FTUV/95-23
IFIC/95-23
June 9, 1995

Group Theoretical Foundations of Fractional Supersymmetry

J. A. de Azcárraga and A. J. Macfarlane[1]

Departamento de Física Teórica and
IFIC (Centro Mixto Universidad de Valencia-CSIC)
46100 Burjasot (Valencia), Spain

HEP-TH-9506177

Abstract

Fractional supersymmetry denotes a generalisation of supersymmetry which may be constructed using a single real generalised Grassmann variable, $\theta = \bar{\theta}$, $\theta^n = 0$, for arbitrary integer $n = 2, 3, ...$. An explicit formula is given in the case of general n for the transformations that leave the theory invariant, and it is shown that these transformations possess interesting group properties. It is shown also that the two generalised derivatives that enter the theory have a geometric interpretation as generators of left and right transformations of the fractional supersymmetry group. Careful attention is paid to some technically important issues, including differentiation, that arise as a result of the peculiar nature of quantities such as θ.

[1]On leave of absence from DAMTP, Cambridge University, Silver Street, Cambridge CB3 9EW, UK, and from St. John's College, Cambridge, UK.

1 Introduction

Supersymmetry has been a popular and fruitful area of research for at least twenty years. Study of it in space-time of one dimension, time, has given rise to the important topic of supersymmetric quantum mechanics (see [1]- [3] for reviews). The most primitive version of supersymmetric quantum mechanics is one that involves use of a single real Grassmann number θ such that

$$\theta = \bar{\theta} \quad , \quad \theta^2 = 0 \quad . \tag{1.1}$$

As a result the theory possesses a natural $\mathcal{Z}_2-$grading and a single generator Q of its supersymmetry transformations which obeys $Q^2 = -\partial_t$. The distinctive features of supersymmetric theories which possess such a $\mathcal{Z}_2-$grading can be seen by reference to various papers [4]-[9]. The term fractional supersymmetry is currently being applied to a class of generalisations of supersymmetry in one dimension. Our work on fractional supersymmetry can be presented most straightforwardly by creating theories with $\mathcal{Z}_n-$grading by generalisation of theories with $\mathcal{Z}_2-$grading. Thus we consider theories involving a single real (generalised) Grassmann number θ which obeys

$$\theta = \bar{\theta} \quad , \quad \theta^n = 0 \quad , \quad n = 2, 3, 4, \ldots \quad , \tag{1.2}$$

in which the generator Q of the generalised ('fractional') supersymmetry transformations that leave such a theory invariant obeys

$$Q^n = -\partial_t \quad . \tag{1.3}$$

The last result accounts loosely for the use of the term 'fractional' as an identifier of the theory.

The generalisation from ordinary to fractional supersymmetry not only has intrinsic interest but may also be expected to produce interesting new models in classical and quantum mechanics. There have been a large number of studies of fractional supersymmetry in recent years [10] - [18]. Some of these deal with a complex Grassmann variable θ such that $\theta^n = 0$. Others employ N different copies of θ which obey (1.1), thus developing N-extended fractional supersymmetry. Fractional supersymmetry is contrasted below with a distinct class of generalisations of basic, or $\mathcal{Z}_2$ -graded, supersymmetry, those which possess parasupersymmetry. There has been a great deal of attention given recently to work in this field [19] - [26], and often these papers contain thinking relevant also to fractional supersymmetry. We belive that the whole area promises both activity and progress in the future.

This paper discusses two important aspects of fractional supersymmetry, Firstly, we discuss the fact that the fractional supersymmetry transformations that describe the invariance properties of the $\mathcal{Z}_n$-graded theory form a group G_n. Secondly, we elucidate certain fundamental technical matters stemming from unfamiliar features of the algebra. Two areas need attention. One concerns differentiation with respect to θ; the other is the situation surrounding families of multiplicative rules of the type

$$\epsilon\theta = q^{-1}\theta\epsilon \quad , \quad q = \exp(2\pi i/n) \quad , \tag{1.4}$$

involving a Grassmann number θ, its associated transformation parameter ϵ and dynamical variables of Grassmann type. In the former there are difficulties of principle, which we treat; in the latter it is a matter of demonstrating a coherent rationale behind the formulation and the consistency of results like (1.4) within it.

We give explicit formulas for the elements of the group G_n of transformations that should leave any $\mathcal{Z}_n$-graded theory invariant, and related proofs. Once the status of derivatives with respect to θ is established, we turn to the objects of generalised covariant derivative type that enter (up to now in an *ad hoc* way) into a theory possessing fractional symmetry. We show that these have the interesting geometrical interpretation of being the generators of the left and right actions of the fractional supersymmetry group G_n (as is the case for ordinary supersymmetry [27]).

We have introduced into our discussion a quantity $q = \exp(2\pi i/n)$ which obeys

$$q^n = 1 \quad . \tag{1.5}$$

To provide some appropriate comment, we recall that fractional symmetry aims at a generalisation of supersymmetry. The latter when quantized, involves one boson and one fermion variable, and requires use of a 2×2 matrix representation of the fermion. We plan a generalisation, (see (4.3), (4.1) for $n = 3$ or (7.2) below), which retains the boson and replaces the fermion by some more general object, cf. (1.1) and (1.2). Two classes of variables, which can be represented by matrices in an n-dimensional vector space are known to us. The parafermions [28][29] are one of these; use of them leads down a path of interest, but not the one we are able usefully to follow at the moment, toward parasupersymmetry. We follow the other path. The q-deformed harmonic oscillator [30][31][32] possesses commutation relations in terms of its a and $a^\dagger$ variables that make sense, not only for $q \in R$, but also $q \in C$, when (1.5) applies. In this situation, a and $a^\dagger$ are represented (for each n) by $n \times n$ matrices. Since for $n = 2$ we get back in this way to a description of fermions, it is clear that we are talking about generalisations of these. By looking at the $n = 3$ case and beyond one can see that the generalisations are distinct from parafermions. We are not yet in a position to push satisfactorily the quantization of our theory to a point where the implied interpretation is present in a consistent well-understood way, but we are certainly describing a plausible scenario for it. We plan to report on this soon.

For reasons of notational simplicity and clarity, we present first our ideas for the $\mathcal{Z}_3$-graded case, the first non-trivial generalisation of the basic supersymmetry. This already requires that most of the central issues of the $\mathcal{Z}_n$ case be treated seriously. The paper contains seven sections. Section 2 contains introductory material for G_3 including its group law, and the reasons behind expressions such as (1.4). Section 3 derives the formula for the transformations of G_n. Section 4 discusses the problem of defining derivatives with respect to θ, leading into section 5 which shows how the usual derivatives Q and D enter crucially into the construction of a Lagrangian theory with G_3 invariance. Section 6 establishes Q and D as the generators of the left and right actions of G_3 by introducing a suitable exponentiation of the first order formulas. Section 7 extends our results for $n = 3$ to the general case and includes the proof of the exponentiation for general n.

2 Fractional supersymmetry transformations: the case of G_3

The simplest version of ordinary supersymmetry deals with the transformation

$$t' = t + \tau + i\epsilon\theta \quad , \quad \theta' = \theta + \epsilon \quad . \tag{2.1}$$

This $\mathcal{Z}_2$-graded theory contains a time variable t and a parameter τ of grade zero, and a real Grassmann number θ and parameter ϵ of grade one. Thus

$$\theta = \bar{\theta} \quad , \quad \theta^2 = 0 \quad ; \quad \epsilon = \bar{\epsilon} \quad , \quad \epsilon^2 = 0 \quad ; \quad \theta\epsilon = -\epsilon\theta \quad . \tag{2.2}$$

We consider generalisation to a situation involving a single real Grassmann variable θ, such that $\theta = \bar{\theta}$, $\theta^n = 0$, $n = 2, 3, 4...$, within a theory that possesses $\mathcal{Z}_n$-grading; the case $n = 3$ provides the simplest non-trivial generalisation of ordinary supersymmetry. Without loss of generality, we take θ to have grade one in the $\mathcal{Z}_3$-grading, and to obey

$$\theta = \bar{\theta} \quad , \quad \theta^3 = 0 \quad . \tag{2.3}$$

The $\mathcal{Z}_3$-generalisation of (2.1) is then given by

$$t \to t' = t + \tau + \xi(\epsilon, \theta) \quad , \quad \theta \to \theta' = \theta + \epsilon \quad , \tag{2.4}$$

where

$$\xi(\epsilon, \theta) = q(\epsilon\theta^2 + \epsilon^2\theta) \quad , \tag{2.5}$$

t and τ are as in (2.1), ϵ is a real grade one parameter, such that $\epsilon = \bar{\epsilon}$, $\epsilon^3 = 0$,

$$\epsilon\theta = q^{-1}\theta\epsilon \quad . \tag{2.6}$$

and q is a complex cube-root of unity. For definiteness we take $q = \exp(2\pi i/3)$; replacing q by q^{-1} in (2.5) and (2.6) would modify only slightly the appearance of the expressions written below, but not their content. Eq. (2.6) ensures that the two terms of (2.5), in addition to being of overall grade zero, are real, *e.g.*

$$\overline{q\epsilon\theta^2} = q^{-1}\theta^2\epsilon = q^{-1}q^2\epsilon\theta = q\epsilon\theta^2 \quad .$$

The fact that (2.4) describes a group G_3 of transformations is easy to check. Applying two transformations $g = (\tau, \epsilon)$ and $g' = (\tau', \epsilon')$ to (t, θ) we find $g'' = g'g$ with parameters

$$\epsilon'' = \epsilon' + \epsilon \quad , \quad \tau'' = \tau' + \tau + q(\epsilon'\epsilon^2 + \epsilon'^2\epsilon) \equiv \tau' + \tau + \xi(\epsilon', \epsilon) \tag{2.7}$$

where, in analogy with (2.6), we have

$$\epsilon'\epsilon = q^{-1}\epsilon\epsilon' \quad . \tag{2.8}$$

In fact, we may view (2.4) as the (left) action of the element $g \in G_3$ on a $\mathcal{Z}_3$-graded physical 'manifold' M, of 'coordinates' (t, θ), given by

$$g : (t, \theta) \mapsto (t'\theta') \quad , \quad \theta' = \theta + \epsilon \quad , \quad t' = t + \tau + \xi(\epsilon, \theta) \quad ; \tag{2.9}$$

likewise, we may view (2.7) as describing the left action of $g' = (\tau', \epsilon')$ on the G_3 group itself.

The unit and inverse elements are given by (0,0) and $(-\tau, -\epsilon)$. The associativity of the group law $g''(g'g) = (g''g')g$ is easily checked, and, in fact, it follows from two-cocycle condition

$$\xi(\epsilon'', \epsilon') + \xi(\epsilon'' + \epsilon', \epsilon) = \xi(\epsilon'', \epsilon' + \epsilon) + \xi(\epsilon', \epsilon) \quad , \tag{2.10}$$

in which $\epsilon, \epsilon', \epsilon''$ are the grade one parameters of three transformation performed in succession, and which holds for the $\xi(\epsilon', \epsilon)$ given in (2.7) provided that

$$\epsilon''\epsilon' = q^{-1}\epsilon'\epsilon \quad , \quad \epsilon'\epsilon = q^{-1}\epsilon\epsilon' \quad . \tag{2.11}$$

As is well known (see, *e.g.* [33]), two-cocycles are associated with central extensions of a Lie group . In their Lie algebra formulation they correspond to a curvature two-form (which is symmetric rather than antisymmetric in the case of supersymmetry, see [27]). The structure of the fractional supersymmetry group opens the possibility of extending these concepts to a (here) ternary algebra by introducing a 'curvature' three-form (*cf.* [12]).

To exhibit the origin of (2.6), (2.8) and (2.11), we observe that in any context where such results arise, there is a natural ordering of the numbers of non-zero grading that enter it. It will further be seen that this ordering determines consistently (and always according to the same pattern) the powers of q that enter the required multiplicative relations. In the case of group multiplication, the above ordering (in *symbolic* notation $\epsilon' > \epsilon > \theta$) requires

$$\epsilon'\epsilon = q^{-1}\epsilon\epsilon' \quad , \quad \epsilon\theta = q^{-1}\theta\epsilon \tag{2.12}$$

used above. To these, we add the result $\epsilon'\theta = q\theta\epsilon'$ (see below). For all three, one passes from the lexical order to the opposite one by using relations that use the same power of q, here q^{-1}, in the same places. In the discussion of associativity, the ordering $\epsilon'' > \epsilon' > \epsilon$ similarly implies the results (2.11) used above, and in addition $\epsilon''\epsilon = q^{-1}\epsilon\epsilon''$.

Similarly, if we had elect to write our fractional supersymmetry transformation as

$$\theta' = \theta + \eta \quad , \quad t' = t + \tau + q^2(\eta\theta^2 + \eta^2\theta) \quad , \tag{2.13}$$

the reality of t' would now imply

$$\eta\theta = q\theta\eta \quad , \tag{2.14}$$

and, for the ordering $\eta' > \eta > \theta$, the same rule would govern matters but with the power q as in (2.14), and in $\eta'\eta = q\eta\eta'$. However, (2.13) is equivalent to

$$\theta' = \theta + \eta \quad , \quad t' = t + \tau + q(\theta^2\eta + \theta\eta^2) \quad , \tag{2.15}$$

so that we prefer the ordering $\theta > \eta > \eta'$, and write

$$\theta\eta = q^{-1}\eta\theta \quad , \quad \eta\eta' = q^{-1}\eta'\eta \quad , \quad \theta\eta' = q^{-1}\eta'\theta \quad . \tag{2.16}$$

This is now in full conformity with the other examples discussed Further, just as our discussion related to $\epsilon' > \epsilon > \theta$ is appropriate to the case of left transformations, the passage involving (2.15) and $\theta > \eta > \eta'$ is seen to be similarly suited to the discussion of right translations. The results (2.15) and (2.16) are indeed so employed in section six.

One consequence of results of the type (2.8) is in the form of q-deformed binomial expansions. For example,

$$(\epsilon' + \epsilon)^m = \sum_{t=0}^{r} \begin{bmatrix} m \\ t \end{bmatrix} \epsilon'^t \epsilon^{m-t} \, . \tag{2.17}$$

The braced object here is the q-analogue of the ordinary binomial coefficient, in which ordinary factorials, *e.g.* $m!$, are replaced by

$$[m]! = [m][m-1]\cdots[1] \quad , \quad [m] \equiv \frac{1-q^m}{1-q} = 1 + q + \cdots + q^{m-1} \quad . \tag{2.18}$$

It is easy to see and well-known that (2.17) indeed follows by use of (2.8). Results such as (2.17) are employed in section three.

We append our notation for q-deformed exponentials for use in sections six and seven. We write

$$\exp(q^k; X) = \sum_{m=0}^{\infty} \frac{1}{[m; q^k]!} X^m \quad , \tag{2.19}$$

for suitable k, where

$$[m; q^k]! = [m; q^k] \cdots [2; q^k][1] \quad , \quad [m; q^k] \equiv \frac{1-q^{km}}{1-q^k} \quad . \tag{2.20}$$

In this notation $[m]$ in (2.18) is $[m; q]$.

3 The transformation formula for G_n

We now extend the work done in the previous section on G_3 to the $\mathcal{Z}_n$-graded case which employs a single real Grassmann number θ and an associated parameter ϵ with the properties

$$\theta = \bar{\theta} \quad , \quad \theta^n = 0 \quad ; \quad \epsilon = \bar{\epsilon} \quad , \quad \epsilon^n = 0 \quad ; \quad \epsilon\theta = q^{-1}\theta\epsilon \quad . \tag{3.1}$$

It is understood that no power of θ or ϵ lower than the n-th can vanish.

We retain the general structure (2.4) for G_n but seek, for the cocycle ξ, a formula of the type

$$\xi(\epsilon', \epsilon) = \sum_{r=1}^{n-1} c_r \epsilon'^{\,r} \epsilon^{n-r} q^{\omega(r)} \quad ,$$

$$q = \exp(2\pi i/n) \quad , \quad n = 2, 3, 4... \quad , \tag{3.2}$$

so that $q^n = 1$ replaces $q^3 = 1$ in previous work. Also, the exponent of q shown in (3.2) namely

$$\omega(r) = \frac{1}{2} r(n-r) \quad , \tag{3.3}$$

ensures using (2.8) that each term of (3.2) is real if c_r is real. We set $c_1 = 1$, and rewrite (3.2) as

$$\xi(\epsilon', \epsilon) = \sum_{r=1}^{n-1} d_r \epsilon'^{\,r} \epsilon^{n-r} \quad . \tag{3.4}$$

We must determine the numbers d_r in such a way that (2.10) is satisfied, so that when ξ is given by (3.2) and (3.3), eq. (2.4) has the required G_n group multiplication properties. First we note that the terms on the two sides of (2.10) that are independent of ϵ agree. Then, with the aid of results like (2.17), we can show that consideration of the terms of (2.10) linear in ϵ allow us to determine all the d_r as multiples of d_1. Explicitly we find $d_r = d_{n-r}$ and

$$\frac{d_r}{d_1} = \frac{[n-1]!}{[r]![n-r]!} \quad , \quad r = 1, 2, \ldots, n-1. \tag{3.5}$$

Now that (3.4) is fully determined by (3.5) we must prove that (2.10) is identically satisfied. Thus, we use (3.4), (3.5) and (2.17) to obtain

$$\xi(\epsilon'', \epsilon' + \epsilon) = \sum_{r=1}^{n-1} \sum_{s=0}^{n-r} \frac{\epsilon''^{\,r} \epsilon'^{\,s} \epsilon^{n-r-s} d_1 [n-1]!}{[r]![s]![n-r-s]!} \quad . \tag{3.6}$$

We now observe that $\xi(\epsilon', \epsilon)$ differs only slightly from what will provide the r=0 of the r.h.s. of (3.6). In fact, we can write

$$\xi(\epsilon'', \epsilon' + \epsilon) + \xi(\epsilon', \epsilon) = \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{\epsilon''^{\,r} \epsilon'^{\,s} \epsilon^{n-r-s} d_1 [n-1]!}{[r]![s]![n-r-s]!} - d_1(\epsilon''^{n} + \epsilon'^{n} + \epsilon^n)/[n] \quad . \tag{3.7}$$

Here, in order to make a tractable double sum we have added and subtracted certain ill-defined terms. The procedure is necessary to expedite the key step of our proof. In this, we reverse the order of summations in (3.7) obtaining

$$\sum_{r=0}^{n}\sum_{s=0}^{r} = \sum_{s=0}^{n}\sum_{r=s}^{n} = \sum_{s=0}^{n}\sum_{u=0}^{n-s} \quad ,$$

where a shift in the variable of summation r to $u = r - s$ has also been made. The result so obtained for the left side of (2.10) can now be shown to agree exactly with the analogue of (3.7) obtained by direct calculation of the r.h.s. of (2.19), completing the required demonstration.

The remaining ingredients of the group multiplication laws for G_n are attended to immediately. Indeed, an additional calculation to prove associativity is not needed, since the cocycle property guarantees it.

4 Derivatives with respect to θ

We want to move from the description of the group properties of the fractional supersymmetry transformation towards the construction of actions and dynamical systems that possess invariance properties relative to them. This requires a geometrical understanding of the derivatives $\partial/\partial\theta$, and of objects in the theory of covariant derivative type. Let us go back to G_3, again as a good example, aiming in particular to expose and treat the conceptual difficulties that occur in discussing reality properties of $\partial/\partial\theta$. It is sufficient for the purposes of this section, although not of course for the eventual construction of Lagrangian theories, to work with scalar, *i.e.* grade zero real superfields f, whose expansion in powers of θ involves three real terms (see *e.g.* [11, 15, 16, 18])

$$f = x + q\alpha\theta + q\beta\theta^2 \quad = \quad \bar{f} = x + q^2\theta\alpha + q^2\theta^2\beta \quad , \tag{4.1}$$

in which x is a grade zero (bosonic) variable, and the variables α and β are of grades two and one. The reality of f expressed by (4.1) implies the properties

$$\theta\beta = q\beta\theta \quad , \quad \theta\alpha = q^{-1}\alpha\theta \quad . \tag{4.2}$$

Comparing (4.1) with (2.5) now seen to be of scalar superfield nature, we see that β, of grade one, is related like ϵ to θ, so that $\beta > \theta$, and α, of grade two, is likewise related to ϵ^2, so that $\alpha < \theta$. The latter implies that we should adopt the rule $\beta\alpha = q^{-1}\alpha\beta$, although in this section no call for any such result is made.

To prepare the ground for our discussion of derivatives in the $\mathcal{Z}_3$ case, we recall briefly the case of basic supersymmetry and $\mathcal{Z}_2$-grading (eq. (11)) for which a the real scalar field has the expansion

$$f = x + i\theta\phi = x - i\phi\theta \quad . \tag{4.3}$$

It is normal to use the left spinorial derivative so that

$$\frac{\partial f}{\partial \theta} = \partial f \equiv \partial_L f = i\phi \quad , \tag{4.4}$$

and to employ $\frac{\partial \theta}{\partial \theta} = 1$ and

$$\theta \partial + \partial \theta = 1 \tag{4.5}$$

to do routine manipulations. Since (4.4) is not real for real f there is no case for viewing ∂ as a real entity. However one did not consider using such an idea as a guide towards (4.5). Eq. (4.5) is valid because it holds applied to an arbitrary superfield f. In fact, the right spinorial derivative ∂_R can be consistently viewed as a conjugate to ∂_L via

$$\overline{\partial_L f} \equiv \overline{\frac{\partial f}{\partial \theta}} \equiv f \frac{\overleftarrow{\partial}}{\partial \theta} = \partial_R f \quad , \tag{4.6}$$

which agrees trivially with

$$\partial_L f = i\phi \quad , \quad \partial_R f = -i\phi \quad .$$

Similarly, by application to arbitrary f it follows that the conjugate of (4.5),

$$\partial_R \theta + \theta \partial_R = 1 \quad ,$$

makes good sense.

Returning to the $\mathcal{Z}_3$ case, we see that to compute $\frac{\partial f}{\partial \theta} \equiv \partial f \equiv \partial_L f$ and $\partial_R f$, we need the $\mathcal{Z}_3$-analogue of (4.5) to treat the θ^2 terms of (4.1). We begin by postulating

$$\frac{\partial \theta}{\partial \theta} = 1 \tag{4.7}$$

and a result of the type

$$\partial \theta = a \theta \partial + b \quad , \tag{4.8}$$

in which $a, b \in C$. Eq. (4.7) is certainly natural. We discuss whether it can or needs to be modified (it doesn't) below. When (4.8) is applied to 1, then (4.7) implies $b = 1$. Applied to θ, eq. (4.8) yields

$$(\partial \theta^2) = (1 + a)\theta \quad , \tag{4.9}$$

Then, using (4.1), we get

$$\frac{\partial f}{\partial \theta} = \partial_L f = q^2 \alpha + q^2 (1 + a) \theta \beta \quad , \tag{4.10}$$

which is not real for real f. To complete the specification of (4.8), we stipulate that it must be a true when applied to an arbitrary scalar superfield. It is easy to see that it does so if

$$1 + a + a^2 = 0 \quad . \tag{4.11}$$

Thus we find two solutions for a; the two corresponding candidates for the derivative with respect θ are both used in the literature and, as we see below, essential. If $a = q$, we shall write ∂ for the derivative that obeys

$$\frac{\partial\theta}{\partial\theta} = 1 \quad , \quad \partial\theta = q\theta\partial + 1 \quad , \quad [\partial, \theta]_q = 1 \quad . \tag{4.12}$$

If $a = q^{-1}$, we write δ, and

$$\frac{\delta\theta}{\delta\theta} = 1 \quad , \quad \delta\theta = q^{-1}\theta\delta + 1 \quad , \quad [\delta, \theta]_{q^{-1}} = 1 \quad . \tag{4.13}$$

Also $\partial\delta = q^{-1}\delta\partial$ (or $[\partial, \delta]_{q^{-1}} = 0$). Both derivatives hereby introduced are acting from the left. Neither has any natural reality properties that can be uncovered without reference to their partner right derivatives. The above is sufficient for our own intended applications. However, variations in the literature exist, and are often associated with implicit assumptions hinting at reality properties of ∂. If one uses (4.8) with or without (4.7) and without reference to the requirement that, applied to an arbitrary superfield, it holds good, one might try to complete specification of (4.8) by demanding that its correctness ensures the correctness of its adjoint. However, if one assumes ∂ is in some sense real (which we do not believe to be a tenable view) then (4.8) implies successively

$$\begin{aligned} \theta\partial &= \bar{a}\partial\theta + \bar{b} \quad , \\ a\theta\partial &= a\bar{a}\partial\theta + a\bar{b} \quad , \text{ calling for } \quad a\bar{a} = 1 \quad , \\ \partial\theta &= a\theta\partial - a\bar{b} \quad , \end{aligned}$$

reproducing (4.8) when $b = -a\bar{b}$. The choice $b = 1$, the natural choice, implies $a = -1$, and we are forced back to the $\mathcal{Z}_2$-supersymmetry result as the only non-trivial possibility. If one tries a choice like $a = q$, then $b = iq^{1/2}r$, $r \in R$, so that

$$\partial\theta = q\theta\partial + irq^{1/2} \quad .$$

Application of this result to an arbitrary real scalar f fails to give an identity. So also does *any* attempt to view $c\partial$ as a conjugate to ∂, for $c \in C$.

In fact, it is sensible to view ∂_R as the conjugate of ∂. Since doing so is independent of whether one is looking at ∂ or δ, it is sufficient to give details for the former. Thus we shall employ here (4.12) and (4.10). We take $\partial_R\theta = (\theta\frac{\overleftarrow{\partial}}{\partial\theta}) = 1$ and, from (4.9), by conjugation, deduce

$$(\partial_R\theta^2) \equiv (\theta^2\frac{\overleftarrow{\partial}}{\partial\theta}) = (q^2 + 1)\theta \quad .$$

Then, from (4.1), we obtain

$$\partial_R f = (f \frac{\overleftarrow{\partial}}{\partial\theta}) = q\alpha + q\beta(1+q^2)\theta = \overline{\partial_L f} \tag{4.14}$$

where (4.2) and (4.10) for $\partial f \equiv \partial_L f$ have been used. Similarly, a consistent picture for δ, δ_R emerges. So, in summary, if on rare occasions one needs a conjugate for ∂, one may not use any multiple of ∂, although ∂_R serves perfectly well. Neither δ nor δ_R are satisfactory candidates for the rôle of the conjugate of ∂. We note in passing that the fact that a variable and the derivative with respect to it cannot be made *simultaneously* real (or hermitian) is a known feature of non-commutative geometry and has been discussed in completely different contexts (see, *e.g.* [34]).

We note that the result

$$q\partial_L\partial_R f = \partial_R\partial_L f \quad ,$$

treated with care, also makes sense, but forbear from appending any remark about ordering.

We return finally to (4.7). It is not obviously wrong to let $(\partial\theta) = c, c \in C$, but (4.7) clearly remains the natural choice. With ∂_R, rather than any multiple of ∂, seen as the true conjugate of ∂, we are not aware of any compelling reason for using $c \neq 1$.

5 Covariant derivative objects

The derivatives ∂ and δ discussed feature in the literature on fractional supersymmetry in the definition (see *e.g.* [11, 15, 18]) of the important quantities

$$Q = \partial_\theta + q\theta^2\partial_t \quad , \tag{5.1}$$

$$D = \delta_\theta + q^2\theta^2\partial_t \quad ; \tag{5.2}$$

Q produces the first order generalised supersymmetry transformation. We then write

$$\delta_{(\epsilon)} f = \epsilon Q f \quad . \tag{5.3}$$

Eq. (5.3) implies the superfield component transformations

$$\begin{aligned} \delta_{(\epsilon)} x &= q^2\epsilon\alpha \quad , \\ \delta_{(\epsilon)} \alpha &= -q\epsilon\beta \quad , \\ \delta_{(\epsilon)} \beta &= \epsilon\dot{x} \quad . \end{aligned} \tag{5.4}$$

We note that the 'θ^2 component' of f changes by a total time derivative. Proceeding from this remark towards the construction of actions, we realise that D has been defined in (5.2) and in relation to (5.1) in such a way that Df has the same transformation law as f. It follows that the same philosophy as worked for supersymmetry will enable us to construct actions with the correct invariance properties under generalised supersymmetry transformations. We just take the 'θ^2 component' of a suitable product, of the correct dimensions, of superfields such as $f\,,\dot{f}\,,Df$ etc. For example (*cf.* [15, 16, 18])

$$S=\int dt\,\frac{1}{2}\,i\,f\,D\,f\,|_{\theta^2}=\int dt\,L \quad , \tag{5.5}$$

$$L=\frac{1}{2}\dot{x}\dot{x}+\frac{1}{2}q^2\dot{\beta}\alpha-\frac{1}{2}q\dot{\alpha}\beta \quad . \tag{5.6}$$

Exposition of the canonical formalism that stems from (5.6) is neither problem free in quantum mechanics, nor in existence at all at the present time to our knowledge in classical mechanics. We may expect, as is the case in $\mathcal{Z}_2$-supersymmetry where symmetric Poisson brackets are associated with anticommutators, that both formalisms, classical (fractional pseudomechanics) and quantum, are closely related. We intend to present a discussion of these questions elsewhere.

The important rôles of Q and D having been put into evidence, we note that Df will transform like f provided that

$$\delta_{(\epsilon)}D \;=\; D\delta_{(\epsilon)} \quad , \tag{5.7}$$

or

$$\epsilon\,Q\,D \;=\; D\,\epsilon\,Q \quad . \tag{5.8}$$

Since (2.6) implies $\theta^2\,\epsilon=q^2\,\epsilon\,\theta^2$, and hence

$$q\,D\,\epsilon=\epsilon\,D\,,$$

we deduce that (5.7) requires

$$D\,Q=q\,Q\,D \quad , \quad [D,Q]_q=0 \quad . \tag{5.9}$$

The consistency of (5.8) as an operator identity demands that, when applied to an arbitrary f, it gives a superfield identity. The choice (5.2) shows this to be satisfied.

To conclude, we note the further well-known results (*cf.* [11, 15, 18])

$$D^3=-\partial_t \quad , \quad Q^3=-\partial_t \quad , \tag{5.9}$$

which are most easily seen as identities by applying them to arbitrary f.

6 Left and Right Transformations

What governed the choices (5.1), (5.2)? In the case of (5.1) the application of ϵQ to (t,θ) does reproduce (2.4) to *first* order in ϵ. This however does not allow ∂ to be preferred to δ in (5.1), nor conversely. Once (5.1) has been chosen, as seems sensible enough, it is quite easy to find a derivative in the form (5.2) that satisfies (5.8). However, this choice has a deep geometrical interpretation. In fact, we now show that Q and D can be regarded as the generators of the left and right actions of the group G_3 on the physical 'manifold' M of (2.9), and that (5.8) expresses the fact that left and right actions commute. This geometrical picture is a nontrivial generalisation of one that applies to ordinary supersymmetry where, of course, both actions are linear.

Let us denote the parameters of the left and right transformations of G_3 as ϵ and η. Thus $\epsilon > \theta > \eta$, as discussed in section two, and hence

$$\epsilon\theta = q^{-1}\theta\epsilon \quad , \quad \theta\eta = q^{-1}\eta\theta \quad . \tag{6.1}$$

We define the left and right actions $L_{(\epsilon)}$ and $R_{(\eta)}$ by

$$L_{(\epsilon)} : \theta \mapsto \theta' = \epsilon + \theta \quad , \quad t \mapsto t' = t + \tau + q(\epsilon\theta^2 + \epsilon^2\theta) \quad , \tag{6.2}$$

$$R_{(\eta)} : \theta \mapsto \theta' = \theta + \eta \quad , \quad t \mapsto t' = t + \tau + q(\theta^2\eta + \theta\eta^2) \quad , \tag{6.3}$$

which agree with (2.4) and (2.15). It is a non-trivial result is that these transformations may written as exponentials of the generators Q and D respectively

$$L(\epsilon)\, t = \exp\,(q^{-1}\,; \epsilon Q)\, t \quad , \tag{6.4}$$

$$R_{(\eta)}\, t = \exp\,(q\,; \eta D)\, t \quad , \tag{6.5}$$

where we have used the notation of (2.19). The proof, which due to the ordering necessarily involves distinct deformed exponentials, is given below. The commutativity of the two actions implies

$$[\epsilon Q\ ,\ \eta D] = 0 \quad . \tag{6.6}$$

This requires the consequences

$$D\epsilon = q^{-1}\epsilon D \quad , \quad \eta Q = q^{-1}Q\eta \tag{6.7}$$

of (6.1), and a hitherto unused relation

$$\epsilon\eta = q^{-1}\eta\epsilon \quad . \tag{6.8}$$

Then (6.6) is seen to imply (5.8).

To prove (6.4), we use (2.19) in the form

$$\exp(q^{-1}; \epsilon Q) = 1 + \epsilon Q + \epsilon Q\epsilon Q/[2\,;\ q^{-1}] \quad . \tag{6.9}$$

A simple computation using $\epsilon\theta = q^{-1}\theta\epsilon$ and $(\partial\theta^2) = [2; q]\theta$ gives us (6.4); the proof depends crucially on the former and on the occurrence of ∂ rather than δ in the definition (5.1) of Q.

We prove (6.5) in the same way, noting again how critically the success of the proof depends on the actual arrangement of details involving θ, δ, η and $\exp\,(q\ ;\ \eta D)$.

7 Superfields, derivatives and q-exponentiation for G_n

We have given already in section three, the definition of the transformation of the group G_n when

$$\theta = \bar{\theta} \quad , \quad \theta^n = 0 \quad ; \quad \epsilon = \bar{\epsilon}\ , \epsilon^n = 0 \quad ; \quad \theta\epsilon = q\epsilon\theta \quad . \tag{7.1}$$

In general we expect most features of the $\mathcal{Z}_3$ - graded theory that are discussed above allow fairly direct extension to the $\mathcal{Z}_n$ theory. We will indicate some of these briefly in this section, without examining in much detail how the general case may yield a theory significantly richer in content.

In place of (4.1), we have the expansion of the real scalar superfield

$$f = x + \sum_{r=1}^{n-1} q^{\omega(r)}\psi_r\theta^{n-r} = \bar{f} = x + \sum_{r=1}^{n-1} q^{-\omega(r)}\theta^{n-r}\psi_r \quad , \tag{7.2}$$

where the power $\omega(r)$ of $q = \exp(2\pi i/n)$ is chosen to make all terms in f all real; it is given by (3.3). Moreover, in place of (4.2), we now have

$$\theta\psi_r = q^r\psi_r\theta\ ,\ r = 1, \ldots, n-r \quad ; \tag{7.3}$$

in the $\mathcal{Z}_3$ case, α and β of (4.1) would be written as ψ_2 and ψ_1 to conform with (7.2).

The discussion of derivatives, via (7.1) and

$$\partial\theta = a\theta\partial + 1 \quad , \tag{7.4}$$

yields more possibilities, for a must now obey

$$1 + a + a^2 + \ldots + a^{n-1} = 0 \quad . \tag{7.5}$$

We thus write ∂_r for the derivative which obeys

$$\partial_r\ \theta = q^r\ \theta\ \partial_r + 1 \quad , \quad r = 1, 2 \ldots n-1 \quad . \tag{7.6}$$

Our previous ∂ and δ correspond to ∂_1 and ∂_{n-1}. We will continue to use the former notation because we do not describe any context that involves crucial use of ∂_r for $r \neq 1$ or $r \neq n-1$. We note, in particular, the direct consequences of (7.6)

$$(\partial\,\theta^s) = [s]\theta^{s-1} \quad , \quad (\delta\,\theta^s) = q^{1-s}[s]\theta^{s-1} \quad , \quad s = 1, 2, \ldots, n-1 \quad . \tag{7.7}$$

Here δ directly involves $(1-q^{-s})/(1-q^{-1})$, which we have expressed in terms of $[s]$, defined by (2.18).

The definitions (5.1) and (5.2) of Q, D in section five are now modified to read

$$Q = \partial_1 + q^{\omega(1)}\theta^{n-1}\partial_t \equiv \partial + q^{\omega(1)}\theta^{n-1}\partial_t \quad , \tag{7.8}$$

$$D = \partial_{n-1} + q^{-\omega(1)}\theta^{n-1}\partial_t \equiv \delta + q^{-\omega(1)}\theta^{n-1}\partial_t \quad , \tag{7.9}$$

where $\omega(1)$ is given by (3.3). We note that Q, so defined, does generate correctly the first order term of the G_n-transformation for a parameter ϵ related to θ via (7.1). Also

$$\delta_{(\epsilon)}\psi_1 = \epsilon\dot{x} \tag{7.10}$$

indicates that we can still follow the usual way of obtaining invariant actions from the θ^{n-1} components of suitable superfields. Further (5.8) again holds. But in place of (5.9), we use n-th powers: the $\mathcal{Z}_n$ theory is of fractional supersymmetry with fractions $1/n$, of course.

Finally, it is to be expected that the exponentiation results of section six carry over into the general theory. We rewrite then in general notation

$$L_{(\epsilon)}t \mapsto t' = \exp(q^{-1}; \epsilon Q)\, t \quad , \tag{7.11}$$

$$R_{(\eta)}t \mapsto t' = \exp(q; \eta D)\, t \quad , \tag{7.12}$$

where

$$\theta\epsilon = q\epsilon\theta \quad , \quad \eta\theta = q\theta\eta \quad , \quad \eta\epsilon = q\epsilon\eta \quad . \tag{7.13}$$

To prove the extension (7.11) of (6.4) to the $\mathcal{Z}_n$-graded theory we need to recover

$$L_{(\epsilon)} : t \to t' = t + \sum_{r=1}^{n-1} d_r \epsilon^r \theta^{n-r} \quad , \tag{7.14}$$

where d_r is given by (3.5), as the expansion of (7.11)

$$L_{(\epsilon)}t = t + \sum_{r=1}^{n} \frac{1}{[r; q^{-1}]!}(\epsilon Q)^r t \quad , \tag{7.15}$$

where Q is given by (7.8). The first order term, which comes from the action of ϵQ on t is clearly correct:

$$d_1 \epsilon\theta^{n-1} = (\epsilon Q)t \quad , \tag{7.16}$$

since $d_1 = c_1 q^{\omega(1)} = q^{\omega(1)}$.

This is first key element of a proof, by induction, that the individual terms of (7.14) and (7.15) coincide. We therefore assume this for $r = 1, 2, ..., k$ and seek, on the basis of that assumption, to prove it for $r = k + 1$. This requires us to show that

$$d_{k+1}\epsilon^{k+1}\theta^{n-k-1} = \frac{1}{[k+1;q^{-1}]}(\epsilon Q)d_k\epsilon^k\theta^{n-k} \quad . \tag{7.17}$$

Only the term $\partial_1 \equiv \partial$ of Q contributes. The formula $\partial\epsilon^k = q^{-k}\epsilon^k\partial$ then prepares for the use of (7.7), and we can see that (7.17) is an equality provided that

$$\frac{1}{[k+1;q]} = \frac{q^{-k}}{[k+1;q^{-1}]} \quad . \tag{7.18}$$

It is easy to show that (7.18) is true, and the proof is complete.

Proof of (7.12) proceeds similarly. We remark that the exponentials in (7.11) and (7.12) are necessarily different because of the use of the different derivatives ∂ and δ in the definitions (7.8) and (7.9) of Q and D.

Acknowledgements. This research has been supported in part by a grant from the CICYT, Spain. A.J. Macfarlane also wishes to thank the Department of Theoretical Physics of Valencia University for its hospitality.

References

[1] E. Witten, Nucl. Phys. **B138**, 513-554 (1981).

[2] M. De Crombrugghe and V. Rittenberg, Ann. Phys. **151**, 99-126 (1983).

[3] L.É. Gendenshtein and I.V. Krive, Sov. Phys. Usp. **28**, 645-666 (1985).

[4] M. Sakamoto, Phys. Lett. **151 B** 115-118 (1985).

[5] R. Floreanini and L. Vinet, Lett. Math. Phys. **8** 439-449 (1984).

[6] H. Braden, Nucl. Phys. **B 291** 516-534 (1982).

[7] R. A. Coles and G Papadopoulos; Class. Quantum Grav. **11** 427-438 (1990).

[8] R. H. Rietdijk, Ph. D. Thesis, Amsterdam 1992, *Applications of Supersymmetric Quantum Mechanics.*

[9] A. J. Macfarlane, Nucl. Phys. **438** 455-468 (1995).

[10] L. Baulieu and E. G Floratos, Phys. Lett. **B258** 171-8 (1991).

[11] C. Ahn, D. Bernard and A. Leclair, Nucl. Phys. **B346** 409-439 (1990).

[12] R. Kerner, J. Math. Phys. **33**, 403-411 (1992); *$\mathcal{Z}_3$-Grading and ternary algebraic structures*, in *Symmetries in science*, B. Gruber ed. Plenum, (1993) p. 373-388.

[13] A.T. Filippov, A. P. Isaev and R. D. Kurdikov, Mod. Phys. Lett. **A 7** 2129-2141 (1992).

[14] S. Durand, Phys. Lett. **B312** 115-120 (1993).

[15] S. Durand, Mod. Phys. Lett. **A8** 2323-2334 (1993) and *ibid.* 1795-1804 (1993).

[16] N. Debergh, J. Phys. **26** 7219-7226 (1993).

[17] N. Mohammedi, *Fractional Supersymmetry*, hep-th/9412133.

[18] L. P. Colatto and J.L. Matheus-Valle, *On q-deformed Supersymmetric Classical Mechanical Models*, ICTP-Trieste, preprint, April 1995.

[19] V.A. Rubakov and V.P. Spirodonov, Mod. Phys. Lett. **A3** 1332-1347 (1988).

[20] R. Floreanini and L Vinet, Phys. Rev. **D44** 3851-6 (1991).

[21] J. Beckers and N. Debergh, Nucl. Phys. **B 340** 767-776 (1991).

[22] A. Khare, J. Phys. **25** L1749-L1754 (1992).

[23] A. Khare, J. Math. Phys. **34** 1274-1294 (1993).

[24] J. Beckers, N. Debergh and A.G. Nitikin, J. Math. Phys. **33** 3387-3392 (1992); Fortschr. Phys. **43**, 67-80, 81-96 (1995).

[25] N. Debergh, J. Math. Phys. **34** 1270-1276 (1992).

[26] N. Debergh, J. Phys.**A27** L 213-217 (1994).

[27] V. Aldaya and J. A. de Azcárraga, J. Math. Phys. **26**, 1818-1821 (1985).

[28] H. S. Green, Phys. Rev. **90** 270-273 (1951).

[29] Y. Ohnuki and S. Kamefuchi, *Quantum Field Theory and Parastatistics*, Springer, Berlin, (1982).

[30] M. Arik and D.D. Coon, J. Math. Phys. **17** 524- 7 (1976).

[31] A. J. Macfarlane, J. Phys. **22** 4581-4588 (1989).

[32] L. C. Biedenharn, J. Phys. **22** L 873-8 (1989).

[33] J. A. de Azcárraga and J.M. Izquierdo, *Lie groups, Lie algebras, cohomology and some applications in physics*, CUP (1995).

[34] O. Ogievetsky, W. B. Schmidke, J. Wess and B. Zumino, Commun. Math. Phys. **150**, 495-518 (1992); O. Ogievetsky and B. Zumino, Let. Math. Phys. **25**, 121-130 (1992).