Quantum Jacobi-Trudi and Giambelli Formulae for $U_q(B_r^{(1)})$ from Analytic Bethe Ansatz

by

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Abstract

Analytic Bethe ansatz is executed for a wide class of finite dimensional $U_q(B_r^{(1)})$ modules. They are labeled by skew-Young diagrams which, in general, contain a fragment corresponding to the spin representation. For the transfer matrix spectra of the relevant vertex models, we establish a number of formulae, which are $U_q(B_r^{(1)})$ analogues of the classical ones due to Jacobi-Trudi and Giambelli on Schur functions. They yield a full solution to the previously proposed functional relation (T-system), which is a Toda equation on discrete space-time.

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1. Introduction

In [KS1] analytic Bethe ansatz was worked out for all the fundamental representations of the Yangians $Y(X_r)$ of classical types $X_r = B_r, C_r$ and D_r. Namely, for any $a \in \{1, 2, \ldots, r\}$, a rational function $\Lambda_1^{(a)}(u)$ of the spectral parameter u has been constructed, which should describe the spectrum of the transfer matrices of the corresponding solvable vertex models. It is a Yangian analogue of the character of the auxiliary space and satisfies a couple of conditions required for it. In particular $\Lambda_1^{(a)}(u)$ has been shown pole-free provided that the Bethe ansatz equation (BAE) holds. These results are also valid for $U_q(X_r^{(1)})$ case after replacing the rational functions by their natural q-analogues. See [R,KS1] for general accounts on the analytic Bethe ansatz.

In this paper we extend such analyses beyond the fundamental representations for $X_r = B_r$. We introduce skew-Young diagrams $\lambda \subset \mu$ [M] and a set of tableaux on them obeying a certain semi-standard like conditions. Then we construct the corresponding function $T_{\lambda\subset\mu}(u)$ in terms of a sum over such tableaux via a certain rule. The $T_{\lambda\subset\mu}(u)$ is to be regarded as the spectrum of the commuting transfer matrix with auxiliary space labeled by $\lambda \subset \mu$. It has a dressed vacuum form (DVF) in the analytic Bethe ansatz. We shall rewrite $T_{\lambda\subset\mu}(u)$ in several determinantal forms, where the matrix elements are only those $T_\mu(u)$ for the usual Young diagrams with shapes $\mu = (1^a), (m)$ or $(m+1, 1^a)$. They can be viewed as $U_q(B_r^{(1)})$ analogues of the classical Jacobi-Trudi and Giambelli formulae on Schur functions [M]. Pole-freeness of the $T_{\lambda\subset\mu}(u)$ under BAE follows immediately from these formulae and our previous proof for the case $\mu = (1^a)$ [KS1]. These results correspond to the case where the auxiliary space is even with respect to the tensor degree of the spin representation. We shall simply refer to such a case spin-even and spin-odd otherwise. See the remark after (3.12) for a precise definition. We will also treat the spin-odd case by using a modified skew-Young diagrams and semi-standard like conditions on them. Combining these results, we obtain a full solution in terms of the DVF to the transfer matrix functional relation (T-system) proposed in [KNS]. This substantially achieves our program raised in [KS1] for B_r.

A natural question here is, what is the finite dimensional auxiliary space labeled by those skew-Young diagrams as a representation space of $U_q(B_r^{(1)})$ or $Y(B_r)$? We suppose that it is an irreducible one in view that all the terms in $T_{\lambda\subset\mu}(u)$ are coupled to make the associated poles suprious under BAE. Moreover we specify, in the Yangian context, the Drinfeld polynomial explicitly based on some empirical procedure. We shall also determine how the irreducible $Y(B_r)$ module decomposes as a B_r module through the embedding $B_r \hookrightarrow Y(B_r)$ for the spin-even case.

The paper is organized as follows. In the next section we recall the results in [KS1] on $U_q(B_r^{(1)})$. We then introduce the basic functions $T^a(u)$ and $T_m(u)$ for all $a, m \in \mathbf{Z}_{\geq 0}$. These are analogues of a-th anti-symmetric and m-th symmetric fusion transfer matrices (or its eigenvalues), respectively. For $1 \leq a \leq r-1$, we have $T^a(u) = \Lambda_1^{(a)}(u) = T_{(1^a)}(u)$ in the above. The introduction of $T^a(u)$ with $a \geq r$ is a key in this paper and we point out a new functional relation (2.14) among them. In section 3 and 4, we treat the spin-even and odd cases, respectively. In terms of the DVFs in these sections, we give, in section 5, a full solution to the T-system [KNS] with an outline of the proof. Until this point we will exclusively consider the situation where the quantum space is formally trivial. This means that the vacuum part in DVF is always 1 as well as the "left hand side" of the BAE. Section 6 includes a discussion on how to recover the vacuum part for the non-trivial quantum spaces. A prototype of them is a tensor product of irreducible finite dimensional modules such as (6.1). The problem is essentially equivalent to specifying the left hand side of the BAE (cf section 2.4 in [KS1]) for such a general quantum space. For the Yangian $Y(X_r)$, we propose quite generally for any X_r that it is just given by a ratio of the relevant Drinfeld polynomials.† See (6.2). Then we shall briefly indicate a way to recover the vacuum parts.

Many formulae in section 3 are formally valid also for $U_q(A^{(1)}_r)$ under a suitable condition. In particular $\lambda = \phi$ case of (3.5) has appeared in [BR], for which a representation theoretical background is available in [C].

We hope to report similar results for C_r and D_r cases in near future.

2. Review of the results on fundamental representations

Here we shall recall the B_r case of the results in [KS1]. Let $\{\alpha_1, \ldots, \alpha_r\}$ and $\{\Lambda_1, \ldots, \Lambda_r\}$ be the set of the simple roots and fundamental weights of B_r $(r \geq 2)$. Our normalization is $t_1 = \cdots = t_{r-1} = \frac{1}{2}t_r = 1$ for $t_a = 2/(\alpha_a|\alpha_a)$. Then $(\alpha_a|\alpha_b) = \frac{2}{t_a}\delta_{a,b} - \delta_{a,b-1} - \delta_{a,b+1}$ and $(\alpha_a|\Lambda_b) = \delta_{ab}/t_a$. The $U_q(B^{(1)}_r)$ BAE for the trivial quantum space reads [RW]

$$-1 = \prod_{b=1}^{r} \frac{Q_b(v^{(a)}_k + (\alpha_a|\alpha_b))}{Q_b(v^{(a)}_k - (\alpha_a|\alpha_b))} \quad \text{for } 1 \leq a \leq r,\ 1 \leq k \leq N_a, \tag{2.1}$$

$$Q_a(u) = \prod_{j=1}^{N_a} [u - v^{(a)}_j], \tag{2.2}$$

where $[u] = (q^u - q^{-u})/(q - q^{-1})$ and $N_1, \ldots, N_r$ are some positive integers. Throughout the paper we assume that q is generic. The LHS of (2.1) is just -1 as opposed to the non-trivial quantum space case (6.2), which will be discussed in section 6. Until then we shall focus on the dress parts in the analytic Bethe ansatz.

Following [KS1] we introduce the set J and the order $\prec$ in it as

$$J = \{1, 2, \ldots, r, 0, \bar{r}, \ldots, \bar{1}\}, \tag{2.3a}$$

$$1 \prec 2 \prec \cdots, \prec r \prec 0 \prec \bar{r} \prec \cdots, \prec \bar{1}. \tag{2.3b}$$

For $a \in J$, define the function $z(a;u)$ by

$$\begin{aligned} z(a;u) &= \frac{Q_{a-1}(u+a+1)Q_a(u+a-2)}{Q_{a-1}(u+a-1)Q_a(u+a)} \qquad 1 \leq a \leq r, \\ z(0;u) &= \frac{Q_r(u+r-2)Q_r(u+r+1)}{Q_r(u+r)Q_r(u+r-1)}, \\ z(\bar{a};u) &= \frac{Q_{a-1}(u+2r-a-2)Q_a(u+2r-a+1)}{Q_{a-1}(u+2r-a)Q_a(u+2r-a-1)} \qquad 1 \leq a \leq r, \end{aligned} \tag{2.4}$$

† We thank E.K. Sklyanin and V.O. Tarasov for a discussion on this point.

where we have set $Q_0(u)=1$. $z(a,u)$ is the dress part of the box $\boxed{a}$ in (4.4a) of [KS1], which corresponds to a weight in the vector representation. For $(\xi_1,\ldots,\xi_r)\in\{\pm\}^r$, define the function $sp(\xi_1,\ldots,\xi_r;u)$ by the following recursion relation with respect to r and the initial condition $r=2$.

$$\begin{aligned}
sp(+,+,\xi_3,\ldots,\xi_r;u) &= \tau^Q sp(+,\xi_3,\ldots,\xi_r;u),\\
sp(+,-,\xi_3,\ldots,\xi_r;u) &= \frac{Q_1(u+r-\frac{5}{2})}{Q_1(u+r-\frac{1}{2})}\tau^Q sp(-,\xi_3,\ldots,\xi_r;u),\\
sp(-,+,\xi_3,\ldots,\xi_r;u) &= \frac{Q_1(u+r+\frac{3}{2})}{Q_1(u+r-\frac{1}{2})}\tau^Q sp(+,\xi_3,\ldots,\xi_r;u+2),\\
sp(-,-,\xi_3,\ldots,\xi_r;u) &= \tau^Q sp(-,\xi_3,\ldots,\xi_r;u+2).
\end{aligned} \tag{2.5a}$$

$$\begin{aligned}
sp(+,+;u) &= \frac{Q_2(u-\frac{1}{2})}{Q_2(u+\frac{1}{2})},\\
sp(+,-;u) &= \frac{Q_1(u-\frac{1}{2})Q_2(u+\frac{3}{2})}{Q_1(u+\frac{3}{2})Q_2(u+\frac{1}{2})},\\
sp(-,+;u) &= \frac{Q_1(u+\frac{7}{2})Q_2(u+\frac{3}{2})}{Q_1(u+\frac{3}{2})Q_2(u+\frac{5}{2})},\\
sp(-,-;u) &= \frac{Q_2(u+\frac{7}{2})}{Q_2(u+\frac{5}{2})}.
\end{aligned} \tag{2.5b}$$

In (2.5a) τ^Q is the operation $Q_a\to Q_{a+1}$, namely,

$$\begin{aligned}
&\tau^Q F(Q_1(u+x^1_1),Q_1(u+x^1_2),\ldots,Q_2(u+x^2_1),Q_2(u+x^2_2),\ldots)\\
&\quad = F(Q_2(u+x^1_1),Q_2(u+x^1_2),\ldots,Q_3(u+x^2_1),Q_3(u+x^2_2),\ldots)
\end{aligned} \tag{2.6}$$

for any function F. $sp(\xi_1,\ldots,\xi_r;u)$ is the dress part of the box $\overbrace{\boxed{\xi_1,\xi_2,\cdots,\xi_r}}^{r}$ in (4.25,26) of [KS1].

Now we introduce the meromorphic functions $T^a(u)$ and $T_m(u)$ of u for any $a,m\in\mathbf{Z}_{\geq 0}$ by the following "non-commutative generating series"

$$\begin{aligned}
&(1+z(\bar{1};u)X)\cdots(1+z(\bar{r};u)X)(1-z(0;u)X)^{-1}(1+z(r;u)X)\cdots(1+z(1;u)X)\\
&= \sum_{a=0}^{\infty} T^a(u+a-1)X^a, \qquad (2.7a)\\
&(1-z(1;u)X)^{-1}\cdots(1-z(r;u)X)^{-1}(1+z(0;u)X)(1-z(\bar{r};u)X)^{-1}\cdots(1-z(\bar{1};u)X)^{-1}\\
&= \sum_{m=0}^{\infty} T_m(u+m-1)X^m, \qquad (2.7b)
\end{aligned}$$

where X is a difference operator with the commutation relation

$$XQ_a(u)=Q_a(u+2)X \quad \text{for any } 1\leq a\leq r. \tag{2.8}$$

Thus $Xz(a;u) = z(a;u+2)X$ for any $a \in J$. We set $T^a(u) = T_m(u) = 0$ for $a, m < 0$. An immediate consequence of the above definition is

$$\delta_{ij} = \sum_{k=0}^{N} (-)^{i-k} T_{i-k}(u+i+k) T^{k-j}(u+k+j) \tag{2.9a}$$

$$= \sum_{k=0}^{N} (-)^{i-k} T_{i-k}(u-i-k) T^{k-j}(u-k-j) \tag{2.9b}$$

for any $N \geq 0$ and $0 \leq i, j \leq N$. Define $T_1^{(a)}(u)$ for $1 \leq a \leq r$ by

$$\begin{aligned} T_1^{(a)}(u) &= T^a(u) \quad \text{for } 1 \leq a \leq r-1, \\ T_1^{(r)}(u) &= \sum_{\xi_1, \ldots, \xi_r = \pm} sp(\xi_1, \ldots, \xi_r; u). \end{aligned} \tag{2.10}$$

Then $T_1^{(a)}(u)$ coincides with the dress part of $\Lambda_1^{(a)}(u)$ in [KS1] for all $1 \leq a \leq r$.

Theorem 2.1. *$T_1^{(r)}(u), T^a(u)$ and $T_m(u) (\forall a, m \in \mathbf{Z})$ are pole-free provided that the BAE (2.1) holds.*

For $T_1^{(r)}(u)$ and $T^a(u)$ with $a \leq r-1$, this was proved in [KS1] in the more general setting including the vacuum parts. The other cases can be verified quite similarly. $T_1^{(1)}(u)$ and $T_1^{(r)}(u)$ was considered earlier [R].

The functions $z(a;u)$ and $sp(\xi_1, \ldots, \xi_r; u)$ are related as follows. Given two sequences $(\xi_1, \ldots, \xi_r)$ and $(\eta_1, \ldots, \eta_r) \in \{\pm\}^r$, we define $i_1 < \cdots < i_k, I_1 < \cdots < I_{r-k}$ $(0 \leq k \leq r)$ and $j_1 < \ldots < j_l, J_1 < \ldots < J_{r-l}$ $(0 \leq l \leq r)$ by the following.

$$\begin{aligned} &\xi_{i_1} = \cdots = \xi_{i_k} = +, \ \xi_{I_1} = \cdots = \xi_{I_{r-k}} = -, \\ &\eta_{j_1} = \cdots = \eta_{j_l} = -, \ \eta_{J_1} = \cdots = \eta_{J_{r-l}} = +. \end{aligned} \tag{2.11}$$

Then we have

Proposition 2.2. *For any $a \in \mathbf{Z}_{\geq 0}$,*

$$\begin{aligned} &sp(\xi_1, \ldots, \xi_r; u-r+a+\frac{1}{2}) sp(\eta_1, \ldots, \eta_r; u+r-a-\frac{1}{2}) \\ &= \prod_{n=1}^{a} z(b_n; u+a+1-2n) \qquad \text{if } k+l \leq a, \end{aligned} \tag{2.12a}$$

where

$$b_n = \begin{cases} i_n & \text{for } 1 \leq n \leq k \\ 0 & \text{for } k < n \leq a-l \\ \overline{j_{a+1-n}} & \text{for } a-l < n \leq a \end{cases} . \tag{2.12b}$$

For any $a \in \mathbf{Z}_{\le 2r-1}$,

$$
\begin{aligned}
&sp(\xi_1,\ldots,\xi_r; u-r+a+\frac{1}{2}) sp(\eta_1,\ldots,\eta_r; u+r-a-\frac{1}{2}) \\
&= \prod_{n=1}^{2r-1-a} z(b'_n; u+2r-a-2n) \qquad \textit{if } k+l \ge a+1,
\end{aligned}
\tag{2.13a}
$$

where

$$
b'_n = \begin{cases} J_n & \textit{for } 1 \le n \le r-l \\ 0 & \textit{for } r-l < n \le r+k-1-a \\ \overline{I_{2r-a-n}} & \textit{for } r+k-1-a < n \le 2r-1-a \end{cases} . \tag{2.13b}
$$

This enables the evaluation of the product $sp(\xi_1,\ldots,\xi_r; u-r+a+\frac{1}{2}) sp(\eta_1,\ldots,\eta_r; u+r-a-\frac{1}{2})$ for any $\{\xi_i\},\{\eta_i\}$ and $a \in \mathbf{Z}$ in terms of z (2.4). For $1 \le a \le r-1$, (2.12) is theorem A.1 in [KS1]. It is straightforward to extend it to any $a \in \mathbf{Z}_{\ge 0}$. Eq. (2.13) can be derived from (2.12) by replacing a by $2r-1-a$. Note in (2.12b) that $b_1 \prec \cdots \prec b_k \prec b_{k+1} = \cdots = b_{a-l} = 0 \prec b_{a-l+1} \prec \cdots \prec b_a \in J$. A similar inequality holds also for b'_n. Comparing them with (2.7a) and (2.10) we get

Theorem 2.3.

$$
T^a(u) + T^{2r-1-a}(u) = T^{(r)}_1(u-r+a+\frac{1}{2}) T^{(r)}_1(u+r-a-\frac{1}{2}) \quad \forall a \in \mathbf{Z}. \tag{2.14}
$$

This is invariant under the exchange $a \leftrightarrow 2r-1-a$. If $a<0$ or $a>2r-1$, there is in fact only one term on the LHS. The new functional relation (2.14) will play an important role in this paper. It is also valid after including the vacuum parts. See section 6.

3. Spin-even case

Let $\mu = (\mu_1, \mu_2, \ldots)$, $\mu_1 \geq \mu_2 \geq \cdots \geq 0$ be a Young diagram and $\mu' = (\mu'_1, \mu'_2, \ldots)$ be its transpose. We let d_μ denote the length of the main diagonal of μ. By a skew-Young diagram we mean a pair of Young diagrams $\lambda \subset \mu$. It is depicted by the region corresponding to the subtraction $\mu - \lambda$. See the Fig.3.1 for example.

Fig.3.1

For definiteness, we assume that $\lambda'_{\mu_1} = \lambda_{\mu'_1} = 0$. A Young diagram μ is naturally identified with a skew-Young diagram $\phi \subset \mu$. By an *admissible* tableau b on a skew-Young diagram $\lambda \subset \mu$ we mean an assignment of an element $b(i,j) \in J$ to the (i,j)-th box in $\lambda \subset \mu$ under the following rule: (We locate $(1,1)$ at the top left corner of μ, $(i+1,j)$ and $(i,j+1)$ to the below and the right of (i,j), respectively.)

$$\begin{aligned}
&b(i,j) \preceq b(i,j+1), \qquad b(i,j) \prec b(i+1,j) \quad \text{with the exception that} \\
&b(i,j) = b(i,j+1) = 0 \text{ is forbidden}, \quad b(i,j) = b(i+1,j) = 0 \text{ is allowed.}
\end{aligned} \tag{3.1}$$

Without the exception this coincides with the usual definition of the semi-standard Young tableau. Denote by $Atab(\lambda \subset \mu)$ the set of admissible tableaux on $\lambda \subset \mu$.

Given a skew-Young diagram $\lambda \subset \mu$, we define the function $T_{\lambda\subset\mu}(u)$ as the following sum over the admissible tableaux.

$$T_{\lambda\subset\mu}(u) = \sum_{b \in Atab(\lambda\subset\mu)} \prod_{(i,j)\in(\lambda\subset\mu)} z(b(i,j); u + \mu'_1 - \mu_1 - 2i + 2j). \tag{3.2}$$

Comparing this with (2.7) we have

$$T^a(u) = T_{(1^a)}(u): \text{ single column of length } a, \tag{3.3a}$$

$$T_m(u) = T_{(m)}(u): \text{ single row of length } m. \tag{3.3b}$$

We also prepare a notation for the single hook,

$$T_{k,l} = T_{(l+1,1^k)}(u). \tag{3.3c}$$

Our main result in this section is

Theorem 3.1.

$$T_{\lambda\subset\mu}(u) = det \begin{pmatrix} 0 & \cdots & 0 & R_{11} & \cdots & R_{1d_\mu} \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & R_{d_\lambda 1} & \cdots & R_{d_\lambda d_\mu} \\ C_{11} & \cdots & C_{1d_\lambda} & H_{11} & \cdots & H_{1d_\mu} \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ C_{d_\mu 1} & \cdots & C_{d_\mu d_\lambda} & H_{d_\mu 1} & \cdots & H_{d_\mu d_\mu} \end{pmatrix}, \tag{3.4a}$$

where

$$\begin{aligned}
R_{ij} &= T_{\mu_j - \lambda_i + i - j}(u + \mu'_1 - \mu_1 + \mu_j + \lambda_i - i - j + 1), \\
C_{ij} &= -T^{\mu'_i - \lambda'_j - i + j}(u + \mu'_1 - \mu_1 - \mu'_i - \lambda'_j + i + j - 1), \\
H_{ij} &= T_{\mu'_i - i, \mu_j - j}(u + \mu'_1 - \mu_1 - \mu'_i + \mu_j + i - j).
\end{aligned} \tag{3.4b}$$

Two particular cases corresponding to the formal choices $\mu_i = \lambda_i$ or $\mu'_i = \lambda'_i$ for $1 \le i \le d_\lambda = d_\mu$ yield simpler formulae. In these cases, redefining μ_i, μ'_i, λ_i and λ'_i so that $\lambda'_{\mu_1} = \lambda_{\mu'_1} = 0$, we have

$$T_{\lambda\subset\mu}(u) = det_{1\le i,j\le\mu_1}(T^{\mu'_i-\lambda'_j-i+j}(u+\mu'_1-\mu_1-\mu'_i-\lambda'_j+i+j-1)), \qquad (3.5a)$$
$$= det_{1\le i,j\le\mu'_1}(T_{\mu_j-\lambda_i+i-j}(u+\mu'_1-\mu_1+\mu_j+\lambda_i-i-j+1)). \qquad (3.5b)$$

Eq.(3.5a) can be verified, for example, by induction on μ_1, i.e., by showing the same recursive relation for the tableau sum (3.2) as an expansion of the determinant. Then (3.5b) follows from (2.9). Theorem 3.1 is proved from these results by applying Sylvester's theorem on determinants. From (3.5a) and Theorem 2.1 one has

Corollary. *$T_{\lambda\subset\mu}(u)$ is pole-free provided the BAE (2.1) holds.*

The admissibility condition (3.1) leads to the above conclusion although it is by no means obvious in the defining expression (3.2). Despite the exception in (3.1), our formulae (3.4) and (3.5) formally coincide with the classical ones due to Giambelli and Jacobi-Trudi on Schur functions [M] if one drops the u-dependence (or in the limit $|u| \to \infty$). If $\mu'_{i+1} - \lambda'_i > 2r$ for some i, $Atab(\lambda \subset \mu) = \phi$. Correspondingly, one can show that the determinant (3.5a) is vanishing using the fact that $T^a(u)$ factorizes for $a \ge 2r$ due to Theorem 2.3. Henceforth we assume that $\mu'_{i+1} - \lambda'_i \le 2r$ for $1 \le i \le \mu_1$. (We set $\mu'_{\mu_1+1} = -\infty$.)

The $T_{\lambda\subset\mu}(u)$ (3.2) describes the spectrum of the transfer matrix whose auxiliary space is labeled by the skew-Young diagram $\lambda \subset \mu$ and u. Denote the space by $W_{\lambda\subset\mu}(u)$. We suppose it is an irreducible finite dimensional module over $Y(B_r)$ (or $U_q(B_r^{(1)})$ in the trigonometric case) in view that all the terms in (3.2) seem coupling to make the apparent poles suprious under BAE. Now we shall specify the Drinfeld polynomial $P_a(\zeta)$ [D] that characterizes $W_{\lambda\subset\mu}(u)$ based on some empirical procedure. Our convention slightly differs from the original one in Theorem 2 of [D] in such a way that

$$1 + \sum_{k=0}^{\infty} d_{ik}\zeta^{-k-1} = \frac{P_i(\zeta + \frac{1}{t_i})}{P_i(\zeta - \frac{1}{t_i})}. \qquad (3.6)$$

For any $b \in Atab(\lambda \subset \mu)$, the corresponding summand (3.2) has the form

$$\prod_{a=1}^{r} \frac{Q_a(u+x_1^a)\cdots Q_a(u+x_{i_a}^a)}{Q_a(u+y_1^a)\cdots Q_a(u+y_{i_a}^a)}, \qquad (3.7)$$

where x_j^a, y_j^a and i_a are specified from b. This summand carries the B_r-weight

$$wt(b) = \sum_{a=1}^{r}\Bigl(\frac{t_a}{2}\sum_{j=1}^{i_a}(y_j^a - x_j^a)\Bigr)\Lambda_a \qquad (3.8)$$

in the sense that $\lim_{q^u\to\infty}(3.7) = q^{-2(wt(b)|\sum_{a=1}^r N_a\alpha_a)}$. From $Atab(\lambda \subset \mu)$, take such b_0 that $wt(b_0)$ is highest, which corresponds to the "top term" in section 2.4 of [KS1]. In our case, such b_0 is unique and given as follows. Fill the left most column of $\lambda \subset \mu$ from the top to the bottom by assigning the first $\mu'_1 - \lambda'_1$ letters from the sequence $1, 2, \ldots, r, 0, 0, \ldots$. Given the $(i-1)$-th column, the i-th column is built from the top to the bottom by taking the first $\mu'_i - \lambda'_i$ letters from the sequence $1, 2, \ldots, r, \overbrace{0, \ldots, 0}^{k}, \overline{r}, \overline{r-1}, \ldots, \overline{1}$, where $k = \max(0, \min(\lambda'_{i-1} - \lambda'_i, \mu'_i - \lambda'_i - r))$. (We set $\lambda'_0 = +\infty$.) See the example in Fig.3.2.

Fig.3.2.

It turns out that (3.7) for the top term b_0 can be expressed uniquely in the form

$$\prod_{a=1}^{r} \prod_{j=1}^{M_a} \frac{Q_a(u + z_j^a - \frac{1}{t_a})}{Q_a(u + z_j^a + \frac{1}{t_a})} \tag{3.9}$$

for some M_a and $\{z_j^a | 1 \le j \le M_a\}$ up to the permutations of z_j^a's for each a. We then propose that the Drinfeld polynomial $P_a^{W_{\lambda\subset\mu}(u)}(\zeta)$ for $W_{\lambda\subset\mu}(u)$ is given by

$$P_a^{W_{\lambda\subset\mu}(u)}(\zeta) = \prod_{j=1}^{M_a} (\zeta - u - z_j^a) \quad 1 \le a \le r. \tag{3.10}$$

In our case, it reads explicitly as follows.

$$\begin{aligned} P_a^{W_{\lambda\subset\mu}(u)}(\zeta) = & \prod_{\substack{1 \le i \le \mu_1 \\ \mu'_i - \lambda'_i = a}} (\zeta - u - \mu'_1 + \mu_1 + 1 + a + 2\lambda'_i - 2i) \\ & \times \prod_{\substack{1 \le i \le \mu_1 - 1 \\ \mu'_{i+1} - \lambda'_i = 2r - a}} (\zeta - u - \mu'_1 + \mu_1 + 2 + a + 2\lambda'_i - 2i) \quad 1 \le a \le r-1, \end{aligned} \tag{3.11a}$$

$$\begin{aligned} P_r^{W_{\lambda\subset\mu}(u)}(\zeta) = & \prod_{\substack{1 \le i \le \mu_1 \\ \lambda'_i + r \le \mu'_i \le \lambda'_{i-1} + r}} (\zeta - u - \mu'_1 + \mu_1 + 2\mu'_i - 2i - r + \frac{3}{2}) \\ & \times \prod_{\substack{1 \le i \le \mu_1 \\ \mu'_{i+1} \le \lambda'_i + r \le \mu'_i}} (\zeta - u - \mu'_1 + \mu_1 + 2\lambda'_i - 2i + r + \frac{1}{2}), \end{aligned} \tag{3.11b}$$

where we have set

$$\mu'_{\mu_1+1} = -\infty, \quad \lambda'_0 = \infty. \tag{3.12}$$

We will call the irreducible finite dimensional $Y(B_r)$ module spin-even (resp. spin-odd) if and only if the characterizing Drinfeld polynomial $P_r(\zeta)$ is even (resp. odd) degree. The one in (3.11b) is even for any skew-Young diagram $\lambda \subset \mu$. For example, in the case of the single column or row (3.3), (3.11) reads

$$P_a^{W_{(1^c)}(u)}(\zeta) = \begin{cases} (\zeta - u)^{\delta_{ac}} & 1 \le c < r \\ \left((\zeta - u + c - r + \frac{1}{2})(\zeta - u - c + r - \frac{1}{2})\right)^{\delta_{ar}} & c \ge r \end{cases}, \tag{3.13a}$$

$$P_a^{W_{(m)}(u)}(\zeta) = \left((\zeta - u + m - 1)(\zeta - u + m - 3) \cdots (\zeta - u - m + 1)\right)^{\delta_{a1}}. \tag{3.13b}$$

As a B_r module, the $Y(B_r)$ module $W_{\lambda\subset\mu}(u)$ decomposes as

$$W_{\lambda\subset\mu}(u) \simeq \sum_{\eta}(\sum_{\kappa,\nu} LR^{\mu}_{\lambda\nu} LR^{\nu}_{(2\kappa)'\eta})\pi_{O(2r+1)}(V_\eta), \qquad (3.14)$$

which is u-independent. Here $LR^{\mu}_{\lambda\nu}$ etc denote the Littlewood-Richardson coefficients for the universal character ring Λ of GL type introduced in [KT]. The sums run over all the Young diagrams η, ν and $\kappa = (\kappa_1, \kappa_2, \ldots)$, where $(2\kappa)'$ stands for the transpose of $2\kappa = (2\kappa_1, 2\kappa_2, \ldots)$. $\pi_{O(2r+1)}(V_\eta)$ is the image of the specialization homomorphism [KT]. It is equal to (± 1 or 0) "times" the irreducible B_r module V_{η^*} with the highest weight labeled by the Young diagram η^* with $(\eta^*)'_1 \leq r$. They are determined according to the equality $\pi_{O(2r+1)}(\chi(\eta)) = (\pm 1 \text{ or } 0) \times \chi(\eta^*)$ at the character level [KT].

4. Spin-odd case

Consider the following subset $Spin \subset Atab((1^r))$.

$$\begin{array}{|c|}\hline i_1 \\ \hline \vdots \\ \hline i_r \\ \hline\end{array} \in Spin \Leftrightarrow \begin{cases} i_1 \prec \cdots \prec i_r \in J, \\ 0 \text{ is not contained,} \\ \text{only one of } i \text{ and } \overline{i} \text{ is contained for any } 1 \leq i \leq r. \end{cases} \qquad (4.1)$$

There is a bijection $\iota : Spin \to \{(\xi_1, \ldots, \xi_r) \mid \xi_j = \pm\}$ sending (4.1) with $1 \preceq i_1 \prec \cdots \prec i_k \preceq r \prec \overline{r} \preceq i_{k+1} \prec \cdots \prec i_r \preceq \overline{1}$ to such $(\xi_1, \ldots, \xi_r)$ that $\xi_{i_1} = \cdots = \xi_{i_k} = +$, $\xi_{\overline{i_{k+1}}} = \cdots = \xi_{\overline{i_r}} = -$, where we interpret $\overline{k} = i$ if $k = \overline{i}$. Thus the latter of (2.10) can also be written as $T^{(r)}_1(u) = \sum_{b\in Spin} sp(\iota(b); u)$. This type of ι has also been utilized in [KN].

For a skew-Young diagram $\lambda \subset \mu$ with $\mu'_1 - \lambda'_1 \geq r$, hatch the bottom r boxes in the leftmost column, which we call an L-hatched skew-Young diagram $\lambda \subset \mu$. See Fig.4.1.

Fig.4.1.

Consider a tableau b on it, namely, a map $b :$ L-hatched $\lambda \subset \mu \to J$. We call a tableau b on an L-hatched $\lambda \subset \mu$ *L-admissible* if and only if all of the following three conditions are valid. ($n = \mu'_2 - (\mu'_1 - r)$ and see Fig. 4.2 for the definitions of i_l and j_l.)

$$\begin{aligned} &\text{(i) hatched part } \in Spin, \text{ and (3.1) for non-hatched part,} \\ &\text{(ii) } j_0 \prec i_1, \\ &\text{(iii) } i_1 \preceq j_1, \ldots, i_n \preceq j_n \text{ or there exists } k \in \{1, \ldots, n\} \text{ such that} \\ &\qquad i_1 \preceq j_1, \ldots, i_{k-1} \preceq j_{k-1} \text{ and } \overline{r} \preceq j_k \prec i_k \preceq \overline{1}. \end{aligned} \qquad (4.2)$$

Here (ii) is void when $\mu'_1 - \lambda'_1 = r$ and so is (iii) for $n = 0$.

Fig.4.2.

Denote by $Atab_L(\lambda \subset \mu)$ the set of L-admissible tableaux on the L-hatched $\lambda \subset \mu$. We note that $Atab(\lambda \subset \mu) \not\subseteq Atab_L(\lambda \subset \mu)$ nor $Atab(\lambda \subset \mu) \not\supseteq Atab_L(\lambda \subset \mu)$. Given an L-hatched skew-Young diagram $\lambda \subset \mu$, we define the function $S^L_{\lambda\subset\mu}(u)$ by

$$\begin{aligned} S^L_{\lambda\subset\mu}(u) = &\sum_{b\in Atab_L(\lambda\subset\mu)} sp(\iota(\text{hatched part}); u) \\ &\times \prod_{(i,j)\in \text{ non hatched part of } (\lambda\subset\mu)} z(b(i,j); u + 2\mu'_1 - r - 2i + 2j - \frac{3}{2}). \end{aligned} \qquad (4.3)$$

We have an L $\leftrightarrow$ R (left vs. right) dual of these definitions as follows. For a skew-Young diagram $\lambda \subset \mu$ with $\mu'_{\mu_1} \geq r$ (remember we assumed $\lambda'_{\mu_1} = 0$), hatch the top r boxes in the rightmost column, which we call an R-hatched skew-Young diagram $\lambda \subset \mu$. See Fig.4.3.

Fig.4.3.

Consider a tableau b : R-hatched $\lambda \subset \mu \to J$. We call a tableau b on an R-hatched $\lambda \subset \mu$ *R-admissible* if and only if all of the following three conditions are valid. ($n = r - \lambda'_{\mu_1 - 1}$ and see Fig. 4.4 for the definitions of i_l and j_l.)

$$\begin{aligned} &\text{(i) hatched part } \in Spin, \text{ and (3.1) for non-hatched part,} \\ &\text{(ii) } i_1 \prec j_0, \\ &\text{(iii) } j_1 \preceq i_1, \ldots, j_n \preceq i_n \text{ or there exists } k \in \{1, \ldots, n\} \text{ such that} \\ &\qquad j_1 \preceq i_1, \ldots, j_{k-1} \preceq i_{k-1} \text{ and } 1 \preceq i_k \prec j_k \preceq r, \end{aligned} \tag{4.4}$$

where (ii) is void when $\mu'_{\mu_1} = r$ and so is (iii) for $n = 0$.

Fig.4.4.

Denoting by $Atab_R(\lambda \subset \mu)$ the set of R-admissible tableaux on the R-hatched $\lambda \subset \mu$, we define

$$\begin{aligned} S^R_{\lambda\subset\mu}(u) = &\sum_{b \in Atab_R(\lambda\subset\mu)} sp(\iota(\text{hatched part}); u) \\ &\times \prod_{(i,j)\in \text{ non hatched part of } (\lambda\subset\mu)} z(b(i,j); u - 2\mu_1 + r - 2i + 2j + \frac{3}{2}). \end{aligned} \tag{4.5}$$

Our first main results in this section is

Theorem 4.1.

$$S^L_{\lambda\subset\mu}(u) = det_{1\leq i,j \leq \mu_1}(\mathcal{S}^L_{ij}) \tag{4.6a}$$

$$= det_{1\leq i,j \leq \mu'_2}(\overline{\mathcal{S}}^L_{ij}), \tag{4.6b}$$

where

$$\mathcal{S}^L_{ij} = \begin{cases} T^{\mu'_j - \lambda'_i + i - j}(u + 2\mu'_1 - \mu'_j - \lambda'_i + i + j - r - \frac{5}{2}) \ \ j \geq 2 \\ T^{(r)}_1(u + 2i - 2 + 2(\mu'_1 - \lambda'_i - r)) \ \ j = 1 \end{cases}, \tag{4.7a}$$

$$\overline{\mathcal{S}}^L_{ij} = \begin{cases} T_{\mu_i - \lambda_j - i + j}(u + 2\mu'_1 + \mu_i + \lambda_j - i - j - r - \frac{1}{2}) \ 1 \leq j \leq \lambda'_1 \\ \mathcal{H}^L_{\mu_i + \lambda'_1 - i}(u + 2\mu'_1 - 2\lambda'_1 - 2r) \ \ j = \lambda'_1 + 1 \\ T_{\mu_i - i + j - 1}(u + 2\mu'_1 + \mu_i - i - j - r + \frac{1}{2}) \ \ j > \lambda'_1 + 1 \end{cases}, \tag{4.7b}$$

$$\mathcal{H}^L_m(u) = \sum_{l=0}^{m} (-1)^l T^{(r)}_1(u + 2l) T_{m-l}(u + m + r + l - \frac{1}{2}). \tag{4.7c}$$

From (4.6a), (4.7a,c) and Theorem 3.1, $\mathcal{H}^L_m(u)$ is equal to the L-hatched hook $S^L_{(m+1,1^{r-1})}(u)$. $\blacksquare$

For an R-hatched diagram $\lambda \subset \mu$, let $\xi \subset \eta$ be the sub-diagram obtained by removing the rightmost column of $\lambda \subset \mu$. See Fig. 4.5.

Fig. 4.5.

Thus for example $\eta_i = \mu_1 - 1$ for $1 \leq i \leq \mu'_{\mu_1} - \lambda'_{\mu_1 - 1}$. Then another main result in this section is the R-hatched version of the previous theorem as follows.

Theorem 4.2.

$$S^R_{\lambda\subset\mu}(u) = det_{1\le i,j\le \mu_1}(\mathcal{S}^R_{ij}) \quad (4.8a)$$

$$= det_{1\le i,j\le \eta'_1}(\overline{\mathcal{S}}^R_{ij}), \quad (4.8b)$$

where

$$\mathcal{S}^R_{ij} = \begin{cases} T^{\mu'_j-\lambda'_i+i-j}(u-2\mu_1-\mu'_i-\lambda'_j+i+j+r+\frac{1}{2}) & j\le\mu_1-1 \\ T^{(r)}_1(u-2\mu_1-2\mu'_i+2i+2r) & j=\mu_1 \end{cases}, \quad (4.9a)$$

$$\overline{\mathcal{S}}^R_{ij} = \begin{cases} T_{\eta_i-\xi_j-i+j}(u-2\lambda'_{\mu_1-1}-2\eta_1+\eta_i+\xi_j-i-j+r+\frac{1}{2}) & i\ne\eta'_1-\mu'_1+\mu'_{\mu_1} \\ \mathcal{H}^R_{\eta_i-\xi_j-i+j}(u-2\mu'_{\mu_1}+2r) & i=\eta'_1-\mu'_1+\mu'_{\mu_1} \end{cases} \quad (4.9b)$$

$$\mathcal{H}^R_m(u) = \sum_{l=0}^{m}(-1)^l T^{(r)}_1(u-2l)T_{m-l}(u-m-r-l+\frac{1}{2}). \quad (4.9c)$$

From (4.8a), (4.9a,c) and Theorem 3.1, one sees that $\mathcal{H}^R_m(u)$ is equal to the R-hatched "dual hook" $S^R_{(m^{r-1})\subset((m+1)^r)}(u)$. From Theorems 2.1, 4.1 and 4.2, we have

Corollary. *$S^L_{\lambda\subset\mu}(u)$ and $S^R_{\lambda\subset\mu}(u)$ are pole-free provided that the BAE (2.1) holds.*

Following a similar argument to the previous section, we propose the Drinfeld polynomials corresponding to the auxiliary spaces $W^L_{\lambda\subset\mu}(u)$ and $W^R_{\lambda\subset\mu}(u)$ of $S^L_{\lambda\subset\mu}(u)$ and $S^R_{\lambda\subset\mu}(u)$, respectively.

$$P_a^{W^L_{\lambda\subset\mu}(u)}(\zeta) = P_a^{W_{\lambda\subset\mu}(u+\mu'_1+\mu_1-r-\frac{3}{2})}(\zeta) \quad 1\le a\le r-1, \quad (4.10a)$$

$$\begin{aligned} P_r^{W^L_{\lambda\subset\mu}(u)}(\zeta) &= \frac{1}{\zeta-u+1}P_r^{W_{\lambda\subset\mu}(u+\mu'_1+\mu_1-r-\frac{3}{2})}(\zeta) \\ &= \prod_{\substack{2\le i\le\mu_1 \\ \lambda'_i+r\le\mu'_i\le\lambda'_{i-1}+r}}(\zeta-u+3+2(\mu'_i-i-\mu'_1)) \\ &\times \prod_{\substack{1\le i\le\mu_1 \\ \mu'_{i+1}\le\lambda'_i+r\le\mu'_i}}(\zeta-u+2+2(\lambda'_i-i-\mu'_1+r)), \end{aligned} \quad (4.10b)$$

$$P_a^{W^R_{\lambda\subset\mu}(u)}(\zeta) = P_a^{W_{\lambda\subset\mu}(u-\mu'_1-\mu_1+r+\frac{3}{2})}(\zeta) \quad 1\le a\le r-1, \quad (4.11a)$$

$$\begin{aligned} P_r^{W^R_{\lambda\subset\mu}(u)}(\zeta) &= \frac{1}{\zeta-u-1}P_r^{W_{\lambda\subset\mu}(u-\mu'_1-\mu_1+r+\frac{3}{2})}(\zeta) \\ &= \prod_{\substack{1\le i\le\mu_1 \\ \lambda'_i+r\le\mu'_i\le\lambda'_{i-1}+r}}(\zeta-u+2(\mu'_i-i+\mu_1-r)) \\ &\times \prod_{\substack{1\le i\le\mu_1-1 \\ \mu'_{i+1}\le\lambda'_i+r\le\mu'_i}}(\zeta-u-1+2(\lambda'_i-i+\mu_1)), \end{aligned} \quad (4.11b)$$

where we assume (3.12).

In $Atab_L(\lambda \subset \mu)$ and $Atab_R(\lambda \subset \mu)$, we have considered the hatched part ($Spin$ (4.1)) only in the bottom left or top right position. A natural question may be whether it is possible to define a tableau sum that becomes pole-free and contains $Spin$ simultaneously in various places in a skew-Young diagram $\lambda \subset \mu$. It is indeed possible to include $Spin$ both at the bottom left and the top right. However, we have found only few examples beyond that so far.

5. Solution to the T-system

The functions $T_{\lambda\subset\mu}(u)$ (3.2), $S^L_{\lambda\subset\mu}(u)$ (4.3) and $S^R_{\lambda\subset\mu}(u)$ (4.5) provide the solution to the T-system for B_r, one of the functional relations proposed in [KNS] for any X_r. (See [KS2] for the T-system of twisted quantum affine algebras.) For $m \in \mathbf{Z}_{\geq 0}$, put

$$T^{(a)}_m(u) = T_{(m^a)}(u) \quad 1 \leq a \leq r-1, \tag{5.1a}$$

$$T^{(r)}_{2m}(u) = T_{(m^r)}(u), \tag{5.1b}$$

$$T^{(r)}_{2m+1}(u) = S^L_{((m+1)^r)}(u-m) = S^R_{((m+1)^r)}(u+m). \tag{5.1c}$$

The latter equality in (5.1c) can be shown easily by using (2.14), (4.7a) and (4.9a). The definition (5.1) includes (2.10). Moreover, from (3.11) and (4.10,11), the Drinfeld polynomial corresponding to $T^{(a)}_m(u)$ is given by $P_b(\zeta) = \left(\prod_{i=1}^m(\zeta - u + \frac{m+1-2i}{t_a})\right)^{\delta_{ba}}$ for $1 \leq b \leq r$, in agreement with (2.3) of [KS1]. Thus $T^{(a)}_m(u)$ here is the DVF for the transfer matrix $T^{(a)}_m(u)$ considered in [KS1].

Theorem 5.1. *$T^{(a)}_m(u)$ defined above satisfies the following functional relations.*

$$\begin{aligned}
T^{(a)}_m(u-1)T^{(a)}_m(u+1) &= T^{(a)}_{m+1}(u)T^{(a)}_{m-1}(u) + T^{(a-1)}_m(u)T^{(a+1)}_m(u) \\
&\qquad\qquad \text{for } 1 \leq a \leq r-2, \\
T^{(r-1)}_m(u-1)T^{(r-1)}_m(u+1) &= T^{(r-1)}_{m+1}(u)T^{(r-1)}_{m-1}(u) + T^{(r-2)}_m(u)T^{(r)}_{2m}(u), \\
T^{(r)}_{2m}(u-\tfrac{1}{2})T^{(r)}_{2m}(u+\tfrac{1}{2}) &= T^{(r)}_{2m+1}(u)T^{(r)}_{2m-1}(u) \\
&\quad + T^{(r-1)}_m(u-\tfrac{1}{2})T^{(r-1)}_m(u+\tfrac{1}{2}), \\
T^{(r)}_{2m+1}(u-\tfrac{1}{2})T^{(r)}_{2m+1}(u+\tfrac{1}{2}) &= T^{(r)}_{2m+2}(u)T^{(r)}_{2m}(u) + T^{(r-1)}_m(u)T^{(r-1)}_{m+1}(u).
\end{aligned} \tag{5.2}$$

Outline of the proof. We use the determinantal expressions (3.5a) and (4.6a). Then the first two equations in (5.2) reduce to the Jacobi identity. (cf. [KNS] eqs.(2.20)-(2.22).) To prove the third equation, substitute (4.6a) into $T^{(r)}_{2m+1}(u)T^{(r)}_{2m-1}(u)$. Expanding the determinants with respect to the first column, we have

$$\begin{aligned}
T^{(r)}_{2m+1}(u)T^{(r)}_{2m-1}(u) &= \sum_{i=0}^{m-1}\sum_{j=0}^{m}(-1)^{i+j}R^{(m)}_j R^{(m-1)}_i \\
&\times \left(T^{r+j-i-1}(u-m+i+j+\tfrac{1}{2}) + T^{r+i-j}(u-m+i+j+\tfrac{1}{2})\right).
\end{aligned}$$

Here, $R^{(m)}_j$ denotes the cofactor of $T^{(r)}_1(u-m+2j)$ in $T^{(r)}_{2m+1}(u)$ and we have used (2.14). On taking the j-sum, the $T^{r+j-i-1}(u-m+i+j+\frac{1}{2})$ term vanishes. After taking the i-sum, the $T^{r+i-j}(u-m+i+j+\frac{1}{2})$ term is non-zero only for $j=0$ or $j=m$. Noting that $R^{(m)}_0 = T^{(r)}_{2m}(u+\frac{1}{2})$ and $R^{(m)}_m = T^{(r-1)}_m(u-\frac{1}{2})$, one has the third equation. The last equation in (5.2) can be verified quite similarly.

The functional relation (5.2) is the unrestricted T-system for B_r, (3.20) in [KNS] (in a different normalization). There was a factor $g^{(a)}_m(u)$ in each equation as $T^{(a)}_m(u+\frac{1}{t_a})T^{(a)}_m(u-\frac{1}{t_a}) = T^{(a)}_{m+1}(u)T^{(a)}_{m-1}(u) + g^{(a)}_m(u)(\cdots)$. The $g^{(a)}_m(u)$ is 1 here because we are considering the case where vacuum part = 1. The choice (5.1a) has been conjectured in (4.20) of [KS1] including the vacuum parts. The case $r=2$ had been proved earlier [K].

It may be interesting to regard u and m as discrete space-time variables and consider (5.2) as a discretized Toda equation. Actually, a "continuum limit" of (5.2) (with $g^{(a)}_m(u)$) under an appropriate rescaling of u, m and $g^{(a)}_m(u)$ leads to

$$(\partial_u^2 - \partial_m^2) \log \phi_a(u,m) = \text{const} \prod_{b=1}^{r} \phi_b(u,m)^{-A_{ab}},$$

where $\phi_a(u,m)$ is a scaled $T^{(a)}_m(u)$ and $A_{ab} = \frac{2(\alpha_a|\alpha_b)}{(\alpha_a|\alpha_a)}$ is the Cartan matrix. The constant above can be made arbitrary by choosing the $g^{(a)}_m(u)$ suitably. We remark that the T-system proposed in [KNS] has this aspect for all the classical simple Lie algebra X_r.

6. On vacuum parts and BAE in terms of Drinfeld polynomial

So far we have treated the case where the quantum space is formally trivial. This corresponds to choosing the LHS of the BAE (2.1) to be just -1 and the vacuum parts in the DVFs $T_{\lambda\subset\mu}(u)$, $S^L_{\lambda\subset\mu}(u)$ and $S^R_{\lambda\subset\mu}(u)$ to be 1. To recover the vacuum parts for the non-trivial quantum space

$$\otimes_{i=1}^{N} W^{(i)}, \tag{6.1}$$

one needs to know the corresponding BAE. Assuming that each $W^{(i)}$ in (6.1) is a finite dimensional irreducible $Y(B_r)$ module characterized by the Drinfeld polynomial $P^{(i)}_a(\zeta)\,(1 \le a \le r)$, we conjecture the BAE:

$$\begin{aligned} -\frac{P_a(v^{(a)}_k + \frac{1}{t_a})}{P_a(v^{(a)}_k - \frac{1}{t_a})} &= \prod_{b=1}^{r} \frac{Q_b(v^{(a)}_k + (\alpha_a|\alpha_b))}{Q_b(v^{(a)}_k - (\alpha_a|\alpha_b))} \quad 1 \le a \le r,\ 1 \le k \le N_a, \\ P_a(\zeta) &= \prod_{i=1}^{N} P^{(i)}_a(\zeta). \end{aligned} \tag{6.2}$$

Here we understand that $q \to 1$ in (2.2) for $Y(B_r)$. (On the other hand, for generic q, we suppose that (6.2) is the correct BAE for $U_q(B^{(1)}_r)$ if $P^{(i)}_a(\zeta)$ is replaced by a natural q-analogue.) The equation (6.2) has been formulated purely from the representation theoretical data, the root system and the Drinfeld polynomial. Thus we suppose that it is the BAE for any $Y(X_r)$ (or $U_q(X_r^{(1)})$ in the trigonometric case). This is actually true for all the known examples in which alternative derivations of the BAE are known such as the algebraic Bethe ansatz. It is also agreed in [ST]. Once (6.2) is admitted, the vacuum parts are determined uniquely up to an overall scalar by requiring that the pole-freeness is ensured by (6.2). This is a straightforward task and here we shall only indicate the initial step concerning Theorems 2.1 and 2.3.

Redefine $z(a;u)$ (2.4) and $sp(\xi_1,\ldots,\xi_r;u)$ (2.5) by multiplying the vacuum parts $vac(\cdots)$ (cf. (2.9a) in [KS1]):

$$\begin{aligned}
vac\, z(a;u) &= \prod_{j=1}^{a-1} P_j(u+j-1) \prod_{j=a}^{r-1} P_j(u+j+1) P_r(u+r+\frac{1}{2}) P_r(u+r-\frac{1}{2}) \\
&\quad \times \prod_{j=1}^{r-1} P_j(u+2r-j)\Phi(u) \quad 1\le a\le r, \\
vac\, z(0;u) &= \prod_{j=1}^{r-1} P_j(u+j-1) P_r(u+r-\frac{1}{2})^2 \prod_{j=1}^{r-1} P_j(u+2r-j)\Phi(u), \qquad (6.3a) \\
vac\, z(\bar{a};u) &= \prod_{j=1}^{r-1} P_j(u+j-1) P_r(u+r-\frac{1}{2}) P_r(u+r-\frac{3}{2}) \\
&\quad \times \prod_{j=1}^{a-1} P_j(u+2r-j) \prod_{j=a}^{r-1} P_j(u+2r-j-2)\Phi(u) \quad 1\le a\le r,
\end{aligned}$$

$$\Phi(u) = \prod_{b=1}^{r}\prod_{j=1}^{b-1} P_b(u+b-2j-\frac{1}{t_b}) P_b(u+2r-b+2j-1+\frac{1}{t_b}). \qquad (6.3b)$$

$$vac\, sp(\xi_1,\ldots,\xi_r;u) = \psi^{(1)}_{n_1}(u)\cdots\psi^{(r)}_{n_r}(u), \qquad (6.4a)$$

$$n_b = \sharp\{j \mid \xi_j = -, 1\le j\le b\}, \qquad (6.4b)$$

$$\begin{aligned}
\psi^{(b)}_n(u) &= \prod_{j=0}^{n-1} P_b(u+r-b+2j+\frac{1}{2}-\frac{1}{t_b}) \\
&\quad \times \prod_{j=n}^{b-1} P_b(u+r-b+2j+\frac{1}{2}+\frac{1}{t_b}). \qquad (6.4c)
\end{aligned}$$

In terms of $z(a;u)$ involving the above vacuum parts, redefine $T^a(u)$ by (2.7a) assuming $XP_b(u) = P_b(u+2)X$ $(1\le b\le r)$ and modifying the RHS into

$$\sum_{a=0}^{\infty} F_a(u+a-1) T^a(u+a-1) X^a,$$

$$F_a(u) = \prod_{b=1}^{r}\prod_{j=1}^{a-1} \psi^{(b)}_0(u+r-a-\frac{1}{2}+2j)\psi^{(b)}_b(u-r+a+\frac{1}{2}-2j).$$

It is easily seen that this $T^a(u)$ is of positive order $2b$ with respect to the P_b function (6.2). One can check that Theorem 2.1 is still valid (for $T_1^{(r)}(u)$ and $T^a(u)$) for the BAE (6.2). Relations (2.12a) and (2.13a) also hold if the right hand sides are divided by $F_a(u)$ and $F_{2r-1-a}(u)$, respectively. Thus Theorem 2.3 remains valid without any changes. Along these lines, one can proceed further to include the vacuum parts for general $T_{\lambda\subset\mu}(u)$, $S^L_{\lambda\subset\mu}(u)$ and $S^R_{\lambda\subset\mu}(u)$ so that they become pole-free under the BAE (6.2).

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Figure Captions.

Figure 3.1: An example of a skew-Young diagram $\lambda \subset \mu$. Here $\mu = (5, 4^2, 1), \lambda = (2, 1), \mu' = (4, 3^3, 1)$ and $\lambda' = (2, 1)$, respectively. The lengths of the main diagonal are given by $d_\mu = 3$ and $d_\lambda = 1$.
Figure 3.2: The way to assign the letters to each box is explained in the text. This is an example for $r = 3, \mu' = (9, 7, 2), \lambda' = (3, 1)$. Notice that zeros are arranged lest they are adjacent horizontally.
Figure 4.1: An example of an L-hatched skew-Young diagram $r = 4, \mu = (4^3, 3, 2, 1^2), \lambda = (3, 1)$.
Figure 4.2: The bottom left part of an L-hatched skew-Young tableau and the assignment of the letters $\{i_l\}$ and $\{j_l\}$ in (4.2).
Figure 4.3: An example of an R-hatched skew-Young diagram $r = 4, \mu = (4^5, 3, 1), \lambda = (3^2, 2, 1)$.
Figure 4.4: The top right part of an R-hatched skew-Young tableau and the assignment of the letters $\{i_l\}$ and $\{j_l\}$ in (4.4).
Figure 4.5: An R-hatched skew-Young diagram for $r = 3$ with $\mu = (5^4, 3, 2, 1), \lambda = (4, 3, 1^2)$. Broken lines are guides to eyes for defining $\eta = (4^3, 3, 2, 1), \xi = (3, 1^2)$.