C = \frac{\Gamma(k-2j_N+1)x^{-\sum_i k_i}}{\Gamma(k-2j_N+\sum_i k_i+1)} = \frac{\Gamma(k-2j_N+1)}{\Gamma(k-\sum_{i=1}^N j_i+1)} x^{\sum_{i=1}^N j_i - 2j_N} \tag{16}
$$

This is the simple proof of the statement presented in the beginning of this paper.

The aforementioned subtleties are associated with the fact that considerable care is needed in the above manipulations, since at any one step in removing adjacent factors of

$$e^{\pm x J_1^-}$$

we should examine how these exponentials are defined. Indeed as discussed in [7] the expansions of exponentials and other functions involving the β- and γ-fields, depend on which monodromy the problem at hand requires one to select. All these subtleties are dealt with using the following two lemmas:

Lemma 1: If the fractional part in powers of x is α, then we can expand the last expression in eq.(9)

$$e^{xJ_0^-} : e^{-j\sqrt{2/t}\varphi(z)} : e^{-xJ_0^-} = \sum_{n\in\mathbb{Z}} \frac{(xJ_0^-)^{\alpha-\beta+n}}{(\alpha-\beta+n)!} : e^{-j\sqrt{2/t}\varphi(z)} : \sum_{m\in\mathbb{Z}} \frac{(-xJ_0^-)^{\beta+m}}{(\beta+m)!} \tag{17}$$

for arbitrary complex number β.

Lemma 2:

$$1 = \sum_{n\in\mathbb{Z}} \frac{(xJ_0^-)^{\alpha+n}}{(\alpha+n)!} \sum_{m\in\mathbb{Z}} \frac{(-xJ_0^-)^{-\alpha+m}}{(-\alpha+m)!} = e^{xJ_0^-} e^{-xJ_0^-} \tag{18}$$

Before proving these lemmas we make the following remarks: Define

$$\begin{aligned} \phi_j^{[n]}(z,0) &= [J_0^-, \phi_j^{[n-1]}(z,0)] \\ \phi_j^{[0]}(z,0) &= \phi_j(z,0) \end{aligned} \tag{19}$$

When x is integrally powered, it is clear that we can expand $\phi_j(z,x)$ as

$$\begin{aligned} & e^{xJ_0^-} : e^{-j\sqrt{2/t}\varphi(z)} : e^{-xJ_0^-} \\ = & \sum_{n\geq 0} \frac{\phi_j^{[n]}(z,0)x^n}{n!} \\ = & \sum_{n\geq 0} \frac{(D_y^-)^n \phi_j(z,y)x^n}{n!}|_{y=0} \\ = & e^{xD_y^-}\phi_j(z,y)|_{y=0} \\ = & \phi_j(z,x) \end{aligned} \tag{20}$$

However, when x is fractionally powered, we can no longer Taylor expand $\phi_j(z,x)$, and the definition for both $\phi_j^{[n]}(z,0)$ in eq.(19) and $(D_y^-)^n\phi_j(z,y)|_{y=0}$ in eq.(20) requires specification. It is still possible to generalize eq.(19) and eq.(20) by defining

$$\frac{\phi_j^{[N+\alpha+\beta]}(z,0)}{(N+\alpha+\beta)!} = \sum_{\substack{n+m=N \\ n,m\in\mathbb{Z}}} \frac{(J_0^-)^{\alpha+n}}{(\alpha+n)!} : e^{-j\sqrt{2/t}\varphi(z)} : \frac{(-J_0^-)^{\beta+m}}{(\beta+m)!} \tag{21}$$

Although it looks like that the r.h.s. of eq.(21) depends on both α and β, lemma 1 essentially means that it only depends on the combination $\alpha+\beta$.

The fractional derivatives at the origin may also be considered as analytical continuations of their integral counterparts. Now $\phi_j(z,x) = (1+\gamma(z)x)^{2j} : e^{-j\sqrt{2/t}\varphi(z)} :$, so for non-negative integer n we have

$$\frac{(D_y^-)^n}{\Gamma(n+1)}\phi_j(z,y)|_{y=0} = \left(\begin{array}{c} 2j \\ n \end{array}\right) \gamma^n(z) : e^{-j\sqrt{2/t}\varphi(z)} : \tag{22}$$

We can analytically continue the variable n in the above equation from integers to complex numbers. Therefore n could be any fractional number and we have

$$\frac{(D_y^-)^{n+\alpha}}{\Gamma(n+\alpha+1)}\phi_j(z,y)|_{y=0} = \left(\begin{array}{c} 2j \\ n+\alpha \end{array}\right) \gamma^{n+\alpha}(z) : e^{-j\sqrt{2/t}\varphi(z)} : \tag{23}$$

Proof of Lemma 1:

$$\begin{aligned}
& e^{xJ_0^-} : e^{-j\sqrt{2/t}\varphi(z)} : e^{-xJ_0^-} \\
= & \sum_{n\in\mathbb{Z}} \frac{(xJ_0^-)^{\alpha-\beta+n}}{(\alpha-\beta+n)!} : e^{-j\sqrt{2/t}\varphi(z)} : \sum_{m\in\mathbb{Z}} \frac{(-xJ_0^-)^{\beta+m}}{(\beta+m)!} \\
= & \sum_{n\in\mathbb{Z}} \frac{(xJ_0^-)^{\alpha-\beta+n}}{(\alpha-\beta+n)!} \sum_{m\in\mathbb{Z}} \frac{(-xJ_0^- + xD_y^-)^{\beta+m}}{(\beta+m)!}\phi(z,y)|_{y=0} \\
= & \sum_{N\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} \frac{(xJ_0^-)^{\alpha-\beta+N-m}(-xJ_0^- + xD_y^-)^{\beta+m}}{(\alpha-\beta+N-m)!(\beta+m)!}\phi(z,y)|_{y=0} \\
= & \sum_{N\in\mathbb{Z}} \frac{(xD_y^-)^{\alpha+N}}{(\alpha+N)!}\phi(z,y)|_{y=0} \\
= & \sum_{N\in\mathbb{Z}} \left(\begin{array}{c} 2j \\ N+\alpha \end{array}\right) (\gamma(z)x)^{N+\alpha} : e^{-j\sqrt{2/t}\varphi(z)} : \\
= & \phi_j(z,x)
\end{aligned} \tag{24}$$

This proves lemma 1.

Proof of Lemma 2:

$$\begin{aligned}
e^{xJ_0^-}e^{-xJ_0^-} &= \sum_{n\in\mathbb{Z}} \frac{(xJ_0^-)^{\alpha+n}}{(\alpha+n)!} \sum_{m\in\mathbb{Z}} \frac{(-xJ_0^-)^{-\alpha+m}}{(-\alpha+m)!} \\
&= \sum_{N\in\mathbb{Z}} \sum_{n+m=N} \frac{(xJ_0^-)^N}{(\alpha+n)!(-\alpha+m)!}(-1)^{-\alpha+m} \\
&= \sum_{N\in\mathbb{Z}} (xJ_0^-)^N \delta_{N,0} \\
&= 1
\end{aligned} \tag{25}$$

This proves lemma 2. Thus the manipulations in the proof in eq.(15) are justified.

3 Comparison with standard formulations of hamiltonian reduction

Having proved the equivalence of the two apparently different kinds of correlators, we now want to understand this equivalence from the point of view of quantum hamiltonian reduction. We briefly review the background. Setting the Kac-Moody current

$$J^+(z) = 1 \tag{26}$$

in the the equation of motion derived from the $SL(2)$ WZNW theory, one recovers the classical equation of motion for Liouville theory. In order to implement the constraint, eq.(26), at the quantum level, one introduces a lagrangian multiplier field, $A(z)$, and follows the standard procedure for hamiltonian reduction [2], where $A(z)$ is treated as a gauge field. The final theory, after gauge fixing, involves Faddeev-Popov ghost fields, which are supposed to cancel out unwanted degrees of freedom in the original WZNW theory. The BRST quantization has now become a standard approach to constrained hamiltonian systems. As far as correlation functions on the sphere are concerned, the BRST quantization is equivalent to imposing the constraint eq.(26) on the correlators. Suppose one writes the correlation function on the sphere as an operator insertion

$$\langle 0|\hat{O}|0\rangle \tag{27}$$

then for the constrained system satisfying eq.(26), we have

$$\langle 0|\hat{O}(J^+(z) - 1)|0\rangle = 0 \tag{28}$$

Eq.(28) is equivalent to the following condition

$$\begin{aligned} J^+_n|0\rangle &= 0 \qquad n \geq 0 \\ \langle 0|\hat{O}(J^+_{-n} - \delta_{n,1}) &= 0 \qquad n \geq 1 \end{aligned} \tag{29}$$

In order not to confuse the notations in this paper, $J^+(z)$ is always considered to be a conformal spin 1 field to fit the WZNW theory, so that one has the expansion

$$J^+(z) = \sum_{n\in\mathbb{Z}} J^+_n z^{-n-1} \tag{30}$$

As usual, to fix $J^+(z)$ to be a constant value would require $J^+(z)$ be a scalar field. In other words, the energy momentum tensor must be improved from the Sugawara construction by adding a term $\partial_z J^3(z)$. In that context, one should rename $J^+_n \to J^+_{n+1}$.

Eq.(29) is called the physical state condition. In BRST quantization the physical state space is the same as the BRST cohomology space $Ker(Q)/Im(Q)$, where Q is the BRST charge defined by

$$Q = \oint \frac{dw}{2\pi i}(J^+(w) - 1)c(w) \tag{31}$$

Here $c(w)$ is a conformal spin 1 fermionic ghost field with respect to the improved energy momentum tensor. Its conjugate field $b(w)$ is the antighost field of spin 0 satisfying

$$b(w)c(z) = \frac{1}{w - z} \tag{32}$$

Eq.(29) is equivalent to the BRST condition, in which one requires that the vacuum states $\langle 0|$ and $|0\rangle$ be physical states, and $\hat{O}$ be a physical operator which maps physical states into physical states. In other words

$$\langle 0|Q = [Q, \hat{O}] = Q|0\rangle = 0 \tag{33}$$

Now consider the most general form for a class of conformal blocks in $SL(2)$ WZNW theory, which are proportional to those in the Virasoro minimal models. They can be written in the following form

$$\begin{aligned} &\langle j_{r_N,s_N}|F(J_1^-)R\phi_{j_{r_{N-1},s_{N-1}}}(z_{N-1},0)...\phi_{j_{r_2,s_2}}(z_2,0)\prod_i \oint \frac{dw_i}{2\pi i} S_{k_i}(w_i)|j_{r_1,s_1}\rangle \\ = \quad & C\langle h_{r_N,s_N+1}|R\phi_{r_{N-1},s_{N-1}+1}(z_{N-1})...\phi_{r_2,s_2}(z_2)\prod_i \oint \frac{dw_i}{2\pi i} V_{\alpha_i}(w_i)|h_{r_1,s_1+1}\rangle \end{aligned} \tag{34}$$

where the normalization constant C is found to be

$$\begin{aligned} C &= \frac{\Gamma(k-2j_N+1)}{\Gamma(k-2j_N+\sum_i k_i+1)}(\partial_y)^{-\sum_i k_i}F(y)|_{y=0} \\ &= \frac{\Gamma(k-2j_N+1)}{\Gamma(k-\sum_{i=1}^N j_i+1)}(\partial_y)^{-2j_N+\sum_{i=1}^N j_i}F(y)|_{y=0} \end{aligned} \tag{35}$$

In general C depends on t and the j_i's. If for some values of t and j_i's, C vanishes, then the correlation functions in the Virasoro minimal models can only be obtained by dividing out C. Strictly speaking, simply taking the limit $x_i \to z_i$ is not equivalent to quantum hamiltonian reduction eq.(26). Rather it is in accord with the constraint

$$J^+(w) = J^+_{-1} \tag{36}$$

To go to the minimal model we must further impose the condition

$$J^+_{-1} = 1 \tag{37}$$

To see this, let us consider the BRST charge for quantum hamiltonian reduction eq.(36)

$$\tilde{Q} = \oint \frac{dw}{2\pi i}(J^+(w) - J^+_{-1})c(w) \tag{38}$$

The physical state space now becomes the BRST cohomology space $Ker(\tilde{Q})/Im(\tilde{Q})$. It is clear that $\phi_j(z,0)$ commutes with $\tilde{Q}$, hence maps a physical state into another physical state. Now consider the ket and the bra states. Notice that the ket state $|j_1\rangle$ is a highest weight state and the bra state $\langle j_{r_N,s_N}|$ is a lowest weight state

$$J^+_n|j_1\rangle = \langle j_{r_N,s_N}|J^+_{-n-1} = 0, \qquad n \geq 0 \tag{39}$$

For the b, c ghost fields, we have the following condition

$$c_n|j_1\rangle = b_{n+1}|j_1\rangle = \langle j_{r_N,s_N}|c_{-n-1} = \langle j_{r_N,s_N}|b_{-n} = 0, \qquad n \geq 0 \tag{40}$$

It can be verified that with respect to the BRST charge $\tilde{Q}$ in eq.(38), $|j_1\rangle$ is a physical state, and the bra state $\langle j_{r_N,s_N}|F(J_1^-)$ is a physical state for any arbitrary function $F(J_1^-)$. However, this extra degree of freedom is removed if we further impose the condition eq.(37), which would fix the function $F(J_1^-)$ uniquely, and we recover exactly the conformal blocks in the Virasoro minimal models

$$\langle j_{r_N,s_N}|F(J_1^-) = \langle j_{r_N,s_N}|e^{-\gamma_1} = \langle j_{r_N,s_N}| \sum_{n\in\mathbb{Z}} \frac{\Gamma(k-2j_N-n-\alpha+1)}{\Gamma(k-2j_N+1)\Gamma(n+\alpha+1)}(J_1^-)^{n+\alpha} \quad (41)$$

where γ_n is conjugate to J_{-n}^+

$$[J_{-n}^+, \gamma_m] = \delta_{n,m} \quad (42)$$

If we were to use $\phi_j(z,z)$ to represent a primary field in the hamiltonian reduced system (strictly speaking, $\phi_j(z,z)$ does not transform as a primary field for the Virasoro algebra in the reduced system), then we should normalize the correlation function by dividing out the normalization constant C. Then in the limit C goes to zero, the conformal block in the reduced system would remain finite.

In conclusion, the constraint $J^+(z) = 1$ completely freezes the degrees of freedom of the $J^+(z)$ field. However we could proceed in two steps in putting the constraint on the correlation functions. First set $J^+(z) = J_{-1}^+$ and then let $J_{-1}^+ = 1$. The first step would result in a class of correlation functions which are proportional to that of the completely constrained system, like the ones considered in the previous section. However, the remaining degrees of freedom of the J_{-1}^+ mode is reflected by the arbitrariness of the proportionality. If we normalize the correlation function by dividing out the normalization constant, which is equivalent to setting $J_{-1}^+ = 1$, then we recover the corresponding correlators in the completely reduced system.

4 Analysis of the explicit correlators

Finally we want to verify explicitly that the conformal blocks for the WZNW model evaluated in ref.[7] satisfy the above result. To this end we consider the interpolating correlator

$$\langle j_N|e^{-J_1^- x_N} \prod_{\ell=2}^{N-1} \phi_{j_{N-1}}(z_{N-1},x_{N-1})...\phi_{j_2}(z_2,x_2) \prod_{i=1}^{M} \oint \frac{dw_i}{2\pi i} S_{k_i}(w_i)|j_1\rangle \equiv \langle j_N|\mathcal{O}|j_1\rangle \quad (43)$$

with

$$\begin{aligned} x_\ell &= z_\ell x, \quad \ell = 1,...,N-1 \\ x_N &= x-1 \end{aligned} \quad (44)$$

Thus for $x=1$ we get the WZNW model with all x_i's put equal to the z_i's. For $x=0$ we should get the minimal model correlator. We wish to show that when this interpolating correlator is evaluated according to ref.[7] then indeed it is independent of x. Using ref.[7] we find

$$\langle\mathcal{O}\rangle = W_B W_N^{\varphi} F \quad (45)$$

where

$$\begin{aligned}
B(w) &= \sum_{\ell=1}^{N-1} \frac{x_\ell/u_\ell}{w-z_\ell} - x_N/u_N \\
W_B &= \prod_{i=1}^{M} B(w_i)^{-k_i} \\
F &= \Gamma(k-2j_N+1)u_N^{k-2j_N-1}e^{1/u_N}\prod_{\ell=2}^{N-1}\Gamma(2j_\ell+1)u_\ell^{2j_\ell-1}e^{1/u_\ell} \\
W_N^\varphi &= \prod_{m<n}(z_m-z_n)^{2j_mj_n/t}\prod_{i=1}^{M}\prod_{m=1}^{N-1}(w_i-z_m)^{2k_ij_m/t}\prod_{i<j<M}(w_i-w_j)^{2k_ik_j/t} \qquad (46)
\end{aligned}$$

Here we used that

$$\begin{aligned}
\langle j_N|e^{-x_NJ_1^-} &= \langle j_N|(1+x_N\gamma_1)^{k-2j_N} \\
&= \lim_{z'\to\infty}\langle j_N|(1+x_N\gamma(z'))^{k-2j_N} \qquad (47)
\end{aligned}$$

Consider now

$$\begin{aligned}
G^-(w) &= \langle J^-(w)\mathcal{O}\rangle \\
&= \Bigg\{-\sum_{i,j}B(w)\frac{D_{B_i}D_{B_j}}{(w-w_i)(w-w_j)} + (t-2)\sum_i\frac{D_{B_i}}{(w-w_i)^2} \\
&- 2\sum_{i,j}\frac{k_iD_{B_j}}{(w-w_i)(w-w_j)} - 2\sum_{m,j}\frac{j_mD_{B_j}}{(w-z_m)(w-w_j)}\Bigg\}W_BW_N^\varphi F \qquad (48)
\end{aligned}$$

This function has simple poles as a function of w. It is a rather simple matter to evaluate the pole residues along the lines described in ref.[7]. The result is

$$\begin{aligned}
\oint_{z_m}\frac{dw}{2\pi i}G^-(w)w &= z_m\partial_{x_m}W_BFW_N^\varphi \\
\oint_{\infty}\frac{dw}{2\pi i}G^-(w)w &= \partial_{x_N}W_BFW_N^\varphi \\
\oint_{w_i}\frac{dw}{2\pi i}G^-(w)w &= t\partial_{w_j}\left(w_j\frac{W_N^\varphi W_BF}{B(w_j)}\right) \qquad (49)
\end{aligned}$$

After integration over the w_i's we see that we precisely produce the total derivative of the original correlator with respect to x:

$$\sum_m z_m\partial_{x_m}W_B + \partial_{x_N}W_B = \partial_xW_B \qquad (50)$$

This expression will vanish since this merely is the condition that the sum of pole residues vanishes (when the pole at infinity is included as it is here).

Acknowledgement

We are grateful to A.Ch. Ganchev and V.B. Petkova for explaining several details of the work in refs.[3, 9] to us, and for other illuminating remarks, and we are grateful to O. Andreev for having pointed out his work, [6], to us. M.Y. thanks the Danish research Academy for financial support.

References

[1] A.M. Polyakov, in *Physics and Mathematics of Strings,* Eds. L. Brink, D. Friedan and A.M. Polyakov (World Scientific, 1990)

[2] M. Bershadsky and H. Ooguri, Commun. Math. Phys. **126** (1989) 49

[3] P. Furlan, A.Ch. Ganchev, R. Paunov and V.B. Petkova, Phys. Lett. **B267** (1991) 63;
P. Furlan, A.Ch. Ganchev, R. Paunov and V.B. Petkova, Nucl. Phys. **B394** (1993) 665;
A.Ch. Ganchev and V.B. Petkova, Phys. Lett. **B293** (1992) 56

[4] V.A. Fateev and A.B. Zamolodchikov, Sov. J. Nucl. Phys. **43** (1986) 657

[5] H. Awata, A. Tsuchiya and Y. Yamada, Nucl. Phys. **B365** (1991) 680;
H. Awata, KEK-TH-310/KEK Preprint 91-189, preprint

[6] O. Andreev, HUB-IEB-94/9 hep-th/9407180 preprint, and Landau-95-TMP-1/hep-th/9504082, preprint

[7] J.L. Petersen, J. Rasmussen and M. Yu, NBI-HE-95-16/hep-th/9504127, preprint

[8] V. Knizhnik and A. Zamolodchikov, Nucl. Phys. **B247** (1984) 83

[9] A.Ch. Ganchev and V.B. Petkova, Phys. Lett. **B318** (1993) 74

[10] M. Wakimoto, Commun. Math. Phys. **104** (1986) 605

[11] G. Felder, Nucl. Phys. **B317** (1989) 215 [Erratum: **B324** (1989) 548]

[12] D. Bernard and G. Felder, Commun. Math. Phys. **127** (1990) 145