hep-th/9506178
T95/086

THRESHOLD CORRECTIONS IN ORBIFOLD MODELS AND SUPERSTRING UNIFICATION OF GAUGE INTERACTIONS

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The string one loop renormalization of the gauge coupling constants is examined in abelian orbifold models. The contributions to string threshold corrections independent of the compactification moduli fields are evaluated numerically for several representative examples of orbifold models. We consider cases with standard and non-standard embeddings as well as cases with discrete Wilson lines background fields which match reasonably well with low energy phenomenology. The gap separating the observed grand unified theories scale $M_{GUT} \simeq 2 \times 10^{16} GeV$ from the string unification scale $M_X \simeq 5 \times 10^{17} GeV$ is discussed on the basis of standard-like orbifold models. We examine one loop gauge coupling constants unification in a description incorporating the combined effects of moduli dependent and independent threshold corrections, an adjustable Kac-Moody level for the hypercharge group factor and a large mass threshold associated with an anomalous $U(1)$ mechanism.

PACS numbers: 11.10.Hi, 11.25.Mj, 12.10.Kt

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1. INTRODUCTION

In superstring as in grand unification, the high energy extrapolation of the standard model renormalized gauge coupling constants is described by a one loop scale evolution of familiar form:

$$\frac{(4\pi)^2}{g_a^2(\mu)} = \frac{(4\pi)^2 k_a}{g_X^2} + 2b_a \log\frac{\mu}{M_X} + \tilde{\Delta}_a(M_i, \overline{M}_i). \quad (1)$$

(The index $a = 3, 2, 1$ labels the $SU(3) \times SU(2) \times U(1)$ group factors G_a and b_a are the beta function slope parameters associated with the low energy modes, $\beta_a(g) = -b_a g_a^3/(4\pi)^2 + \cdots$) The superstring case is, however, distinguished by three important features [1]:

(i) Tree level relations [2] involving the gauge and gravitational interactions,

$$g_X^2 = k_a g_a^2 = \frac{4\kappa^2}{\alpha'} = \frac{32\pi}{\alpha' M_P^2}. \quad (2)$$

In addition to the string theory expansion parameter g_X (or 4-d dilaton VEV $< S >= 1/g_X^2$) which is specified by the ratio of the string mass scale $M_S = \frac{2}{\sqrt{\alpha'}}$ to the phenomenological Planck mass $M_P = \frac{\sqrt{8\pi}}{\kappa} = 1.22 \times 10^{19} GeV$, as exhibited in eq.(2), three extra free (positive rational numbers) parameters k_a are introduced in eq.(2), corresponding to the levels of the Kac-Moody algebras for the gauge group factors G_a in the underlying string theory.

(ii) An improved unification scale M_X defined in eq.(1) as the matching scale between the field and string theories renormalized coupling constants at which these obey most closely the tree level relations, eq.(2). For the field theory coupling constants in the $\overline{DR}$ regularization scheme,

$$M_X = \frac{e^{(1-\gamma)/2}}{4\pi\sqrt[4]{27}} g_X M_P = \frac{e^{(1-\gamma)/2}}{\sqrt{2\pi}\sqrt[4]{27}} M_S \simeq g_X 5.27 \times 10^{17} GeV. \quad (3)$$

The field theory (ft) convention in use here is related to the string theory (st) one as, $g_a^{ft} = \sqrt{2} g_a^{st}$, corresponding to the normalization of the Lie algebra generators, $Tr_R(Q_a^2) = \frac{1}{2}c(R)$, where $c(R) = l(R)$ is the Dynkin index of representation R.

(iii) Threshold corrections accounting for the contributions of the infinite set of massive string states at the string (M_S) and compactification (M_C) scales, integrated out by matching the field and string theories scattering amplitudes. These corrections are represented in eq.(1) by the functions $\tilde{\Delta}_a(M_i, \overline{M}_i)$ depending upon the structure of the string mass spectrum and the other characteristic parameters of the compactified space manifold, such as the vacuum expectation values (VEVs) of the compactification moduli fields, $M_i = T_i, U_i$ [3]. Specifically, M_X is defined as the choice of scale which minimizes the threshold corrections contributions. Of course, the perturbative character of formula (1) implies that the size of $\tilde{\Delta}_a$ should be comparable to that of two-loop effects, so that $\tilde{\Delta}_a = O(1)$.

For a quantitative test of superstring unification based on eq.(1) and for a proper identification of the fundamental parameters, M_X and g_X, it is essential to understand the structure and size of threshold corrections. Thus, an additive decomposition such as $\tilde{\Delta}_a = k_a Y - b_a \Delta$, may be exploited to introduce effective unification scale and coupling constant,

$$M_X \rightarrow M'_X = M_X e^{\Delta/2}, \quad g_X \rightarrow g'_X = \frac{g_X}{(1 + \frac{Y g_X^2}{(4\pi)^2})^{\frac{1}{2}}}, \tag{4}$$

so defined as to incorporate the contributions from the above two components,

The toroidal compactification orbifold models prove very helpful in obtaining an information on $\tilde{\Delta}_a$. The contributions from compactification modes admit here a natural additive decomposition into a moduli dependent component arising from the chiral mass F-terms and a moduli independent component arising from the vector mass D-terms [4]. As is well known, the moduli dependent contributions play an essential rôle in the cancellation of sigma model anomalies affecting the target space duality symmetry [5]. These can be represented by general formulas involving the automorphic functions of the compactication manifold accompanied by model dependent coefficients. On the other hand, the moduli independent contributions carry only an implicit dependence on the compactification manifold, such as the orbifold gauge embedding or the discrete Wilson lines. In spite of several attempts in the literature to estimate the size of both components of threshold corrections [1,6-9], one is still lacking a clear physical understanding of their magnitude. Our main goal in this paper is to present results for the moduli independent threshold corrections through an extensive numerical study based on a sample of orbifold models.

The main physical motivation for this paper is, however, the wide gap that separates the improved string unification scale $M_X = 0.216 M_S \simeq 5 \times 10^{17} GeV$, assuming $g_X = O(1)$, from the observed grand unification scale, $M_{GUT} \simeq 2 \times 10^{16} GeV$, as determined by extrapolating the gauge coupling constants up from their experimentally determined values at the $Z-$boson mass [10]. The implications of this order of magnitude discrepancy in scales have been emphasized on several occasions [11]. The conflict for superstring unification can be resolved in two different ways: One can postulate [12] large string threshold corrections such that after becoming equal and joining together at the observed scale M_{GUT} the gauge coupling constants follow diverging flows up to M_X. A matching of the one loop extrapolated values of $g_a(M_X)$ with their predicted values, as obtained by adjusting the moduli dependent threshold corrections, can be successfully achieved in terms of wide classes of solutions for the modular weights of massless modes consistent with the anomaly cancellation constraints [5,12]. Alternatively, one can postulate [13] a Kac-Moody level parameter for the weak hypercharge group $U(1)_Y$ somewhat lower than the standard grand unification group value, $k_1 = \frac{5}{3}$. With such an enhanced starting value for $(k_1 \alpha_1^2(m_Z))^{-1}$ one achieves a delayed joining of the gauge coupling constants flows which can easily raise up the unification scale by one order of magnitude. While either of these possibilities is well motivated by itself and appears sufficient to rescue a superstring grand desert scenario, there remains certain unsatisfactory points. Thus, the VEVs of moduli fields requested in the first possibility, $< T >= 10 - 30$, appear to be somewhat too large, whereas no known realistic orbifold example [14] exists for which the hypercharge group level parameter comes as low as the value $k_1 \simeq 1.4$ favored in the second possibility.

A generic feature of standard-like orbifold models is the occurrence of a rich spectrum of charged massless modes appearing on side of the requested (quarks and leptons) chiral families in vector representations of the color and weak groups. In fact, the matter representations of the observable sectors group factors are generally sizeable enough so that the corresponding beta function parameters β_a arise with either small negative values or large positive values. This suggests that a first stage of slow or non asymptotically free scale evolution may well take place from M_X down to some scale where the extra modes pair up by acquiring mass and decouple. As is well-known [15], in order for the 4-d low energy effective theory to be weakly coupled, so as not to invalidate the use of eq.(1) ($g_X \approx g_{[d=10]} M_C^3 < 1$), and in order to avoid dealing with a strongly coupled 10-d theory ($g_d M_S^3 < 1$) one must require that the compactification and unification scales retain a magnitude comparable to the string scale, $M_X \simeq M_C \simeq M_S$. (The second restriction can be relaxed by allowing, for instance, for an anisotropic compactification manifold (large radius in one out of the six compactified dimensions) in which a weakly coupled effective theory, $g_X < O(1)$, could remain compatible with a strongly coupled string theory (large g_D) [16].) Assuming the above near equality of scales, a natural identification for the decoupling scale of the extra matter is the mass scale, denoted M_A, which is induced by a non-vanishing Fayet-Iliopoulos D-term contribution to some apparently anomalous $U(1)$ group factor occurring on compactification [17]. This suggestion is not new, of course, and appears in several places in the specialized literature. The idea is to cancel the non vanishing one loop string contributions to the D-term scalar potential of an apparently anomalous $U(1)$ factor by judiciously lifting the VEVs of certain scalar fields while restoring a stable supersymmetric vacuum. We shall carry out an analysis of the one loop gauge coupling constants unification which combines together the above ideas of adjustable moduli VEVs and k_1 level parameters together with that of an adjustable intermediate scale M_A, while describing the scale evolution in the interval from M_X to M_A on the basis of orbifold models predictions.

The paper contains 5 sections. In Section 2, we discuss in wide outline the basic formalism involved in the one loop string renormalization of the gauge coupling constants as applied to orbifold models. In Section 3, we present numerical results for the moduli independent threshold corrections for a sample of representative orbifold models. In Section 4, we examine the viability of superstring unification in an extended picture including threshold corrections and an intermediate scale associated with an anomalous $U(1)$ symmetry. In Section 5, we summarize the main conclusions.

2. ONE LOOP STRING RENORMALIZATION

2.1 Threshold corrections to gauge coupling constants

We consider the class of low energy supersymmetric theories descending from 4-d heterotic string theories with a nonsemi-simple gauge group $\prod_a G_a$. The genus zero (unity) world sheet (with Wick-rotated Euclidean metric) of the conformal field theory is a sphere (torus) parametrized by planar coordinates: $\bar{z} = e^{-2\pi i \bar{\zeta}}, z = e^{2\pi i \zeta}$, with corresponding cylindrical coordinates given for the sphere by: $\bar{\zeta} = \sigma - it, \quad \zeta = \sigma + it, \quad \sigma \in [0, 1], \quad t \in [-\infty, \infty]$ and for the torus by: $\zeta = \sigma + \tau t, \quad \bar{\zeta} = \sigma + \bar{\tau} t, \quad \sigma, t \in [0, 1], \tau = \tau_1 + i\tau_2$. The right-moving RNS (Ramond-Neveu-Schwarz) superstring is built with 20 spacetime and spin fields $X^\mu(\bar{z}), \psi^\mu(\bar{z}), \quad [\mu = 0, \cdots, 9]$, associated with $D = 4$ external dimensions of the flat spacetime $[\mu = 0, \cdots, 3]$ and $d - 10 = 10 - D = 6$ internal dimensions $[\mu = 4, \cdots 9]$ of the compactification space manifold, represented in a complex basis as, $X_R^i, X_R^{\bar i}, \quad \psi^i = e^{i\phi_i}, \psi^{\bar i} = e^{-i\phi_i}, [i = 1,2,3]$ where the complex scalar fields $\phi_i(z)$ are coordinates of the $SO(6)$ group Cartan torus. This is tensored by a left-moving bosonic string built with 26 fields $X^\mu(z), [\mu = 0, ..., 25]$, comprising D external space coordinates and $26 - D$ internal space coordinates which are distributed into 6 compactified space coordinates $X_L^i, X_L^{\bar i}$ and 16 gauge coordinates of the $E_8 \times E_8'$ Cartan torus $F^I, F'^I [I = 1, ..., 8]$ generating the currents $J_a(z)$ of the Kac-Moody algebras G_a of levels k_a. At certain places, we refer to these coordinates globally as $F^I, [I = 1, \cdots, 16]$ and also by using their fermionic representation in terms of complex 2-d Weyl spinors, $(\lambda^\alpha, \lambda^{\bar\alpha}) = e^{\pm i F^I}, [I = 1, \cdots, 16; \quad \alpha = 1, \cdots 8]$. Of course, the above covariantly quantized string theory must be supplemented with the conformal ghost fields $c^z(z, \bar z), b_{zz}(z, \bar z)$ and the superconformal ghost fields, $\gamma(\bar z), \beta_z(\bar z)$ [18].

The one loop string threshold corrections are described by a general formula obtained by Kaplunovsky [1],

$$\tilde\Delta_a \equiv k_a Y_0 + \Delta_a, \qquad \Delta_a = -\int_F \frac{d^2\tau}{\tau_2}\Big(k_a B_a(q,\bar q) - b_a\Big), \tag{5}$$

where one has decomposed the total contribution, denoted $\tilde\Delta_a$, into a universal contribution, $k_a Y_0$, independent of the gauge group factor (except for the coefficient k_a), arising from gravitational interactions and oscillator excitations modes, and a contribution solely due to the massive compactification modes, denoted Δ_a. The latter component is expressed as a deformed partition function integrated over the inequivalent complex structures of the genus 1 world sheet,

$$\begin{aligned} B_a(q,\bar q) = -\frac{1}{2}\sum_{even \quad \bar\alpha,\bar\beta} \Bigg[(-1)^{2\bar\alpha+2\bar\beta}\frac{1}{\eta(\tau)^2\eta(\bar\tau)^2} 2\bar q \frac{d}{d\bar q}\Bigg(\frac{r\vartheta\begin{bmatrix}\bar\alpha\\ \bar\beta\end{bmatrix}(\bar\tau)}{\bar\eta(\bar\tau)}\Bigg)\Bigg] \\ \times 2Trace\Bigg((-1)^{2\bar\beta F} Q_a^2 q^{L_0 - \frac{22}{24}} \bar q^{\bar L_0 - \frac{9}{24}}\Bigg), \end{aligned} \tag{6}$$

where the first factor represents the partition function of the external theory inserted with the operator $(-\frac{1}{12} + \chi^2)$, where χ denotes the 4-d helicity or chirality vertex operator and we have introduced the familiar Dedekind function, $\eta(\tau) = q^{\frac{1}{24}}\prod_n (1 - q^n)$, and the Jacobi theta-functions (cf. eq.(10) below)..

The second factor in eq.(6) (with $F =$ fermion number operator, $L_0, \bar L_0 =$ conformal dimensions operators) corresponds to the internal theory partition function inserted with the square Q_a^2 of any one of the gauge group generators for subgroup G_a. The integral over the world sheet torus complex modular parameter, $\tau = \tau_1 + i\tau_2$, with $q = e^{2\pi i\tau}, \bar q = e^{-2\pi i\bar\tau}$, extends over the modular group $SL(2,Z)$ fundamental domain, $F = [|\tau_1| \leq \frac{1}{2}, |\tau| \geq 1]$. Infrared convergence of the integral, eq.(5), is ensured by the subtraction of $b_a = \lim_{\tau_2\to\infty} k_a B_a$, where $b_a = \frac{1}{6}\sum_\alpha [-c_S(R_\alpha) - 2c_F(R_\alpha) + 11 c_V(R_\alpha)]$ ($S =$ complex scalar, $F =$ Weyl or Majorana fermion, $V =$ vector) represent the summed contributions to the beta function slope parameters from the massless string modes α.

The summation in eq.(6) over the subset of even spin structures of the right-moving sector, $(\bar\alpha, \bar\beta) = [(0,0), (0,\frac{1}{2}), (\frac{1}{2}, 0)]$ where $\bar\alpha, \bar\beta = 0 = NS(A)$ (Neveu-Schwarz, Antiperiodic) or $\frac{1}{2} = R(P)$ (Ramond, Periodic) is performed by insertion of the familiar GSO (Gliozzi-Scherk-Olive) projection phase factors leading to the supersymmetric string [18].

2.2 Specialization to orbifolds

To express the second internal space factor in eq.(6) for orbifolds, we recall first that the projection (modding) with respect to the orbifold point symmetry is achieved by summing over the (space and time) twisted subsectors (g,h) by using [19,20],

$$Trace(\cdots)=\frac{1}{|G|}\sum_{g}\sum_{h;[g,h]=0}\chi(g,h)Trace_{g}(h\cdots),$$

where $|G|$ is the orbifold point group order and $\chi(g,h)$ are degeneracy factors. For toroidal compactication, all fields are free so that the torus partition function is obtained by associating to a coordinate field of given chirality, a factor $1/(\pi\sqrt{2\tau_2})$ (flat case) or $(1-e^{2\pi iv})/\eta(\tau)$ (untwisted case with time twist $X(\sigma,t+1)=e^{2\pi iv}X(\sigma,t)$) or $\eta(\tau)/\vartheta[{\frac{1}{2}+v \atop \frac{1}{2}+v}]$ (space twisted case $X(\sigma+1,t)=e^{2\pi iv}X(\sigma,t)$) and to fermionic Majorana-Weyl fields, obeying the twisted boundary conditions:

$$\psi(\sigma+1,t)=-e^{2\pi i\theta'}\psi(\sigma,t),\quad \psi(\sigma,t+1)=-e^{-2\pi i\phi'}\psi(\sigma,t),$$

a factor $[\vartheta[{\theta' \atop \phi'}]/\eta(\tau)]^{\frac{1}{2}}$. The zero modes are associated a factor $q^{p_L^2/2}\bar{q}^{p_R^2/2}$ summed over the winding modes spanning the compactification manifold lattice Λ_6 with basis vectors e_a^i and on the Kaluza-Klein momentum modes spanning its dual lattice $\Lambda^\star$ with basis vectors $e_i^{\star a}$ (cf. eq.(11) below)).

We recall next that a torus R^6/Λ^6, defined by $X^i\equiv X^i+2\pi n^a e_a^i$, having a point symmetry group, $P=Z_N$, of automorphisms of the lattice Λ^6, defines an abelian orbifold endowed with a space symmetry group, $G=P\times\Lambda^6$. The space group action on the string theory fields is described in terms of rotations θ^k and translations $u_{k,f}$ together with their associated gauge group shift embedding elements described by translations V^I and Wilson lines translations $a_a^I,[I=1,\cdots,16;a=1,2,3]$. The space group $G=\{g_k\}=\{\beta_k,w_k\}$ composition laws read: $g_1g_2=(\beta_1\beta_2,\beta_1w_2+w_1),\quad g^{-1}=(\beta^{-1},-\beta^{-1}w)$.

The string Hilbert space of states consists of the untwisted sector ($k=0$) and the twisted ($k=1,\cdots,N-1$) sectors. The twisted sectors g^k are distinguished by the boundary conditions: $(X(\sigma+1,t),\psi(\sigma+1,t))=g^k(X(\sigma,t),-(-1)^{2\bar{\alpha}}\psi(\sigma,t))$. They are organized into conjugacy classes of the space group with representative elements, $g^k=[\theta^k,u_{k,f}]$ and associated classes, $\{g^k\simeq g'g^kg'^{-1}=(\theta^k,u_k),g'=g^p\in Z_N\}$, where the set of shift vectors $u_k=(\theta^p u_{k,f}+(1-\theta^k)u),\quad [u\in\Lambda_6,p=0,\cdots N-1]$, span lattice cosets (labelled by the index f) with representative elements, $u_{k,f}$. The compactified space coordinates, $X^i=X_L^i+X_R^i=x^i+i\pi tp^i+2\pi\sigma w^i+\cdots$ (units $2\alpha'=1$), admit the (zero and oscillators) modes expansion,

$$(X_L^i(z),X_R^i(\bar{z}))=\frac{x^i}{2}-\frac{i}{2}(p_L^i\ln z,p_R^i\ln\bar{z})+\frac{i}{2}\sum_{m_i}(\frac{\alpha_{m_i}^{Li}}{m_i}z^{-m_i},\frac{\alpha_{m_i}^{Ri}}{m_i}\bar{z}^{-m_i}).$$

In twisted sectors, the string center of mass coordinates x^i are not arbitrary real parameters but rather must satisfy: $g^kx=x+\hat{u}_{k,f}+u,\quad [\hat{u}_{k,f},u\in\Lambda_6]$. Therefore, each of the g^k

twisted sectors splits into subsets which can be classified in terms of the corresponding set of fixed points of the space group, $f^{(k)i}$, defined as: $\theta^k f^{(k)} = f^{(k)} + \hat{u}_{k,f}$ where $\hat{u}^i_{k,f} = m^a_{k,f} e^i_a, [m^a =$ integers] are translation vectors of the 6-d toroidal lattice Λ_6 determined by the condition that they return the rotated fixed point $\theta^k f$ back to its original position, so that $f = (1-\theta^k)^{-1}\hat{u}_{k,f} + u$. Specifically, the k-twisted sector fixed points $f^{(k)}_\alpha$ are distinguished by a label, α running over the number of fixed points. The lattice vectors $\hat{u}_{k,f}$ identify with the lattice coset representatives $u_{k,f}$ introduced above only for prime orbifolds. For simply twisted sectors, $k=1$ or $k = N-1 = -1(modN)$, the fixed points $f^{(k)}$ and conjugacy classes $u_{k,f}$ are in one to one correspondence so that $f^{(k)}$ faithfully label these classes and $\hat{u}_{k,f} = u_{k,f}$. This property holds for all the twisted sectors in the prime orbifolds, $Z_{3,7}$. For the multiply twisted sectors, the full set of fixed points $f^{(k)}_\alpha$ decomposes into disjoint subsets $\{f^{(k)}_A, f^{'(k)}_A, \cdots\}$, where the fixed points within each subset (labelled by A) are related as, $\theta^{p_A} f^{(k)}_A = f^{'(k)}_A \neq f^{(k)}_A$ for $p_A < k$, and hence are in one to one correspondence with the same conjugacy classes, $u_{k,f}$. The non trivial subsets $[f^{(k)}_A]$ arise only for the non prime ($N=1$) orbifolds $Z_{4,6,8,12}$ and the direct product orbifolds $Z_N \times Z_M$.

The orbifold space group elements can now be expressed as ,

$$g^k = \{\theta^k, u_{k,f} = m^a_{k,f} e_a; \quad k\tilde{V}^I = kV^I + m^a_{k,f} a^i_a\},$$

$$\theta^k = diag(\theta^k_i) = diag(e^{2\pi i k v_i}). \Big[\sum_i v_i = 0\Big] \tag{7}$$

The orbifold group action on fields and state vectors reads in obvious notations:

$$g^k X^i_{L,R} = \theta^k_i X^i_{L,R} + m^a_{k,f} e^i_a, \quad g^k F^I = F^I + kV^I + m^a_{k,f} a^I_a, \quad g^k \psi_i = \theta^k_i \psi_i, \tag{8}$$

$$g^h [(\alpha^i_{-n_i})^{p_i} [(\alpha^{\bar{j}}_{-m_j})^{q_j}]_{\binom{L}{R}} |p_R, r^i \equiv \alpha^i + kv^i >_R |p_L, P^I \equiv W^I + k\tilde{V}^I >_L$$

$$= e^{2\pi i k h(v\cdot r + \tilde{V}\cdot P) \mp 2\pi i h(n_i+m_j)} [(\alpha^i)^{p_i}_{-n_i} (\alpha^{\bar{j}})^{q_j}_{-m_j}]_{\binom{L}{R}} |p_R, r^i >_R |p_L, P^I >_L . \tag{8'}$$

The above used correspondence between Wilson lines translation vectors and the non contractible loops, $u_{k,f}$, refers to abelian orbifolds. Non abelian orbifolds with shift gauge embeddings can be constructed by extending the definition of Wilson lines to class dependent shift vectors, $k\tilde{V}^I \to V^I_{k,f}$ derived from a gauge embedding matrix of general form [21].

The internal space oscillator operators, $(\alpha^i_{n_i}, \alpha^{\bar{j}}_{m_j})_{\binom{L}{R}}$, where $i, \bar{j}$ are complex conjugate bases indices, (real basis, $\mu = (1+i2)/\sqrt{2}, (1-i2)/\sqrt{2}, \cdots$), enter with the moddings, $n_i \in Z \mp \theta_i, \quad m_j \in Z \pm \theta_j$, where Z designates the set of integers. The translation vectors $\alpha^i = n_i, (n_i + \frac{1}{2}), \quad [n_i \in Z, \sum_i n_i \in 2Z+1$(odd integers)] are elements of the $SO(6)$ group weight lattice Γ_6 and $W^I = n^I, (n^I + \frac{1}{2}), \quad [n^I \in Z, \sum_{I=1}^8 n^I \in 2Z$ (even integers)] are elements of the $E_8 \times E'_8$ group weight lattice, Γ_{8+8}. The translation vectors v^i and V^I, a^I_a with respect to these lattices must obey: $Nv^i \in \Gamma_6, NV^I \in \Gamma_{8+8}, Nm^a a^I_a \in \Gamma_{8+8}$ as well as the level matching (modular invariance under T^N) conditions $N[(kV^I+m^a_{k,f}a^I_a)^2-(kv^i)^2]\in 2Z$.

With the above rules in hand, we can now quote the following more explicit formula derived from eq.(6):

$$B_a(q,\bar q)=-2\frac{1}{|G|}\sum_{m,n}\chi(m,n)\epsilon(m,n)\frac{1}{2}\sum_{even\,\bar\alpha,\bar\beta}\Big[(-1)^{2\bar\alpha+2\bar\beta}\frac{1}{\eta^2(\tau)\eta^2(\bar\tau)}2\bar q\frac{d}{d\bar q}\Big(\frac{\vartheta\begin{bmatrix}\bar\alpha\\ \bar\beta\end{bmatrix}(\bar\tau)}{\bar\eta(\bar\tau)}\Big)\Big]$$
$$\times\prod_{i=1,3}\Big[\frac{\vartheta\begin{bmatrix}\bar\alpha+mv_i\\ \bar\beta+nv_i\end{bmatrix}(\bar\tau)}{\bar\eta(\bar\tau)}\Big]\prod_{i=1,3}\Big[\frac{\bar\eta(\bar\tau)}{\vartheta\begin{bmatrix}\frac12+mv_i\\ \frac12+nv_i\end{bmatrix}(\bar\tau)}\frac{\eta(\tau)}{\vartheta\begin{bmatrix}\frac12+mv_i\\ \frac12+nv_i\end{bmatrix}(\tau)}\Big]$$
$$\times\frac{1}{4}\frac{1}{\eta^{16}(\tau)}\Big[\sum_{\alpha,\beta;\alpha'\beta'}\eta(m,n;\alpha,\beta;\alpha',\beta')\prod_{I=1}^{8}Q^{I2}_a\vartheta\begin{bmatrix}\alpha+m\tilde V_I\\ \beta+n\tilde V_I\end{bmatrix}(\tau)$$
$$\times\prod_{I=1}^{8}Q^{I'2}_a\vartheta\begin{bmatrix}\alpha'+m\tilde V'_I\\ \beta'+n\tilde V'_I\end{bmatrix}(\tau)\Big]\Big[\sum_{\Lambda_6,\Lambda^\star_6}q^{p_L^2/2}\bar q^{p_R^2/2}\Big],\qquad(9)$$

where the second and third factors, recognizable by the brackets, are contributed by the internal space coordinates and spinors, the fourth factor by the gauge coordinates and the last (fifth) factor by the compactified space zero modes. The numerical factors appearing in denominators account for the averaging over the time-like spin structures.

2.3 Classification of threshold corrections

The generalized GSO orbifold projection, which selects the singlet states with respect to the orbifold space symmetry group, is represented by the sum over the various twisted orbifold subsectors, $(g,h)=(m,n)$, performed jointly with the sum over the spin structures (α,β), (α',β') for the fermionized fields associated with the gauge degrees of freedom.

The summations over twisted subsectors $(m,n),(\alpha,\beta),(\alpha',\beta')$ are weighted by phase factors $\epsilon(m,n)$ and $\eta(m,n;\alpha,\beta;\alpha',\beta')$ which are determined by the requirement that $\tau_2 B_a$ be invariant under the modular $SL(2,Z)$ group, generated by $S:\tau\to-\frac{1}{\tau}$ and T: $\tau\to\tau+1$. The set of twisted (g,h) subsectors are mixed together under the action of the modular group according to the transformation law [19,22]: $\tau\to(a\tau+b)/(c\tau+d);(g,h)\to(h^cg^d,h^ag^b),[a,b,c,d\in Z,ad-bc=1]$. (For Z_N orbifolds, $S:(m,n)\to(N-n,m)$, $T:(m,n)\to(m,m+n)$.) The entire set of twisted subsectors can be organized into disjoint subsets (orbits) of subsectors which close under the modular group action. The inter-orbit phase factors $\eta(m,n,...)$ are fixed uniquely by the requirement of modular invariance. The intra-orbits (discrete torsion) phase factors $\epsilon(m,n)$ are independently fixed by constraints derived from higher string loops modular invariance or from unitarity [23]. The additional freedom that might be present when the factors $\epsilon(m,n)$ are non-trivial phases serves then to label distinct string theories constructed from the same orbifold. Orbifolds with no (g,h) fixed 2-d torus (i.e., not simultaneously fixed by both space g and time h twists) possess one modular orbit only. Orbifolds having one simultaneous (g,h) fixed 2-d torus possess several modular orbits which are in correspondence with the distinct $N = 2$ suborbifolds of the initial orbifold.

The multiplicity factors $\chi(m,n) = \chi(g,h)$ count, for twisted subsectors, the number of distinct degenerate subsectors associated with fixed points of the orbifold point group which are simultaneously invariant under both g and h [24]. (Useful information on these factors is provided in refs.[25,26]). For untwisted sectors ($m = 0$), there occurs corresponding non trivial factors $\chi(1,h)$ from the projection on oscillator states symmetric with respect to orbifold point group. These can be explicitly calculated from the formula: $\chi(1,\theta^n) = \prod_i |-2i\sin(\pi n v_i)|^2 = |det'(1-\theta^n)|$, where the product and determinant are understood to extend over the rotated 2-tori planes.

In the presence of Wilson lines, an additional summation must be included over the independent Wilson lines a_a satisfying the property $\theta^k a_a \neq a_a$ and over the independent noncontractible loop parameters labeled by m^a. The overall sum over twisted subsectors in eq.(9) is then replaced as, $\sum_{m,n} = \sum_{a_a}\sum_{m,n,m^a_{m,f}}$. For the abelian direct products orbifolds, $Z_N \times Z_M, [M = pN, p \in Z]$ straightforward extensions of the above rules apply in which one deals with pairs of generators, (θ_1,θ_2), shift vectors, $(v_1,v_2),(V_1,V_2)$, twisted subsectors, $(g_1g_2;h_1h_2) = (m_1m_2;n_1n_2)$, setting the discrete torsion phase factor as [14,23], $\epsilon(m_1m_2,n_1n_2) = e^{2\pi ik(m_1n_2-m_2n_1)/N}, \quad [k = 0,\cdots,N-1]$.

The inter-orbit phases $\eta(m,n;\alpha,\beta;\alpha',\beta')$ depend, of course, on the conventions adopted for the fermionic determinants. The following carefully chosen phase conventions for theta-functions [23,27],

$$\begin{aligned} det\partial_{\alpha\beta}\begin{bmatrix}\theta\\ \phi\end{bmatrix} &= \vartheta_{\alpha\beta}\begin{bmatrix}\theta\\ \phi\end{bmatrix}(\nu=0|\tau) = e^{-i\pi\theta(\phi+2\beta)}\vartheta\begin{bmatrix}\alpha+\theta\\ \beta+\phi\end{bmatrix}(\nu=0|\tau),\\ \vartheta\begin{bmatrix}\theta'\\ \phi'\end{bmatrix}(\nu|\tau) &= \sum_{n\in Z} q^{(n+\theta')^2/2}e^{2\pi i(n+\theta')(\nu+\phi')}, \end{aligned} \tag{10}$$

which describes the determinant of a free complex Weyl field obeying the boundary conditions specified a few paragraphs above, is found to reduce the modular invariance constraints on the coefficients to the remarkably simple solution of unit phases, $\eta(m,n;...) = 1$. To prove this statement in the orbifold case, one can follow the same steps as in [27] involving the use of the identities relating the fermionic and bosonic representations of theta-functions and of the Poisson formula transforming the summation over the compactification lattice to that over its dual. Combining in this way the fourth (gauge sector) and fifth (zero modes) factors in eq.(9) yields an equivalent representation for the product of these factors in terms of a manifestly modular invariant sum over an even, self-dual (shifted) (22,6)-dimensional Lorentzian lattice,

$$Z = \sum_{w\in\Lambda_6, p\in\Lambda_6^\star, W\in\Gamma_{8+8}} q^{P_L^2/2}\bar q^{P_R^2/2};$$

$$P_{L,R} = [p_{L\mu}, W_I; p_{R\mu}], \qquad p_\mu^{L,R} = \pm G_{\mu\nu}w^\nu + \frac{1}{2}(p_\mu - k_\mu),$$

$$k_\mu = 2B_{\mu\nu}w^\nu + W^I A^I_\mu + \frac{1}{2}A^I_\nu w^\nu A^I_\mu, \quad [p^2 = p_\mu G^{\mu\nu}p_\nu], \tag{11}$$

where $w_\mu = \frac{1}{2}G_{\mu\nu}(p^\nu_L - p^\nu_R) = m^a e^\mu_a, p_\mu = p_{L\mu} + p_{R\mu} = n_a e^{\star a}_\mu, [m^a, n_a =$ winding and momentum modes integers], $G^{\mu\lambda}G_{\lambda\nu} = \delta^\mu_\nu$ and the basis vectors norms $\sum_\mu e^\mu_a$ identify with the compactification radii R_a. The background metric and antisymmetric tensor fields, $(G_{\mu\nu}, B_{\mu\nu}) = (G_{ab}, B_{ab})e^{\star a}_\mu e^{\star b}_\nu$, and the Wilson line vector field, $A^I_\mu = a^I_a e^{\star a}_\mu$, represent the generalized coupling constants of the world sheet sigma model of the heterotic string whose action (specialized to the superconformal gauge) is reproduced below, for definiteness,

$$S = -\frac{1}{4\pi\alpha'}\int d\sigma dt\Bigg[\sqrt{h}h^{\alpha\beta}\Big(\partial_\alpha X^\mu \partial_\beta X^\nu + i\bar\psi^\mu_R \rho_\alpha \nabla_\beta \psi^\nu_R\Big)G_{\mu\nu}(X)$$
$$+\epsilon^{\alpha\beta}\Big(\partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu}(X) + \partial_\alpha X^\mu_L \partial_\beta F^I A^I_\mu(X)\Big) - \alpha'\sqrt{h}R^{(2)}D(X)\Bigg], \qquad (12)$$

where $\nabla_\alpha \psi^\nu = \partial_\alpha \psi^\nu + \Omega^\nu_{\lambda\mu}\partial_\alpha X^\lambda \psi^\mu$, $[\Omega =$ generalized spin connection] and $D(X) = -\frac{1}{2}\ln S(X)$ denotes the dilaton field. The $\sigma-$model background fields in orbifolds, as in toroidal manifolds, are $X-$independent constants, due to the vanishing curvature tensor.

The charge generators Q_a in eq.(6) identify with the zero mode components of the Lie algebra G_a gauge current vertex operators, $Q_a = J^0_a \equiv \int \frac{d^2 z}{2\pi i} J_a(z)$. The allowed currents are chosen among the linear combinations of the vertex operators, $\{i\partial F^I(z), e^{iP^I F^I(z)}\}$, invariant under the orbifold group. Any choice of component $Q^\alpha_a[\alpha = 1, \cdots, dim(G_a)]$, is admissible since all the components squared $Q^{\alpha 2}_a$ contribute equally to the trace over string states. It is easiest to work with the Cartan subalgebra generators because of the simpler structure of their representention as linear combinations of the momentum operators, $Q_a = Q_{aI}\int \frac{d^2 z}{2\pi i} i\partial F^I$ with coefficients Q_{aI} such that $Q_{ai} = \sum_I Q_{aI}E^I_i$, $(E^I_i, E^{\star i}_I, [i = 1, \cdots, 16]$ are the moving orthogonal frames basis and its dual for the Γ_{8+8} torus) represent the directions (flat components) in the $E_8 \times E'_8$ weights lattice invariant with respect to the orbifold group subject to the constraints, $Q_{aI}V^I, Q_{aI}a^I_b \in Z$. The weight lattice vector components representing the eigenvalues of the Cartan subalgebra operators, $Q^\alpha_a, [\alpha = 1, \cdots, rank(G)]$, for the momentum eigenstates, $|P^I = W^I + k\tilde V^I >$, are given by the scalar products: $\{Q^\alpha_a \cdot P = Q^\alpha_{aI}P^I\}$.

For non-abelian subgroup factors, the gauge group shift embedding case, to which we have limited our considerations here, always leads to $k_a = 1$. For abelian subgroups, the parameters k_a, which are still called levels for convenience of language, depend on the normalization of the corresponding charge operators Q_a and specified by [14]: $k_a = 2\sum_I (Q^I_a)^2$.

The insertion of the charge squared operators is accounted for, in the notations introduced in eq.(9), by replacing the theta-function factors by modified ones using the following rule:

$$\prod_I \vartheta_I Q^{I2}_a \rightarrow \sum_{I\neq J=1}^{8} Q^I_a Q^J_b \vartheta'_I \vartheta'_J \prod_{K\neq I,J}\vartheta_K + \sum_{I=1}^{8}(Q^I_a)^2 \vartheta''_I \prod_{K\neq I}\vartheta_K, \qquad (13)$$

where the primed and double-primed theta-functions are defined in terms of the sum representation given in eq.(10) by inserting linear and quadratic powers of the lattice momenta according to the prescriptions:

$$\vartheta = \sum_P q^{P^2/2}, \quad \theta'^I = \sum_P P^I q^{P^2/2}, \quad \theta''^I = 2q\frac{d}{dq}\vartheta^I = \sum_P P^{I2} q^{P^2/2}, \qquad (14)$$

using self-evident shorthand notations. Note that the precise definition of the 4-d chirality operator, introduced after equation (6), reads in these notations,

$$\chi^2 = 2q\frac{d}{dq}\ln\frac{\vartheta}{\eta} + \frac{1}{12} = +\frac{1}{12} + \frac{\vartheta''}{\vartheta} + 2\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}.$$

The rules in eqs.(13) and (14) follow directly from a consideration of the bosonic representation of the partition function, as described above in connection with eq.(11). We caution, however, that these rules are not sufficient by themselves in dealing with cases involving massless charged oscillator states. For these fortunately rare cases, one needs to insert proper correction factors in order to ensure a correct normalization of the beta function parameters.

Turning now to the threshold corrections as calculated from eq.(9) we note that the Δ_a have a natural additive decomposition in terms of moduli dependent and independent contributions which we associate to the first and second terms in the formula:

$$\Delta_a(M,\bar{M}) = \delta_a + \Delta_a^{(m)}(M,\bar{M}).$$

This separation arises when one classifies contributions according to the number $N = 4, 2, 1$ of space-time supersymmetries which are realized in terms of disjoint subspaces of the Hilbert space of states [3]. There exists a one to one correspondence between the supersymmetry irreducible representation spaces and the spaces of states of suborbifolds which are constructed from subgroups of the full point symmetry group, themselves identified with the modular orbits. The $N = 4, 2, 1$ supersymmetries are then associated with the suborbifolds leaving fixed 3, 1 or 0 2-tori, respectively. The $N = 4$ supersymmetric subsector arises from the purely toroidal, trivial orbit, $(g, h) = (1, 1)$, which is clearly absent in orbifolds, due to the projection. The moduli dependent terms originate from N=2 suborbifolds (one fixed 2-d torus) subsectors and the moduli independent ones from the N=1 suborbifolds (no fixed 2-d torus) subsectors. The $N = 1, 2$ orbits generally contribute to both b_a or Δ_a while the $N = 4$ toroidal subsector $(g, h) = (1, 1)$ (three fixed 2-tori) contributes to neither.

The moduli dependent $N = 2$ contributions arise necessarily from subsectors having non-vanishing momenta, $p_{L,R}$. Indeed, a non-trivial zero modes factor different from unity occurs only for twisted subsectors (m, n) with a simultaneous fixed 2-d torus. For this case, the factors in the partition function in eq.(9) multiplying the zero mode factor combine into the product of an holomorphic function of τ times an anti-holomorphic function of $\bar{\tau}$ which, being non singular modular functions, must therefore both reduce to constants independent of $\tau, \bar{\tau}$. The modular integral over the zero modes factor can then be expressed by a general formula involving automorphic functions for the moduli fields associated to the fixed 2-d torus. For decomposable 6-d tori, one finds [3]:

$$\Delta_a^{(m)}(T,\bar{T}) = \sum_{G'}\sum_{i=1}^{3}(\tilde{b}_a^{'i})_{G'}\ln[(T_i+\bar{T}_i)|\eta(T_i)|^4], \tag{15}$$

where the sum over G' runs over the distinct $N = 2$ suborbifolds G' or modular orbits and the coefficients $\tilde{b}_a^{'i}$ denote the associated massless modes beta function slope parameters multiplied by the ratios of point groups orders, $\frac{|G'|}{|G|}$. The dependence on the Dedekind function reflects the target space duality symmetry under the $SL(2, Z)$ modular group.

The model dependent coefficients can also be represented as [5]: $\tilde{b}_a^{'i} \equiv b_a^{'i} - k_a\delta_{GS}^i$, with $b_a^{'i} = \frac{1}{2}[c(G_a) - \sum_{R^\alpha}(1+2n_\alpha^i)c(R_\alpha)]$, where n_α^i are the massless modes modular weights and δ_{GS}^i the coefficient of the anomaly cancelling Green-Schwarz counterterm. The splitting $b_a^{'i} = \tilde{b}_a^{'i} + k_a\delta_{GS}^i$ exhibits the characteristic property of the mechanisms responsible for the cancellation of the sigma model duality symmetry anomalies (proportional to $b_a^{'i}$), which involve both threshold corrections ($\tilde{b}_a^{'i}$) and a gauge group independent Green-Schwarz counterterm corresponding to a one loop redefined dilaton field, $S + \bar{S} \to S + \bar{S} + \sum_i \frac{2\delta_{GS}^i}{(4\pi)^2} \ln(T_i + \bar{T}_i)$. For non-decomposable tori, the target space modular symmetry is lowered to subgroups of $PSL(2,Z)$. Similar expressions to eq.(15) continue to hold, differing by a non-trivial dependence on the sets of allowed moduli, in particular, involving rescalings such as $T_i \to T_i/3$ or $T_i/4$ [28].

The moduli independent contributions δ_a are associated with vanishing of all components of the momentum and winding modes, $p_{L,R}^\mu$, yielding therefore a trivial zero modes factor equal to unity. No analytic simplification for the modular integral is known to exist in this case, for which one must resort to a numerical evaluation. This task is the subject of next section and represents the main new result reported in this paper.

3. NUMERICAL RESULTS

Before presenting the results we digress to describe how we deal with the numerical integral over the complex parameter τ. The two dimensional modular integral can be separated in two ways:

$$\begin{aligned} \int_F d^2\tau f(\tau_1,\tau_2) &= \int_0^{\frac{1}{2}} d\tau_1 \int_{(1-\tau_1^2)^{\frac{1}{2}}}^\infty d\tau_2 \Big(f(\tau_1,\tau_2) + f(-\tau_1,\tau_2) \Big) \\ &= \int_{\sqrt{3}/2}^\infty d\tau_2 \int_{Re(1-\tau_2^2)^{\frac{1}{2}}}^{\frac{1}{2}} d\tau_1 \Big(f(\tau_1,\tau_2) + f(-\tau_1,\tau_2) \Big). \end{aligned} \tag{16}$$

The general structure of the integrand is that of an infinite sum of terms involving products of functions of $q, \bar{q}$ reading schematically,

$$B_a(\tau) = \sum_{\lambda,\mu} c_a(\lambda,\mu)\phi_\lambda(q)\phi_\mu(\bar{q}) = \sum_{h_L,h_R} w_a(h_L,h_R) q^{h_L} \bar{q}^{h_R}.$$

The projection on the modular group invariants is an essential element here in cancelling the terms with negative powers of $\binom{q}{\bar{q}} = e^{\pm 2\pi i\tau_1 - 2\pi\tau_2}$, thus leading to non singular expansions with powers identified with the conformal weights, $h_L = N_L + \frac{P^2}{2} + E_0 - 1, \quad h_R = N_R + \frac{r^2}{2} + E_0 - \frac{1}{2}, [E_0 = \frac{1}{2}\sum_i [kv_i](1-[kv_i]), \quad 0 < [kv_i] < 1]$. The functions of τ_2 obtained upon integration over τ_1, as exhibited by the second equation in eq.(16), have discontinuous derivatives at $\tau_2 = 1$, as illustrated in figure 1. When h_L, h_R take integral values, the τ_1 (Fourier) integral for $\tau_2 \geq 1$ extends over one period and so selects the terms $h_L = h_R$.

The untwisted sector contributions have this property and thus reduce for $\tau_2 \geq 1$ to constants. The twisted sectors contributions allow (positive) rational values $\frac{k}{N}$ for h_L, h_R and so result in decreasing exponentials of the form, $e^{-\frac{2\pi k \tau_2}{N}}$. Once the constant parts in the full integrand, which are identified with the massless modes contributions given by b_a, are removed, the subtracted integrands $(k_a B_a - b_a)$ are fastly convergent functions. A cut-off at, say, $\tau_2 = 2.2$ is more than sufficient to retain the dominant part of the quadrature. Nevertheless, the projections involved in the summation over the orbifolds subsectors cause strong cancellations which adversely affect the accuracy of final results. The most appropriate way to organize calculations here would be to express analytically the integrand in power expansions in $q, \bar{q}$ prior to the numerical integration [8,9]. However, this procedure is difficult to implement in a systematic way. We have chosen instead to perform all calculations by brute force numerical means and convinced ourselves by various cross checks that one could maintain a numerical accuracy better than 10^{-2} for orbifolds Z_N or $Z_N \times Z_M$, provided that $N, M \leq 6$, since the rounding errors grow with the orbifold order. The numerical integrations are carried out in the order indicated by the second equation in (16).

Let us also quote useful results concerning the inputs for some of the orbifold parameters. Details can be found by consulting refs.[14,24,25]. For the $Z_{3,7}$ prime orbifolds, the degeneracy factors $\chi(g,h)$ count the number of fixed points. Thus, for twisted sectors, $\chi(g,h) = -27(-3), \quad [g \neq 1]$ for the Z_3 orbifolds and $-7(-1)$ for the Z_7 orbifolds, independently of $[h = 1, \cdots, \theta^N]$, where the first (second) numbers refer to cases without (with two) Wilson lines. In the Z_4 orbifolds, in the absence of Wilson lines, $\chi(\theta, \theta^{[0,1,2,3]}) = -16, \quad \chi(\theta^2, \theta^{[0,1,2,3]}) = [16, 4, 16, 4]$. In the $Z_3 \times Z_3$ orbifold with one Wilson line associated with the first factor, as in the example presented below, $\chi(g,h) = [3,3,3,-9,3,3,3,-9]$ for $g = [\theta_1, \theta_1^2, \theta_2, \theta_1\theta_2, \theta_1^2\theta_2, \theta_2^2, \theta_1\theta_2^2, \theta_1^2\theta_2^2]$, independently of $h = \theta_1^{n_1}\theta_2^{n_2}$. Note that $\chi(1,h) = |\chi(h,1)|$ and $\chi(\theta^m, h) = \chi(\theta^{N-m}, h)$. The minus signs in the degeneracy factors are inserted above in order to account for a twisted sector dependent phase factor associated with the chirality.

The $N=2$ subtwisted sectors associated to given (g,h) simultaneously fixed planes, consist in the Z_4 orbifold case of a single modular orbit $\mathcal{O}$ of (g,h) sectors given by: $\mathcal{O} = \{(1,\theta^2), (\theta^2, 1), (\theta^2, \theta^2)\}$, and in the $Z_3 \times Z_3$ orbifold case of three orbits $\mathcal{O}_i$, associated with the three fixed planes, given by:

$$\mathcal{O}_1 = \{(1, \theta_2^{1,2}), (\theta_2, \theta_2^{0,1,2}), (\theta_2^2, \theta_2^{0,1,2})\}; \quad \mathcal{O}_2 = O_1[\theta_2 \to \theta_1];$$

$$\mathcal{O}_3 = \{(1, \theta_1\theta_2^2), (1, \theta_1^2\theta_2), (\begin{pmatrix}\theta_1\theta_2^2 \\ \theta_1^2\theta_2\end{pmatrix}, 1), (\begin{pmatrix}\theta_1\theta_2^2 \\ \theta_1^2\theta_2\end{pmatrix}, \theta_1\theta_2^2), (\begin{pmatrix}\theta_1\theta_2^2 \\ \theta_1^2\theta_2\end{pmatrix}, \theta_1^2\theta_2)\}.$$

We present our results for three cases associated with standard embedding (2,2) orbifolds in Table 1. Results for four non-standard embedding $(0,2)$ orbifolds are presented in Table 2. Details concerning the gauge group and the massless spectra can be found in the second reference in [20] and in ref.[5]. Finally, to elucidate the rôle of discrete Wilson lines, threshold corrections results for four realistic cases of orbifolds with three chiral matter generations are presented in Table 3. Cases A-C refer to Z_3 orbifolds. Up to extra $U(1)$ factors, the observable sector gauge group for Case A [14] coincides with the standard model gauge group, while that for Case B, also due to Font et al. [14], is a left-right chirally symmetric gauge group extension, $SU(3)_c \times SU(2)_L \times SU(2)_R$ and that of Case C, due to Kim and Kim [29], is an intermediate unification gauge group $SU(3)_c \times SU(3)_w$.

Case D in Table 3 refers to a $Z_3 \times Z_3$ [14] orbifold with an observable sector gauge group $SU(3)_c \times SU(2)_L \times SU(2)_R \times SU(2)$.

One of the first calculation of the moduli independent threshold corrections was that attempted by Kaplunovsky [1] for the simplest case of standard embedding orbifolds. He reported a small gauge group dependent term, $\Delta = \frac{\Delta_a - \Delta_b}{b_a - b_b} \simeq 0.07$. The Z_3 orbifold case with two Wilson lines, designated in Table 3 as Case A, was recently considered by Mayr et al., [8]. Assuming tentatively the following decomposition $\delta_a = -b_a \Delta + k_a Y$, with the first component proportional to the factor groups slope parameters and the second to the Kac-Moody levels, they find: $\Delta \simeq 0.079$, $Y \simeq 4.41$. As for the comparison with the existing estimates made in fermionic constructions of 4-d superstrings, this is not very teaching because the threshold corrections in the models discussed in ref.[6] $(\Delta(SU(5)) - \Delta(U(1)) = 24)$ and in ref.[9] $(\Delta(SU(3)) - \Delta(U(1)) = -2.5)$ arise from moduli dependent contributions in $N = 2$ sectors only. Also, the models obtained in the fermionic construction refer to specific points in the moduli space for which one lumps together the moduli dependent and independent contributions.

The conclusions we draw from Tables 1-3 do not strictly agree with those of refs.[1,6]. In our results the component $-b_a \Delta$ proportional to the slope parameters is much smaller than that quoted above. The coefficient Δ is never larger than a few % and its sign and magnitude change from one group factor to the other and also from case to case. This is clearly seen on the corrections δ_a to $U(1)$ factors where the Δ component is amplified by virtue of the larger value taken there by the slope parameters. The analysis of the structure of δ_a does not quantitatively support the conjecture made in ref.[8] concerning a universal decomposition into two components proportional to b_a and k_a. We do find, in agreement with ref. [8], a large contribution proportional to the Kac-Moody levels of approximate size, $Y \simeq 1 - 3$. This is not universal, however, but shows rather a tendency to increase when including Wilson lines. We remark at this point that the cases in Tables 1 and 2 featuring significantly enhanced values of δ_a for certain group factors are precisely those cases which involve oscillator states charged with respect to these group factors. Thus, charged oscillator states appear as the main responsible for non universal effects.

We have also examined for the $Z_3 \times Z_3$ orbifold, the effect of the discrete torsion factor, $\epsilon(m_1, m_2, n_1, n_2) = e^{2\pi i p(m_1 n_2 - m_2 n_1)/N}$, $[p = 0, \cdots, N]$. The results in Table 3 refer to the case $p = 0$. Although the spectrum and hence the slope parameters b_a depend on thetorsion, we find, however, that the threshold corrections remain remarkably stable with variable $p > 0$.

4. UNIFICATION AND ANOMALOUS $U(1)$ SCALE

4.1 Threshold corrections

In this section we examine the viability of the perturbative superstring unification within the orbifold approach. Let us first discuss the implications of the results obtained in Section 3 for the moduli independent threshold corrections. Assuming the simple formula, $\delta_a = -b_a \Delta + k_a Y$, then as already noted in connection with eq.(4), one can absorb the string threshold corrections into an effective unification scale M'_X and an effective string coupling constant g'_X. Since δ_a are of positive sign, it follows that the moduli independent threshold corrections will always result in reduced effective unified coupling constant and enhanced (reduced) unification scale, depending on whether the beta function slope parameters b_a are positive (negative), or equivalently, gauge (matter) dominated. Using the numerical values for b_a and δ_a in Tables 1-3, we find very small moduli independent corrections to the unification scale and/or coupling constant, which attain at most a few % .

Identifying the string moduli independent threshold corrections obtained here, $\frac{\delta_a}{4\pi} \simeq 0.4$, tentatively with a corresponding field theory threshold correction of typical structure [30], $\delta(\frac{4\pi}{g_a^2}) = \pm O(1) \ln \frac{M_H}{M_X}$, yields for the ratio of the average heavy particle mass to unification mass, $M_H/M_X \simeq \frac{1}{2}$. Thus, one checks that these contributions are of the same order of magnitude as the two loop field theory renormalization corrections [31]. We conclude therefore that the moduli independent threshold corrections should mildly affect the high energy extrapolation of the gauge coupling constants. More quantitatively, one can estimate the corrections to the weak angle and color coupling constant by means of the formulas [5],

$$\sin^2\theta_W(m_Z) = \frac{k_2}{k_1+k_2} + \frac{\alpha(m_Z)}{4\pi}\frac{k_1}{k_1+k_2}\left[A\ln\frac{m_Z^2}{M_X^2} + \Delta_A\right],$$

$$\alpha_s^{-1}(m_Z) = \frac{k_3}{k_1+k_2}\left[\frac{1}{\alpha(m_Z)} + \frac{B}{4\pi}\log\frac{m_Z^2}{M_X^2} + \frac{\Delta_B}{4\pi}\right], \quad (17)$$

where we use the notations: $A = -(b_1k_2/k_1 - b_2)$, $B = -(b_1+b_2-b_3(k_1+k_2)/k_3)$, $\Delta_A = -(\Delta_1 k_2/k_1 - \Delta_2)$, $\Delta_B = -(\Delta_1+\Delta_2-\Delta_3(k_1+k_2)/k_3)$. Evaluating the threshold corrections for Case A in table 3, using $k_1 = 11/3$, yields:

$$\delta(\sin^2\theta_W(m_Z)) \simeq 1.5\times 10^{-4}, \quad \delta(\alpha_s^{-1}(m_Z)) \simeq 2.\times 10^{-2}, \quad [\delta\alpha_s(m_Z) \simeq 2.7\times 10^{-4}]$$

where we have set $\alpha^{-1}(m_Z) = 127.9 \pm 0.1$. We see that the corrections are rather small and lie well within the present experimental uncertainties on these parameters [31], $\alpha_s(m_Z) = 0.120 \pm 0.010$, $\sin^2\theta_W(m_Z) = 0.2324 \pm 0.0006$. The extreme smallnes of the effect here is due to the cancellation of the predominant level dependent component k_aY in δ_a in the linear combinations appearing in $\Delta_{A,B}$.

Turning to the moduli dependent corrections $\Delta_a^{(m)}$, we note that these are generically of opposite sign with respect to δ_a and so have an opposite effect on the effective unification parameters. These contributions become sizeable only to the extent that large moduli VEVs and large ratios $\tilde b'_a/b_a$ are used, as is clearly demonstrated on the following approximate formula, valid for large VEVs,

$$M'_X \simeq M_X\left[\frac{e^{\frac{\pi(T+\bar T)}{6}}}{T+\bar T}\right]^{\frac{\tilde b'_a}{2b_a}}. \quad (18)$$

To estimate the corrections in eqs.(17), one can use the approximate formulas, $\Delta_{A,B} \simeq \binom{A'}{B'}(\ln(2T_R) - \frac{\pi}{3}T_R)$, where $\binom{A'}{B'} = \binom{A-\delta A}{B-\delta B}$ such that $\binom{A}{B} = \binom{28/5}{20}$ for the minimal supersymmetric standard model and $\delta A, \delta B$ depend on the modular weights parameters assignments. The solutions reported in refs. [5,12] give: $\binom{A}{B} \simeq \binom{4\sim 16}{24\sim 40}$, or equivalently,

$\binom{A'}{B'} \simeq \binom{2\sim-10}{0\sim 20}$. In order for these corrections to $\sin^2\theta_W$ and α_s to reach an order of magnitude higher than those found above from the moduli independent corrections, one needs at least, $T_R = Re(T) = O(10)$.

4.2 Standard-like superstring unification scenario

We shall now present an extended analysis of the string unification picture in which the coupling constants scale evolution proceeds through an intermediate threshold at M_A induced by an anomalous $U(1)$ mechanism. A two-stage scale evolution is considered: An initial short evolution from M_S to M_A, described by the slope parameters b_a^A set at the values predicted in the orbifold models, followed by a wide scale evolution from M_A to m_Z described by the minimal supersymmetric standard model slope parameters. The relevant formula reads:

$$\frac{(4\pi)^2}{g_a^2(\mu)} = k_a(\frac{(4\pi)^2}{g_X^2} + \tilde{Y}) + 2b_a \ln\frac{\mu}{M_A} + 2b_a^A \ln\frac{M_A}{M_X} + \Delta_a^{(m)}(T,\bar{T}). \tag{19}$$

We regard the five parameters $[g_X, k_1, T\ ,\ \tilde{Y}\equiv Y_0+Y, M_A]$, which enter explicitly eq.(19), as adjustable parameters. Note that M_X has a fixed linear dependence on g_X which is specified by eq.(3). The compactification scale can be tentatively identified in order of magnitude by writing:

$$M_C = \frac{2\pi}{R} \simeq \frac{M_S}{2}\left(\frac{C_{orb}}{T}\right)^{\frac{1}{2}} \simeq \frac{2\sqrt{C}_{orb}M_X}{\sqrt{T}}, \tag{20}$$

where the compactification radius R and moduli VEV, $< T >= T$ are related as $T = \frac{C_{orb}R^2}{\alpha'(2\pi)^2}$, with C_{orb} a calculable constant of order unity [26]. For, say, the Z_3 orbifold, $C_{orb}=\sqrt{3}/4$. One concludes from eq.(20) that $M_C/M_X \simeq 1/\sqrt{T}$.

A rough order of magnitude estimate for the anomalous $U(1)$ Higgs mechanism scale M_A can be obtained by imposing the condition of a vanishing D-term scalar potential [17], $-D_A/g_A^2 = \sum_\alpha Q_A^\alpha|\phi_\alpha|^2 + \frac{g_X c_A}{4\alpha'\sqrt{k_A}}$, for a group factor $U_A(1)$ distinguished by the index A. (The triangle anomalies coefficient c_A is defined as $48\pi^2 c_A = Tr(Q_A) = 4Tr(Q_A^3)$, where the traces extend over the massless modes. This enters the Green-Schwarz counterterm through the substitution for the dilaton field, $S+\bar{S}\to S+\bar{S}+c_AV_A$, whose function is to cancel the various $U_A(1)$ group factor (gauge and gravitational) triangle anomalies, by assigning to the gauge vector and dilaton chiral supermultiplet fields the transformation laws, $V_A\to V_A-\Lambda_A-\Lambda_A^\star, \quad S\to S+c_A\Lambda_A$.) The predicted magnitude for the scale is:

$$M_A \simeq < \phi >= \frac{M_P}{\sqrt{8\pi}}\frac{g_X}{\sqrt{2}}\left[-\frac{g_X Trace(Q_A)}{192\pi^2 Q_A\sqrt{k_A}}\right]^{\frac{1}{2}}. \tag{21}$$

Using tentatively for the model dependent ratio the estimate $-Tr(Q_A)/(Q_{A\alpha}\sqrt{k_A}) \simeq 10$, one obtains: $M_A \simeq 1.2 g_X^{3/2}\times 10^{17}$ GeV, which indicates that M_A should be of the same order of magnitude as M_X.

We use the known experimental values of the gauge coupling constants at the Z-boson mass, namely, $g_1^2(m_Z)=0.127, g_2^2(m_Z)=0.425, g_3^2(m_Z)=1.44$, as inputs to determine via eq.(19) three among the above quoted adjustable parameters. We choose these to be: $g_X, \tilde{Y}, M_A$. This choice is motivated by the fact that the dependence on these parameters in eq.(19) can be made linear by means of an obvious change of variables. The solutions for $g_X, \tilde{Y}, M_A$ are determined as a function of the remaining free parameters, namely, T and k_1, and the sets of slope parameters, $b_a^A, \tilde{b}'_a$. For a solution to be acceptable it must comply with the perturbation theory constraints that g_X and Y be of order unity and with the obvious inequalities between scales, $\frac{M_A}{M_X} < 1, \quad \frac{M_C}{M_X} < 1$, which we shall eventually supplement by the inequality, $\frac{M_A}{M_C} < 1$, reflecting the assumption that the mechanism inducing the scale M_A is a consequence of compactification.

We shall present the results of numerical applications only for Case A in Table 3, setting $b_a = (-11, -1, 3)$, corresponding to the minimal supersymmetric standard model , $b_a^A = (-71.5, -18, -9)$, as obtained from Table 3, and $b'_a = (18, 8, 6), \delta_{GS} = 7$, where the choice of slope and Green-Schwarz parameters $b'_a = \sum_i b_a^{'i}, \quad \delta_{GS} = \sum_i \delta^i_{GS}$ for the moduli dependent threshold corrections is based on the solutions reported in ref. [5] (see also ref. [32]). Regarding k_1 as a free parameter when this is predicted to be 11/3 and including moduli dependent threshold corrections in a case (such as the Z_3 orbifold) where these are absent, is certainly liable to criticism. However, because the orbifold order appears to have a minor influence on threshold corrections and in view of the wide freedom expected in the hypercharge gauge coupling constant normalization, we hope that these shortcomings do not affect the consistency of our procedure.

Our main purpose is to explain the non trivial interplay between the various parameters which are most significant for string phenomenology. Choosing the particular subset, k_1, T, as our free parameters while adjusting the others $(\tilde{Y}, M_A, g_X)$ to the inputs, $g_a(m_Z^2), [a = 3, 2, 1]$ is only a technical convenience. Let us first discuss some qualitative features of the solutions and, in particular, the correlations among the parameters. The dependence on $\tilde{Y}$ and g_X shows clearly that any change in g_X can be compensated by a negative contribution to $\tilde{Y}$. A decrease of k_1 widens the distance between the quantities $(g_a^2 k_a)^{-1}$ and so can be compensated by decreasing M_A/M_X or g_X. Finally, because the functional dependence on M_A and g_X in eq.(19) involves a logarithm of these quantities, one expects a strong sensitivity of the parameters on the inputs.

The results are displayed in figure 2. These represent a continous two parameters (k_1, T) family of solutions for $g_X, \tilde{Y}, M_A$ consistent with a high energy extrapolation of the gauge coupling constants joining roughly at the common value, $\frac{4\pi}{g_a^2 k_a} \simeq 25$. The physical constraints on $\tilde{Y}, g_X, M_A$ select a reduced domain for the free parameters, $k_1 \in (1.4, 1.8), \quad T \in (1, 30)$. The variations with respect to these parameters are monotonic. For fixed T, increasing k_1 leads to a rapidly (algebraically) increasing $\tilde{Y}$ from large negative to positive values and to less rapidly increasing $M_A/M_X$and g_X. Strong variations are also found for the $T-$dependence. However, as T increases past 25, $\tilde{Y}$ becomes positive and nearly independent of k_1. The values of k_1 on the lower side, $k_1 < 1.4$, are excluded by the constraints on $\tilde{Y}$ and those on the higher side, $k_1 > 1.8$, by the constraints on g_X and M_A/M_X.

A wide class of solutions occur with $g_X << 1$ and $-\tilde{Y} >> 10^3$, independently of T and k_1. These arise through an obvious compensation effect of the moduli independent corrections with g_X in eq.(19). Although the Y_0 component of $\tilde{Y}$ remains uncalculated so far, it appears unlikely that this can much exceed the component Y which was evaluated in

Section to be of $O(1)$. In fact, since large Y_0 is only possible for a strongly coupled string theory involving large g_X, the above must be regarded as an inconsistent class of solutions. (However, because the generic dependence on coupling constant of non perturbative effects is expected to be less suppressed in string theory than in field theory [16], e^{-c/g_X} versus $e^{-(4\pi)^2/g_X^2}$, one could possibly achieve large Y_0 with not too large g_X.) In the following we shall restrict ourselves to the conventional framework where one assumes a smooth connection between string theory and its low energy limit and hence retains the constraints $g_X = O(1)$, $|\tilde{Y}| = O(10)$.

Examining the variation of the solutions with k_1 in figures 2(a-c), we see that these are very rapid, especially that of $\tilde{Y}$. The condition $\tilde{Y} = O(1)$ can be satisfied only through a very careful fine-tuning of k_1 for fixed T, or of T for fixed k_1. This is possible only in cases where $\tilde{Y}$ changes sign in the relevant intervals of k_1, T. The moduli dependent corrections are quite essential to achieve a high energy extrapolation consistent with superstring unification. Incorporating the threshold M_A provides solutions with reduced T. The constraints on $\tilde{Y}$ and g_X require $15 < T < 30$ and $1.5 < k_1 < 1.8$. Incorporating the constraint $\frac{M_A}{M_X} < 1$ restricts this interval to $1.5 < k_1 < 1.7$. (Narrower intervals would be imposed if one also sets lower bounds, say, $\frac{M_A}{M_X} > 10^{-1}$ and $g_X > 10^{-1}$.) If one takes into account the additional constraint $\frac{M_A}{M_C} < 1$, this would lead to the stronger bound, $\frac{M_A}{M_X} \simeq \frac{M_A}{M_C\sqrt{T}} < \frac{1}{\sqrt{T}}$, which would select the narrower interval, $1.5 < k_1 < 1.6$.

For concreteness, we show in figure 2 (d) the scale evolution of the gauge coupling constants for one particular solution as determined by the above procedure. One should not be disturbed by the large value of $|\tilde{Y}|$ used here, since the nearby solution determined with a carefully tuned value of k_1 or T so as to give $\tilde{Y} = O(1)$, would yield nearly identical flows for the gauge coupling constants. This figure illustrates one of the characteristic implications of string unification, namely, that the simultaneous equality at some scale of the extrapolated coupling constants has no special significance. The picture depicted in figure 2 (d) is rather generic. The most favorable situation corresponds then to an approximate joining of the coupling constants flows at a large scale near $5\times10^{16} GeV$, which is to be identified with the anomalous $U_A(1)$ scale M_A, associated with the decoupling of the extra quarks or leptons modes. In the string unification picture, the joining scale M_A can be made larger than M_{GUT} because of the slightly reduced normalization of the hypercharge group coupling constant and of the spread of the coupling constants at M_X which is related to the moduli dependent threshold corrections.

Let us comment briefly on the sensitivity of the solutions to the slope parameters. (Our procedure would obviously break down for $b_a^A \approx b_a$ as this would make the linear system of equations, eq. (19), singular) The slope parameters b_a^A determine the variation of the coupling constants from M_X to M_A. The choice of b_a^A is correlated to that of the moduli dependent slope parameters, $\tilde{b}'_a$, since the latter determine the amount by which the coupling constants are spread at M_X. Consider first the case of fixed $\tilde{b}'_a$. Increasing T implies a wider spread of the coupling constants at M_X which should therefore be compensated by larger slopes b_a^A in order to catch up with the extrapolated coupling constants up to $10^{16} GeV$. Rather than showing new plots, we only mention here that if one performs a uniform reduction of the slopes b_a^A by, say, a factor 2, the solutions would rule out the entire domain in k_1, T except for a narrow region around $T = 15, k_1 = 1.7$. Conversely, enhancing the slopes b_a^A by, say, a factor 2 ameliorates the initial picture without changing qualitatively the character of solutions. One concludes therefore that the cases involving negative slope parameters b_a^A with large absolute values (richer matter spectra), which are generic in orbifolds model building, are more favorable for unification.

The choice of $\tilde{b}'_a = b'_a - k_a\delta_{GS}$ is also quite sensitive. Rather than performing an exhaustive study we have considered two other cases obtained from ref. [5] and further motivated in ref. [32]. Applying the above procedure of solution for these cases, we found a significantly worsened picture. The first case, characterized by $b'_a = (7.5, 2.5, 1.50), \delta_{GS} = 2.5$, admits solutions only for large values of $T > 30$ and correspondingly large $k_1 > 1.8$. It improves slightly if reduced values are used for the slopes b_a^A. The second case, characterized by $b'_a = (-4.67, 4, 5), \delta_{GS} = 6$, admits no solutions at all, mainly on account of an incompatibity between the constraints on Y and M_A/M_X. One concludes therefore that negative or small values for the $N = 2$ slope parameters $\tilde{b}'_a$ do not constitute a favorable option.

Having focussed so far on standard-like compactification models, we briefly discuss the other two possible classes of superstring models. The first refers to compactification models with grand unified groups, $SU(5)$ [6] or $SO(10)$ [21] (up to extra $U(1)$ factors), with a flipped assignment for the matter fields with respect to the standard GUT basis or with a regular GUT assignment involving higher Kac-Moody levels, $k > 1$ [33]. A perturbative weak coupling scenario assuming a smooth evolution from M_{GUT} to M_X can be carried out in the manner described above either by setting the parameters, $b_G, \tilde{b}'_G$ and Y at values specified by the models or by imposing appropriate constraints on them. It should not be difficult to obtain satisfactory solutions for g_X and M_A by following a procedure similar to that used above. An alternative strong coupling scenario could also be envisaged [21] if the slope b_G takes a large (gauge dominated) positive value and g_X is large so as to lead to GUT group G with renormalization group invariant scale comparable to the string scale, $\Lambda_G = M'_X e^{-8\pi^2 k_G/b_G g'^2_X}$. Although such a scenario forbids a smooth connection from string theory to the low energy field theory, it still provides a prediction for the GUT scale, namely, $M_{GUT} \simeq \Lambda_G$.

The second class of compactification models corresponds to intermediate unification on a semisimple gauge group. One interesting example is Case C in Table 3 where the gauge symmetry at compactification, $SU(3)_c \times SU(3)_w \times U(1)_{P_3}$, breaks down to the standard model group at the anomalous $U(1)$ scale according to $SU(3)_w \times U(1)_{P_3} \rightarrow SU(2)_w \times U(1)_Y$, where $Y = T_{8w} + \frac{P_3}{3}$. Using the information supplied in ref.[29], we find a level parameter $k(P_3) = \frac{1}{3}$. This implies a normalization of the hypercharge coupling constant such that $k_1 = 1 + \frac{1}{27} = \frac{28}{27}$. Although this falls well below the favorable interval of k_1 values specified above, it is nevertheless interesting that the present situation for Case C is exactly opposite to that found above for Case A.

5. CONCLUSIONS

Our results indicate that the moduli independent threshold corrections are comparable in size to those for gauge field theories in spite of the fact that infinitely many massive states are integrated out for superstrings. The corrections are marginally relevant at the current precision levels for the low energy gauge coupling constants. The largest contributions reside in a group independent component k_aY of size $Y \simeq 1 \sim 3$ which remains relatively stable with respect to the orbifold order or to the choice of gauge group embedding and Wilson lines. The component $-b_a\Delta$ is much smaller, $|\Delta| < 10^{-2}$ and model dependent.

In order for the large value of the predicted string unification scale M_X not to conflict with observations, one needs both moduli dependent threshold corrections (with associated compactification scale $\frac{M_C}{M_X} \simeq \frac{1}{\sqrt{T}} \approx 0.3$) as well as a weak hypercharge group level parameter varying in the narrow interval, $k_1 = 1.4 \sim 1.7$. The information that the moduli independent corrections are $O(1)$ is useful in providing stronger correlations among the parameters relevant to string phenomenology. Postulating an anomalous $U(1)$ mechanism at a scale $0.1 < M_A/M_X < 1$ significantly eases the above constraints on slope parameters while raising the bound on the allowed values of M_C. The resulting picture is intermediate between a delayed joining of the coupling constants flows, due to the smaller value of k_1, and of a continued flow beyond crossing, consistent with the moduli dependent threshold corrections. Our analysis emphasizes the need of constructing orbifold models combining the property of a low value for the hypercharge group level parameter along with the usual desirable features, namely, three chiral families, low rank gauge group and $N = 2$ subsectors.

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TABLE CAPTIONS

Table 1. Threshold corrections for the $Z_{3,4,7}$ orbifolds with standard gauge embeddings. The entries in the first column are the shift vectors v^i, V^I. The second and subsequent columns correspond to the gauge group factors in the observable and hidden (prime) sectors. For each column, the first line entry gives the levels k_a, the second line gives the beta

function slope parameters b_a or, for the non-prime orbifolds with $N = 2$ suborbifolds, the pairs $(b_a^{N=1}, \tilde{b}'_a)$, such that $b_a = b_a^{N=1} + \tilde{b}'_a$. The third line gives the moduli independent threshold corrections δ_a.

Table 2. Threshold corrections for orbifolds $Z_{3,4}$ with non standard gauge embeddings. The shift vectors V_I are displayed in the first column and the second and subsequent columns correspond to a selection of the gauge group factors in the observable and hidden (prime) sectors. For each column, the first line entry gives the levels k_a, the second line gives the beta function slope parameters b_a or, for the non-prime orbifolds with $N = 2$ suborbifolds, the pairs $(b_a^{N=1}, \tilde{b}'_a)$, such that $b_a = b_a^{N=1} + \tilde{b}'_a$. The third line gives the moduli independent threshold corrections δ_a.

Table 3. Threshold corrections for a selection of three-generations orbifold models with two Wilson lines (Cases A-C) and one Wilson line (Case D). For Z_3 orbifolds, the winding numbers parameters are $m_{k,f} = 0, \pm 1$. Case A is a standard model group Z_3 orbifold model from Font et al., [14] (section 4.2): $3V^I = (1^4 2000)(20^7)', 3a^I_{1,2} = (0^7 2)(0110^5)', 3a_{3,4} = (11121011)(110^6)'$. Case B is a left-right group Z_3 orbifold model from Font et al., [14] (section 4.3): $3V^I = (1^4 2000)(20^7)', 3a^I_1 = (0^7 2)(00110^4)', 3a^I_3 = (1^3 21^3 0)(110^6)'$. Case C is an intermediate unification group Z_3 orbifold model from Kim and Kim [29]: $3V^I = (11211200)(0^8)', 3a^I_1 = (0^3 11211)(1^4 0^4)', 3a^I_3 = (0^7 2)(1^8)'$. Case D is an intermediate unification group $Z_3 \times Z_3$ orbifold model with one Wilson line from Font et al., [14] (section 5): $3v^i_1 = (1, 0, -1), 3v^i_2(0, 1, -1); 3V^I_1 = (2110^5)(110^6)', 3V^I_2 = (020^6)(0 - 1111000)'; 3a^{(1)I}_1 = (0^5 11 - 2)(0^5 11 - 2)'$. (The indices $1, 2$ refer to the two Z_N factors.) For each column, the first line entry gives the levels k_a, the second line gives the beta function slope parameters b_a or, as in Case D, the pairs $\binom{b_a^{N=1}}{\tilde{b}'_a}$ such that $b_a = b_a^{N=1} + \tilde{b}'_a$. The third line gives the moduli independent threshold corrections δ_a.

FIGURES CAPTIONS

Figure 1. The threshold function $-B_a(\tau)$, integrated over τ_1, is plotted as a function of τ_2 for the Z_3 orbifold model group factor $SU(3)_c$ of Case B in Table 3. We show the contributions of the untwisted (continuous line) and of the twisted sectors (double-dashes).

Figure 2. One loop renormalization group analysis of superstring unification parameters based on high energy extrapolation of the gauge coupling constants starting from their experimental values at m_Z. The solutions for $-\tilde{Y}$ (figure a), M_A/M_X (figure b) and g_X (figure c) are plotted as a function of k_1 for a discrete set of values of the moduli VEV, $T = 1$ (continuous), 10 (long dash short double-dashes), 15 (long dash short dash), 20 (dash dot), 30 (dash). The slopes discontinuities exibited by $\tilde{Y}$ in figure (a) arise because of the changes of sign of $\tilde{Y}$ in this semilogarithmic plot. (For the $T = 30$ curve, $\tilde{Y} > 0$.) We display in figure (d) graphs of the gauge coupling constants $(\frac{4\pi}{g_a^2 k_a}, \quad [a = 3, 2, 1])$ variation with renormalization scale for the particular solution characterized by the values, $k_1 = 1.6, T = 20$, yielding the solution $\tilde{Y} = -114, M_A/M_X = 0.38, g_X = 0.63$.

TABLE 1

Orbifold	Gauge Group			
Z_3	SU_3	E_6		E'_8
$(111)/3$	1	1		1
$(11-2)/3$	-72	-72		90
$(1120...0)/3$	2.95	1.55		1.69
Z_4	SU_2	E_6	$U(1)$	E'_8
$(112)/4$	1	1	3	1
$(11-2)/4$	$(-12,-42)$	$(-36,-42)$	$(-231,-94.5)$	$(60,30)$
$(1120...0)/4$	1.22	1.07	7.19	0.77
Z_7	E_6	$U(1)_1$	$U(1)_2$	E'_8
$(124)/7$	1	4	12	1
$(12-3)/7$	-36	-369.3	$-1521.$	90
$(12-30...0)/7$	2.04	15.6	80.8	2.07

TABLE 2

Orbifold	Gauge Group				
Z_3	SU_3	E_6	SU'_3	E'_6	
	1	1	1	1	
$(1120...0)/3$	$-45.$	9.	$-45.$	$-9.$	
$(1120...0)'/3$	1.18	3.66	1.18	3.66	
Z_3	E_7	$U(1)_1$	$U(1)_2$	SO'_{14}	
	1	4	2	1	
$(110...0)/3$	36	-462	-105	-18	
$(20...0)'/3$	3.06	15.6	4.90	3.01	
Z_3	SU_9		SO'_{14}	$U(1)'$	
	1		1	2	
$(11112000)/3$	-18		9	-99	
$(20...0)'/3$	3.64		3.66	3.57	
Z_4	SU_2	E_6	$U(1)$	SU'_2	E'_7
	1	1	12	1	1
$(1120...0)/4$	$(-12,-42)$	$(12,-42)$	$(-2711,-1512)$	$(-104,30)$	$(12,30)$
$(220...0)'/4$	1.13	2.38	100	2.90	2.38

TABLE 3

Case A									
SU_3	SU_2	$U(1)_1$	$U(1)_2$	$U(1)_4$	$U(1)_5$	$U(1)_Y$	$U(1)'_4$	$U(1)'_6$	SO'_{10}
1	1	6	4	2	2	$\frac{11}{3}$	2	4	1
-9	-18	-227	-110.6	-31.2	-16.8	-71.5	-14.4	-69.6	18
3.41	3.41	32.	14.4	3.55	3.51	11.4	1.75	13.8	3.44
Case B									
SU_3	SU_2^L	SU_2^R	$U(1)_1$	$U(1)_2$	$U(1)_3$	$U(1)_4$	SU'_2	SO'_8	U'_1
1	1	1	6	4	4	2	1	1	2
-6	-15	-15	-216	-103	-103	$-12.$	-24	6	-26.4
3.57	3.57	3.57	32	14.6	14.6	-3.57	3.56	1.80	3.61
Case C									
SU_3	SU_3	$U(1)_1$	$U(1)_2$	$U(1)_3$	$U(1)_4$	SO'_{12}	$U(1)'_5$	$U(1)'_6$	
1	1	6	6	2	2	1	8	8	
-18	-18	-349	-284	-35.8	-28.6	27	-490	-453	
3.09	3.09	32.8	31.4	3.65	3.49	3.13	52.8	56.3	
Case D									
SU_2	SU_2^L	SU_2^R	SU_3	$U(1)_1$	$U(1)_2$	$U(1)_3$	SU'_3	SO'_6	U'_1
1	1	1	1	2	4	6	1	1	4
$\binom{2.6}{-24}$	$\binom{-9.4}{-24}$	$\binom{2}{-36}$	$\binom{2}{-12}$	$\binom{2.9}{-58}$	$\binom{-54}{-112}$	$\binom{2.7}{-270}$	$\binom{2}{-12}$	$\binom{2}{-12}$	$\binom{-19.6}{-128}$
1.13	1.13	1.10	1.10	1.13	4.76	9.70	1.10	1.10	4.60