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# THE EQUATIONS OF SOME DISPERSIONLESS LIMIT

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Abstract. These equations are the generalized equations of several dispersionless equations. A complete table for $p \le 10$ is provided.

## 1. Introduction

It is well-known that a lot of nonlinear solitonic equations can be transformed into certain Hirota type bilinear equations [17]. The $\tau$-function of the KP hierarchy can be characterized by the Hirota equations and the Plücker relations are given from these equations. The differential Fay identity which has quasi-classical limit, is a part of the Plücker relations. The leading term of the quasi-classical limit of the differential Fay identity satisfies an identity [22] and from the identity EQUATION$(\cdot,\infty)$ is extracted.* Therefore, EQUATION$(\cdot,\infty)$ is a subset of the dispersionless KP hierarchy. EQUATION$(p,q)$ are derived from EQUATION$(p,\infty)$. EQUATION$(\cdot,2)$ can be regarded as a subset of dispersionless KdV hierarchy. We can easily show that EQUATION$(4,3)$ is a dispersionless Boussinesq equation. Therefore, EQUATION$(\cdot,3)$ can be regarded as a subset of dispersionless Boussinesq hierarchy. EQUATION$(\cdot,q)$ for $q > 3$ is a whole new set of dispersionless equations which can be regarded as a subset of new hierarchy which may have some useful application.

## 2. The formula

Let us use $F_{mn}$ instead of $\dfrac{\partial^2}{\partial t_m \partial t_n}(F(t_1,\dots,t_r,\dots))$.

**Definition 2.1.** EQUATION$(p,q)$:

$$\sum_{\substack{0<i_1<\cdots<i_{k_p} \\ (i_1+1)n_{i_1}+\cdots+(i_{k_p}+1)n_{i_{k_p}}=p}} \left( \left(\sum_{j=1}^{k_p} n_{i_j} - 1\right)! \prod_{j=1}^{k_p} \frac{(-F_{1i_j})^{n_{i_j}}}{n_{i_j}!} \right) + \sum_{m+n=p} \frac{F_{mn}}{mn} = 0.$$

where the terms having $\frac{\partial}{\partial t_q},\dots,\frac{\partial}{\partial t_{kq}},\dots$ vanish.

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* (Caution)

This notation is used only for technical simplicity of expression since the formula was found by the author recently (Fall, 1994) [5]. This is another way of using equation numbers. Therefore, the author strongly recommends careful use of the notation until the formula is well-known. (Of course, given notation has no meaning elsewhere like other usual equation numbers.)

## 3. EQUATION$(4, q)$

One can easily show that EQUATION(4,$\infty$) is a dispersionless KP and EQUATION(4,2) is a dispersionless KdV.

Consider a dispersionless Boussinesq equation

$$(uu_x)_x + \frac{1}{2}u_{yy} = 0. \tag{3.1}$$

If we set $t_1 = x$ and $t_2 = y$, then EQUATION(4,3) is

$$\frac{1}{2}(F_{xx})^2 + \frac{1}{4}F_{yy} = 0. \tag{3.2}$$

Differentiate (3.2) with respect to $x$. Then we get

$$F_{xx}F_{xxx} + \frac{1}{4}F_{xyy} = 0. \tag{3.3}$$

Setting $u = 2F_{xx}$, (3.3) becomes

$$\left(\frac{u}{2}\frac{u_x}{2}\right)_x + \frac{1}{4}\left(\frac{u_{yy}}{2}\right) = 0.$$

which is the same as (3.1).

## 4. Discussion

We could get the useful expression of the generalized equations of the dispersionless limit of KdV, KP and Boussinesq equations. And one can get a specific equation for each $(p, q)$. Furthermore, new hierarchies are derived from EQUATION$(\cdot, q)$ for $q > 3$. For further research, a table of equations are provided. By definition, EQUATION$(p, q)$ is the same as EQUATION$(p, \infty)$ for $q \geq p$.

Table. Equations for $(p, q)$.

| | |
|---|---|
| (4,$\infty$) | $\frac{1}{2}{F_{11}}^2 - \frac{1}{3}F_{13} + \frac{1}{4}F_{22} = 0$ |
| (5,$\infty$) | $F_{11}F_{12} - \frac{1}{2}F_{14} + \frac{1}{3}F_{23} = 0$ |
| (6,$\infty$) | $\frac{1}{3}{F_{11}}^3 - \frac{1}{2}{F_{12}}^2 - F_{11}F_{13} + \frac{3}{5}F_{15} - \frac{1}{9}F_{33} - \frac{1}{4}F_{24} = 0$ |
| (7,$\infty$) | ${F_{11}}^2F_{12} - F_{12}F_{13} - F_{11}F_{14} + \frac{2}{3}F_{16} - \frac{1}{6}F_{34} - \frac{1}{5}F_{25} = 0$ |
| (8,$\infty$) | $\frac{1}{4}{F_{11}}^4 - F_{11}{F_{12}}^2 - {F_{11}}^2F_{13} + \frac{1}{2}{F_{13}}^2 + F_{12}F_{14} + F_{11}F_{15}$ $-\frac{5}{7}F_{17} + \frac{1}{16}F_{44} + \frac{2}{15}F_{35} + \frac{1}{6}F_{26} = 0$ |

(9,∞) $F_{11}{}^3F_{12} - \frac{1}{3}F_{12}{}^3 - 2F_{11}F_{12}F_{13} - F_{11}{}^2F_{14} + F_{13}F_{14}$
$+F_{12}F_{15} + F_{11}F_{16} - \frac{3}{4}F_{18} + \frac{1}{10}F_{45} + \frac{1}{9}F_{36} + \frac{1}{7}F_{27} = 0$

(10,∞) $\frac{1}{5}F_{11}{}^5 - \frac{3}{2}F_{11}{}^2F_{12}{}^2 - F_{11}{}^3F_{13} + F_{12}{}^2F_{13} + F_{11}F_{13}{}^2$
$+2F_{11}F_{12}F_{14} - \frac{1}{2}F_{14}{}^2 + F_{11}{}^2F_{15} - F_{13}F_{15} - F_{12}F_{16} - F_{11}F_{17}$
$+\frac{7}{9}F_{19} - \frac{1}{25}F_{55} - \frac{1}{12}F_{46} - \frac{2}{21}F_{37} - \frac{1}{8}F_{28} = 0$

(4,2) $\frac{1}{2}F_{11}{}^2 - \frac{1}{3}F_{13} = 0$

(5,2) $0 = 0$

(6,2) $\frac{1}{3}F_{11}{}^3 - F_{11}F_{13} + \frac{3}{5}F_{15} - \frac{1}{9}F_{33} = 0$

(7,2) $0 = 0$

(8,2) $\frac{1}{4}F_{11}{}^4 - F_{11}{}^2F_{13} + \frac{1}{2}F_{13}{}^2 + F_{11}F_{15} - \frac{5}{7}F_{17} + \frac{2}{15}F_{35} = 0$

(9,2) $0 = 0$

(10,2) $\frac{1}{5}F_{11}{}^5 - F_{11}{}^3F_{13} + F_{11}F_{13}{}^2 + F_{11}{}^2F_{15} - F_{13}F_{15} - F_{11}F_{17}$
$+\frac{7}{9}F_{19} - \frac{1}{25}F_{55} - \frac{2}{21}F_{37} = 0$

(4,3) $\frac{1}{2}F_{11}{}^2 + \frac{1}{4}F_{22} = 0$

(5,3) $F_{11}F_{12} - \frac{1}{2}F_{14} = 0$

(6,3) $\frac{1}{3}F_{11}{}^3 - \frac{1}{2}F_{12}{}^2 + \frac{3}{5}F_{15} - \frac{1}{4}F_{24} = 0$

(7,3) $F_{11}{}^2F_{12} - F_{11}F_{14} - \frac{1}{5}F_{25} = 0$

(8,3) $\frac{1}{4}F_{11}{}^4 - F_{11}F_{12}{}^2 + F_{12}F_{14} + F_{11}F_{15} - \frac{5}{7}F_{17} + \frac{1}{16}F_{44} = 0$

(9,3) $F_{11}{}^3F_{12} - \frac{1}{3}F_{12}{}^3 - F_{11}{}^2F_{14} + F_{12}F_{15} - \frac{3}{4}F_{18} + \frac{1}{10}F_{45} + \frac{1}{7}F_{27} = 0$

(10,3) $\frac{1}{5}F_{11}{}^5 - \frac{3}{2}F_{11}{}^2F_{12}{}^2 + 2F_{11}F_{12}F_{14} - \frac{1}{2}F_{14}{}^2 + F_{11}{}^2F_{15} - F_{11}F_{17}$
$-\frac{1}{25}F_{55} - \frac{1}{8}F_{28} = 0$

(5,4) $F_{11}F_{12} + \frac{1}{3}F_{23} = 0$

(6,4) $\frac{1}{3}F_{11}{}^3 - \frac{1}{2}F_{12}{}^2 - F_{11}F_{13} + \frac{3}{5}F_{15} - \frac{1}{9}F_{33} = 0$

(7,4) $F_{11}{}^2F_{12} - F_{12}F_{13} + \frac{2}{3}F_{16} - \frac{1}{5}F_{25} = 0$

(8,4) $\frac{1}{4}{F_{11}}^4 - F_{11}{F_{12}}^2 - {F_{11}}^2F_{13} + \frac{1}{2}{F_{13}}^2 + F_{11}F_{15} - \frac{5}{7}F_{17} + \frac{2}{15}F_{35} + \frac{1}{6}F_{26} = 0$

(9,4) ${F_{11}}^3F_{12} - \frac{1}{3}{F_{12}}^3 - 2F_{11}F_{12}F_{13} + F_{12}F_{15} + F_{11}F_{16} - +\frac{1}{9}F_{36} + \frac{1}{7}F_{27} = 0$

(10,4) $\frac{1}{5}{F_{11}}^5 - \frac{3}{2}{F_{11}}^2{F_{12}}^2 - {F_{11}}^3F_{13} + {F_{12}}^2F_{13} + F_{11}{F_{13}}^2 + {F_{11}}^2F_{15}$
$-F_{13}F_{15} - F_{12}F_{16} - F_{11}F_{17} + \frac{7}{9}F_{19} - \frac{1}{25}F_{55} - \frac{2}{21}F_{37} = 0$

(6,5) $\frac{1}{3}{F_{11}}^3 - \frac{1}{2}{F_{12}}^2 - F_{11}F_{13} - \frac{1}{9}F_{33} - \frac{1}{4}F_{24} = 0$

(7,5) ${F_{11}}^2F_{12} - F_{12}F_{13} - F_{11}F_{14} + \frac{2}{3}F_{16} - \frac{1}{6}F_{34} = 0$

(8,5) $\frac{1}{4}{F_{11}}^4 - F_{11}{F_{12}}^2 - {F_{11}}^2F_{13} + \frac{1}{2}{F_{13}}^2 + F_{12}F_{14} - \frac{5}{7}F_{17} + \frac{1}{16}F_{44} + \frac{1}{6}F_{26} = 0$

(9,5) ${F_{11}}^3F_{12} - \frac{1}{3}{F_{12}}^3 - 2F_{11}F_{12}F_{13} - {F_{11}}^2F_{14} + F_{13}F_{14} + F_{11}F_{16}$
$-\frac{3}{4}F_{18} + \frac{1}{9}F_{36} + \frac{1}{7}F_{27} = 0$

(10,5) $\frac{1}{5}{F_{11}}^5 - \frac{3}{2}{F_{11}}^2{F_{12}}^2 - {F_{11}}^3F_{13} + {F_{12}}^2F_{13} + F_{11}{F_{13}}^2 + 2F_{11}F_{12}F_{14}$
$-\frac{1}{2}{F_{14}}^2 - F_{12}F_{16} - F_{11}F_{17} + \frac{7}{9}F_{19} - \frac{1}{12}F_{46} - \frac{2}{21}F_{37} - \frac{1}{8}F_{28} = 0$

(7,6) ${F_{11}}^2F_{12} - F_{12}F_{13} - F_{11}F_{14} - \frac{1}{6}F_{34} - \frac{1}{5}F_{25} = 0$

(8,6) $\frac{1}{4}{F_{11}}^4 - F_{11}{F_{12}}^2 - {F_{11}}^2F_{13} + \frac{1}{2}{F_{13}}^2 + F_{12}F_{14} + F_{11}F_{15} - \frac{5}{7}F_{17} + \frac{1}{16}F_{44}$
$+\frac{2}{15}F_{35} = 0$

(9,6) ${F_{11}}^3F_{12} - \frac{1}{3}{F_{12}}^3 - 2F_{11}F_{12}F_{13} - {F_{11}}^2F_{14} + F_{13}F_{14} + F_{12}F_{15}$
$-\frac{3}{4}F_{18} + \frac{1}{10}F_{45} + \frac{1}{7}F_{27} = 0$

(10,6) $\frac{1}{5}{F_{11}}^5 - \frac{3}{2}{F_{11}}^2{F_{12}}^2 - {F_{11}}^3F_{13} + {F_{12}}^2F_{13} + F_{11}{F_{13}}^2 + 2F_{11}F_{12}F_{14}$
$-\frac{1}{2}{F_{14}}^2 + {F_{11}}^2F_{15} - F_{13}F_{15} - F_{11}F_{17} + \frac{7}{9}F_{19} - \frac{1}{25}F_{55} - \frac{2}{21}F_{37} - \frac{1}{8}F_{28} = 0$

(8,7) $\frac{1}{4}{F_{11}}^4 - F_{11}{F_{12}}^2 - {F_{11}}^2F_{13} + \frac{1}{2}{F_{13}}^2 + F_{12}F_{14} + F_{11}F_{15}$
$+\frac{1}{16}F_{44} + \frac{2}{15}F_{35} + \frac{1}{6}F_{26} = 0$

(9,7) ${F_{11}}^3F_{12} - \frac{1}{3}{F_{12}}^3 - 2F_{11}F_{12}F_{13} - {F_{11}}^2F_{14} + F_{13}F_{14} + F_{12}F_{15}$
$+F_{11}F_{16} - \frac{3}{4}F_{18} + \frac{1}{10}F_{45} + \frac{1}{9}F_{36} = 0$

(10,7) $\frac{1}{5}{F_{11}}^5 - \frac{3}{2}{F_{11}}^2{F_{12}}^2 - {F_{11}}^3F_{13} + {F_{12}}^2F_{13} + F_{11}{F_{13}}^2 + 2F_{11}F_{12}F_{14}$
$-\frac{1}{2}{F_{14}}^2 + {F_{11}}^2F_{15} - F_{13}F_{15} - F_{12}F_{16} + \frac{7}{9}F_{19} - \frac{1}{25}F_{55} - \frac{1}{12}F_{46} - \frac{1}{8}F_{28} = 0$

(9,8) $$F_{11}{}^3F_{12}-\frac{1}{3}F_{12}{}^3-2F_{11}F_{12}F_{13}-F_{11}{}^2F_{14}+F_{13}F_{14}+F_{12}F_{15}$$
$$+F_{11}F_{16}+\frac{1}{10}F_{45}+\frac{1}{9}F_{36}+\frac{1}{7}F_{27}=0$$

(10,8) $$\frac{1}{5}F_{11}{}^5-\frac{3}{2}F_{11}{}^2F_{12}{}^2-F_{11}{}^3F_{13}+F_{12}{}^2F_{13}+F_{11}F_{13}{}^2+2F_{11}F_{12}F_{14}$$
$$-\frac{1}{2}F_{14}{}^2+F_{11}{}^2F_{15}-F_{13}F_{15}-F_{12}F_{16}-F_{11}F_{17}+\frac{7}{9}F_{19}-\frac{1}{25}F_{55}$$
$$-\frac{1}{12}F_{46}-\frac{2}{21}F_{37}=0$$

(10,9) $$\frac{1}{5}F_{11}{}^5-\frac{3}{2}F_{11}{}^2F_{12}{}^2-F_{11}{}^3F_{13}+F_{12}{}^2F_{13}+F_{11}F_{13}{}^2+2F_{11}F_{12}F_{14}$$
$$-\frac{1}{2}F_{14}{}^2+F_{11}{}^2F_{15}-F_{13}F_{15}-F_{12}F_{16}-F_{11}F_{17}-\frac{1}{25}F_{55}-\frac{1}{12}F_{46}$$
$$-\frac{2}{21}F_{37}-\frac{1}{8}F_{28}=0$$

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