

gr-qc/9506052
MIT-CTP-2446
TUTP-95-1
June 22, 1995

AVERAGED ENERGY CONDITIONS AND EVAPORATING BLACK HOLES

L.H. Ford

Center for Theoretical Physics, Laboratory for Nuclear Science
Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

and

Institute of Cosmology, Department of Physics and Astronomy
Tufts University, Medford, Massachusetts 02155*

and

Thomas A. Roman

Institute of Cosmology, Department of Physics and Astronomy
Tufts University, Medford, Massachusetts 02155

and

Department of Physics and Earth Sciences
Central Connecticut State University, New Britain, CT 06050†

*Permanent address; email: lford@pearl.tufts.edu

†Permanent address; email: roman@ccsu.ctstateu.edu

Abstract

In this paper the averaged weak (AWEC) and averaged null energy (ANEC) conditions, together with uncertainty principle-type restrictions on negative energy (“quantum inequalities”), are examined in the context of evaporating black hole backgrounds in both two and four dimensions. In particular, integrals over only half-complete geodesics are studied. We first examine a quantized massless scalar field in two-dimensional Schwarzschild spacetime in the Unruh vacuum state. It is found that AWEC is violated along half-complete timelike geodesics representing observers who are shot outward to infinity at high speed from near the horizon. However, there appears to be a quantum inequality-type bound on the degree of the violation. More specifically, the magnitude of the integrated negative energy density seen by such an observer is less than or equal to the inverse of the characteristic proper time over which the observer sees a significant change in the energy density. Observers who remain close to the horizon, and thus stay in the region of negative energy density for a long period of Schwarzschild time, must follow nearly lightlike trajectories. Hence these observers may experience large averaged negative energy densities but only over very short proper times. For outgoing half-complete null geodesics outside the horizon, we find that ANEC is violated, but a similar quantum inequality bounds the extent of the violation. A careful analysis shows that this result is true for null geodesics on the horizon as well. For ingoing null geodesics or for half-complete outgoing null geodesics inside the horizon, ANEC is satisfied in two dimensions. In four-dimensional Schwarzschild spacetime, we perform a similar analysis for both a conformally-coupled, quantized massless scalar field and a quantized electromagnetic field, in the Unruh vacuum state. For this study, we use previously calculated renormalized vacuum expectation values for the stress-energy tensor components of these fields outside the horizon. Our results for half-complete outgoing timelike and null geodesics outside the horizon are similar to those found in the two-dimensional case. In four-dimensional spacetime, the quantum inequality bounds seem to be even stronger than their two-dimensional counterparts. The possible relevance of these results for the validity of singularity theorems in evaporating black hole spacetimes is discussed.

1 Introduction

It is by now well-known that quantum field theory permits violations of all of the local energy conditions used in classical general relativity. These conditions are employed in a variety of ways, such as in the proofs of theorems on the occurrence of singularities in gravitational collapse and cosmology, or of cosmic censorship. Two of the weakest such local conditions are the “weak energy condition” and the “null energy condition” which state that:

$$T_{\mu\nu} u^\mu u^\nu \geq 0, \quad (1)$$

for all timelike vectors u^μ , and

$$T_{\mu\nu} K^\mu K^\nu \geq 0, \quad (2)$$

for all null vectors K^μ , respectively [1]. Note that the null energy condition follows by continuity if the weak energy condition holds. These two local conditions are satisfied by known forms of classical matter, but there are a variety of states of quantum fields which violate them [2], the most well-known of which is arguably the Casimir vacuum [3].

The extent to which quantum field theory allows violations of the local energy conditions is not yet completely clear, although progress has been made in recent years to answer this question. Two principal approaches for determining the degree of violation have been “averaged energy conditions” and “quantum inequalities.” The first, originally due to Tipler [4], involves a suitable averaging of the local conditions over timelike or null geodesics. It can be shown that many of the standard results of classical general relativity obtained via global techniques can be proved using only the averaged, rather than the local, energy conditions [4]-[9]. Here we take the “averaged weak energy condition” (AWEC) to be

$$\int_{-\infty}^{\infty} T_{\mu\nu} u^\mu u^\nu d\tau \geq 0, \quad (3)$$

where the integral is taken over a timelike geodesic with tangent vector u^μ and parameterized by the proper time τ . Similarly, we take the “averaged null energy condition” (ANEC) to be

$$\int_{-\infty}^{\infty} T_{\mu\nu} K^\mu K^\nu d\lambda \geq 0, \quad (4)$$

where the integral is taken over an affinely parameterized null geodesic with tangent vector K^μ and affine parameter λ [10]. There has been a great deal of effort in the last several years to determine whether quantum field theory enforces averaged energy conditions. Most of this activity has been concentrated primarily on ANEC [11]-[16]. This is in part due to the discovery that violations of ANEC are required to maintain traversable wormholes [17, 18]. It appears that although ANEC holds for a wide class of quantum states in a variety of spacetimes, it does not hold in an arbitrary four-dimensional curved spacetime (see Refs. [13], [14], and [15] for more

detailed discussions). However, it is quite plausible that a suitable generalization of ANEC may hold in more general spacetimes [16, 19].

The second approach involves uncertainty principle-type inequalities, derived from quantum field theory, which restrict the magnitude and duration of negative energy fluxes or densities [19, 20, 21]. For example, one such “quantum inequality” (QI) for negative energy fluxes seen by inertial observers in two-dimensional flat spacetime has the form:

$$|F|(\Delta\tau)^2 \lesssim 1, \quad (5)$$

where $|F|$ is the magnitude of the negative energy flux and $\Delta\tau$ is its duration. This inequality implies that ΔE , the amount of negative energy transmitted in time $\Delta\tau$, is limited by

$$|\Delta E|\Delta\tau \lesssim 1. \quad (6)$$

Therefore, ΔE is less than the quantum uncertainty in the energy, $(\Delta\tau)^{-1}$, on the timescale $\Delta\tau$. Recently, more precise versions of these inequalities have been derived [21]. These involve an integral of the energy flux multiplied by a “sampling function”, i.e., a peaked function of time with a time integral of unity and characteristic width τ_0 . A convenient choice [22] of such a function is $\tau_0/[\pi(\tau^2 + \tau_0^2)]$. If the integrated flux, \hat{F} , is defined by

$$\hat{F} \equiv \frac{\tau_0}{\pi} \int_{-\infty}^{\infty} \frac{F(\tau)}{\tau^2 + \tau_0^2} d\tau, \quad (7)$$

then these inequalities may be written as

$$\hat{F} \gtrsim -\frac{1}{16\pi\tau_0^2}, \quad (8)$$

and

$$\hat{F} \gtrsim -\frac{3}{32\pi^2\tau_0^4}, \quad (9)$$

for all τ_0 , in two- and four-dimensions, respectively. These inequalities have the form required to prevent macroscopic violations of the second law of thermodynamics [20, 21]. It was also discovered that similar inequalities hold for a quantized massless, minimally-coupled scalar field propagating on two- and four-dimensional extreme Reissner-Nordström black hole backgrounds. These inequalities were shown to foil attempts to produce an unambiguous violation of cosmic censorship by injecting a negative energy flux into an extreme charged black hole [23, 24]. The latter results provide perhaps one of the strongest reasons for the belief that the production of large-scale effects via manipulation of negative energy may be forbidden by quantum field theory. Classically, *any* amount of negative energy, no matter how small, injected into an extreme black hole is enough to produce a naked singularity. Therefore, if one had any chance of producing gross effects with negative energy, the scenario discussed above would seem to have offered the best possibility of success. It should also be emphasized that the energy-time uncertainty principle was *not* used as input

in the derivation of any of the QI restrictions. They arise directly from quantum field theory.

In a recent paper, we have shown that there exist deep connections between averaged energy conditions and quantum inequalities, at least in flat spacetime [19]. For a quantized massless, minimally-coupled scalar field in two- and four-dimensional Minkowski spacetime, we have derived analogous QI's to Eqs. (8) and (9) for energy density. In particular, for timelike geodesics in two-dimensional flat spacetime,

$$\frac{\tau_0}{\pi} \int_{-\infty}^{\infty} \frac{\langle T_{\mu\nu} u^\mu u^\nu \rangle d\tau}{\tau^2 + \tau_0^2} \geq -\frac{1}{8\pi\tau_0^2}, \quad (10)$$

for all τ_0 . Here $\langle T_{\mu\nu} u^\mu u^\nu \rangle$ is the renormalized expectation value taken in an arbitrary quantum state $|\psi\rangle$. In the limit as $\tau_0 \rightarrow \infty$, we “sample” the entire geodesic, and our inequality Eq. (10) reduces to AWEC. A similar inequality was also obtained for null geodesics:

$$\frac{\lambda_0}{\pi} \int_{-\infty}^{\infty} \frac{\langle T_{\mu\nu} K^\mu K^\nu \rangle d\lambda}{\lambda^2 + \lambda_0^2} \geq -\frac{1}{16\pi\lambda_0^2}, \quad (11)$$

for all λ_0 . This inequality is the null analog of Eq. (10). If we take the limit of Eq. (11) as $\lambda_0 \rightarrow \infty$, we get ANEC.

For a quantized massless scalar field in a two-dimensional Minkowski spacetime with compactified spatial dimension, it was also discovered that the *difference* between the expectation values of $T_{\mu\nu} u^\mu u^\nu$, or $T_{\mu\nu} K^\mu K^\nu$, in an arbitrary quantum state and in the Casimir vacuum state obey AWEC and ANEC-type inequalities [19]. This is surprising since it is known [11] that AWEC and ANEC are violated for $\langle T_{\mu\nu} u^\mu u^\nu \rangle$ and $\langle T_{\mu\nu} K^\mu K^\nu \rangle$, respectively, in the (renormalized) Casimir vacuum state by itself. Such “difference inequalities” might provide new measures of the degree of energy condition violation in cases where the usual averaged energy conditions and QI's fail. This approach has recently been generalized to arbitrary two-dimensional curved spacetimes by Yurtsever [15].

For a quantized massless, minimally-coupled scalar field in four-dimensional Minkowski spacetime, we also derived the following inequality for timelike geodesics

$$\hat{\rho} = \frac{\tau_0}{\pi} \int_{-\infty}^{\infty} \frac{\langle T_{\mu\nu} u^\mu u^\nu \rangle d\tau}{\tau^2 + \tau_0^2} \geq -\frac{3}{32\pi^2\tau_0^4}, \quad (12)$$

for all τ_0 . From this inequality, it was shown that one can derive AWEC and ANEC [19].

In almost all studies of ANEC for quantum fields, the bounds on the ANEC integral have been taken from $-\infty$ to $+\infty$ [11, 13, 25]. However, a question of some interest is whether ANEC is satisfied in the spacetime of an object collapsing to form a black hole. The answer to this question might determine, for example, whether Penrose's singularity theorem [26] will still hold in the presence of local violations of the energy conditions, such as the Hawking evaporation process [27, 28]. In this case, one is usually concerned with the focusing of null geodesics which generate the boundary of the future of a trapped surface. To prove Penrose's theorem, one would

want ANEC to hold over these half-complete geodesics [7, 8]. This is a stronger condition to impose than to demand that ANEC hold over full (i.e., past and future-complete) geodesics. However, even if this version of ANEC fails in some regions of a given spacetime, it may hold in enough other regions so that the conclusions of Penrose’s theorem will still be valid. In addition, it should be noted that ANEC is a sufficient condition to insure continued focusing of null geodesics. One can also guarantee focusing with other conditions, for example by allowing the right-hand-side of the ANEC integral to be only periodically non-negative [6], or even negative but bounded (see Eq. (5) of Ref. [8], and Ref. [16]).

In the current paper, we examine quantized massless scalar, and electromagnetic fields in the Unruh vacuum state in two- and four-dimensional Schwarzschild black hole backgrounds. We evaluate AWEC and ANEC integrals along half-complete timelike and null geodesics, respectively. Our goal is to determine the regions of these spacetimes in which the averaged energy conditions fail, and whether there exist any bounds on the extent of the violation. We find that AWEC is violated along timelike geodesics which begin close to the horizon and reach infinity. These correspond to timelike observers who are shot outward from near the horizon at high speed. However, there appear to be QI’s which bound the degree of violation. Let $\delta\tau$ be the characteristic proper time lapse over which the observer sees a significant change in the negative energy density. Then, in one form of the QI bound, the AWEC integral is found to be greater than or equal to minus the inverse of $\delta\tau$. (In 4D spacetime, an additional factor of the square of the black hole mass appears in the denominator on the right-hand-side of the inequality.) For null geodesics in 2D spacetime, ANEC is violated along radially outgoing null geodesics which begin just outside the horizon and go to infinity. Once again a QI similar to that found for timelike geodesics bounds the degree of the violation. A careful analysis also shows this to be true for half-complete null geodesics on the horizon. For outgoing null geodesics just inside the horizon and for ingoing null geodesics, ANEC is shown to be satisfied in 2D. Calculations in 4D spacetime for outgoing null geodesics just outside the horizon yield similar results to those in 2D. All of the QI’s are invariant under rescaling of the affine parameter.

The paper is organized as follows. The analysis of 2D evaporating black holes is presented in Sec. (2); the results for 4D black holes are given in Sec. (3). The latter section also includes a discussion of ANEC along orbiting null geodesics. Concluding remarks are given in Sec. (4). Our units are taken to be: $\hbar = G = c = 1$.

2 2D Evaporating Black Holes

2.1 Timelike Observers

In this section, we study AWEC and ANEC for a quantized, massless scalar field in the Unruh vacuum state, along half-complete geodesics in a 2D Schwarzschild spacetime. Our goal is to determine in which regions of the spacetime the averaged

energy conditions hold and, in regions where the conditions are violated, to ascertain whether there exist any bounds on the degree of the violation. The discussion here is a natural extension of the analysis in Ref. [29].

Let u^μ be the two-velocity of an inertial observer. The energy density in this observer's frame is given by

$$U = T_{\mu\nu} u^\mu u^\nu. \quad (13)$$

In the following discussion, we understand $T_{\mu\nu}$ to denote a quantum expectation value in a specified vacuum state. The metric is

$$ds^2 = -C dt^2 + C^{-1} dr^2, \quad (14)$$

where $C = 1 - 2M/r$. A geodesic observer's two-velocity is

$$u^\mu = (u^t, u^r) = \left(\frac{dt}{d\tau}, \frac{dr}{d\tau} \right) = \left(\frac{k}{C}, \pm \sqrt{k^2 - C} \right). \quad (15)$$

Here the $+$ and $-$ refer to outgoing and ingoing observers, respectively. The constant k is the energy per unit rest mass. In our two-dimensional discussion, we will consider observers moving in both the Unruh and Boulware vacua. The stress tensor components in the Unruh vacuum are [30]:

$$T_{tt} = \frac{1}{24\pi} \left(\frac{7M^2}{r^4} - \frac{4M}{r^3} + \frac{1}{32M^2} \right), \quad (16)$$

$$T_{tr} = -\frac{1}{24\pi} \left(1 - \frac{2M}{r} \right)^{-1} \frac{1}{32M^2}, \quad (17)$$

and

$$T_{rr} = -\frac{1}{24\pi} \left(1 - \frac{2M}{r} \right)^{-2} \left(\frac{M^2}{r^4} - \frac{1}{32M^2} \right). \quad (18)$$

The corresponding components in the Boulware vacuum are given by:

$$T_{tt} = \frac{1}{24\pi} \left(\frac{7M^2}{r^4} - \frac{4M}{r^3} \right), \quad (19)$$

$$T_{tr} = 0, \quad (20)$$

and

$$T_{rr} = -\frac{1}{24\pi} \frac{M^2}{r^4} \left(1 - \frac{2M}{r} \right)^{-2}. \quad (21)$$

For an outgoing timelike observer in the Unruh vacuum,

$$T_{\mu\nu} u^\mu u^\nu = \frac{1}{24\pi} C^{-2} \left\{ k^2 \left[\frac{6M^2}{r^4} - \frac{4M}{r^3} + \frac{1}{16M^2} \right] + \frac{CM^2}{r^4} - \frac{C}{32M^2} - \frac{k\sqrt{k^2 - C}}{16M^2} \right\}. \quad (22)$$

With an eye toward the examination of null rays, let us consider the case of an observer who starts out initially very close to the horizon and is shot outward at large velocity, i.e., we are interested in the limits

$$\epsilon \ll M, \quad k \gg 1, \quad (23)$$

where $r = 2M + \epsilon$ is the observer's initial position. In this limit $k \gg C$, so from the geodesic equation, Eq. (15), we have that $d\tau \sim dr/k$. We now wish to multiply $T_{\mu\nu}u^\mu u^\nu$ by a sampling function, i.e., a peaked function of time whose time integral is unity. In Refs. [19] and [21], this function was chosen to be

$$\frac{\tau_0}{\pi} \int_{-\infty}^{\infty} \frac{d\tau}{\tau^2 + \tau_0^2} = 1, \quad (24)$$

where the integral was taken over complete geodesics. However, unlike in Ref. [19], here we are integrating over half-infinite geodesics. Since the function given by Eq. (24) is symmetric about $\tau = 0$, in our case we may choose

$$\frac{2\tau_0}{\pi} \int_0^{\infty} \frac{d\tau}{\tau^2 + \tau_0^2} = 1, \quad (25)$$

where the proper time τ is initialized when the observer starts at $r = 2M + \epsilon$. If we multiply $T_{\mu\nu}u^\mu u^\nu$ by this sampling function, and perform the resulting integral using MACSYMA, we find in the limit of $k \gg 1$ and $\epsilon \ll M$:

$$\begin{aligned} \frac{2\tau_0}{\pi} \int_0^{\infty} \frac{T_{\mu\nu}u^\mu u^\nu d\tau}{\tau^2 + \tau_0^2} &\sim -\frac{k}{24\pi^2 \tau_0 \epsilon} \\ &= -\frac{1}{24\pi^2 \tau_0 \delta\tau}, \end{aligned} \quad (26)$$

where

$$\delta\tau = \frac{\epsilon}{k}. \quad (27)$$

Note that in these limits $\delta\tau \ll M$.

What is the physical significance of $\delta\tau$? The energy density drops off very rapidly with increasing r . In the limit $k \gg 1$ and $\epsilon \ll M$, if we consider the proper time spent by the observer in the region of appreciable negative energy to be

$$\int_0^{\delta\tau} d\tau \sim \frac{1}{k} \int_{2M+\epsilon}^{2M+x\epsilon} dr = \frac{1}{k} \epsilon(x-1), \quad (28)$$

then typically $x-1 \sim O(1)$, so

$$\delta\tau \sim \frac{\epsilon}{k}. \quad (29)$$

To see this a little more explicitly, note that in these limits, Eq. (22) becomes

$$T_{\mu\nu}u^\mu u^\nu \sim -\frac{k^2}{48\pi(r-2M)^2}. \quad (30)$$

If the observer starts at $r = r_0 = 2M + \epsilon$, then at $r = r_1 = 2M + 2\epsilon$,

$$(T_{\mu\nu}u^\mu u^\nu)|_{r=r_1} \approx \frac{1}{4} (T_{\mu\nu}u^\mu u^\nu)|_{r=r_0}, \quad (31)$$

so the energy density drops off to 1/4 of its initial value in a distance $\Delta r = \epsilon$, corresponding to a proper time $\delta\tau = \epsilon/k$. Therefore, we should consider the time interval $\delta\tau$ not to be the entire proper time over which the energy density is negative, but rather a time scale which characterizes a change in the energy density.

Rather than performing the integration in Eq. (26) and then taking the limits $k \gg 1$ and $\epsilon \ll M$, we could have performed these operations in the opposite order. Because the dominant contribution to the integral comes from the lower limit, we may write

$$\begin{aligned} \frac{2\tau_0}{\pi} \int_0^\infty \frac{T_{\mu\nu}u^\mu u^\nu d\tau}{\tau^2 + \tau_0^2} &\approx \frac{2}{\pi\tau_0} \int_0^\infty T_{\mu\nu}u^\mu u^\nu d\tau \\ &\approx \frac{2}{\pi k\tau_0} \int_{2M+\epsilon}^\infty T_{\mu\nu}u^\mu u^\nu dr, \end{aligned} \quad (32)$$

where from Eq. (15), $d\tau \sim dr/k$. Insertion of Eq. (30) into this expression yields Eq. (26).

If we set the width of the sampling function, τ_0 , to be equal to $\delta\tau$, we get

$$\frac{2\tau_0}{\pi} \int_0^\infty \frac{T_{\mu\nu}u^\mu u^\nu d\tau}{\tau^2 + \tau_0^2} \approx -\frac{1}{24\pi^2 \tau_0^2} \quad (33)$$

which has the form of a QI-type bound [19],

$$\frac{2\tau_0}{\pi} \int_0^\infty \frac{T_{\mu\nu}u^\mu u^\nu d\tau}{\tau^2 + \tau_0^2} \gtrsim -\frac{1}{\tau_0^2}. \quad (34)$$

If instead we take the $\tau_0 \rightarrow \infty$ limit of Eq. (26), we “sample” the entire (i.e., half-infinite) geodesic and find

$$\int_0^\infty T_{\mu\nu}u^\mu u^\nu d\tau \approx -\frac{1}{48\pi\delta\tau}, \quad (35)$$

which has the form

$$\int_0^\infty T_{\mu\nu}u^\mu u^\nu d\tau \gtrsim -\frac{1}{\delta\tau}. \quad (36)$$

Let us now turn to the case where the quantum state is the Boulware vacuum. For an outgoing timelike observer in this state,

$$T_{\mu\nu}u^\mu u^\nu = \frac{1}{24\pi} C^{-2} \left\{ k^2 \left[\frac{6M^2}{r^4} - \frac{4M}{r^3} \right] + \frac{CM^2}{r^4} \right\}. \quad (37)$$

Let us again consider the case where $k \gg 1$. Then the k^2 term will dominate and

$$T_{\mu\nu} u^\mu u^\nu \sim \frac{k^2 M}{12\pi C^2 r^3} \left(\frac{3M}{r} - 2 \right). \quad (38)$$

Note that this quantity is negative everywhere for $r > 2M$. Also in this limit $dr/d\tau = \sqrt{k^2 - C} \sim k$, so $\delta r \sim k \delta\tau$.

Assume that we start at $r = 2M + \epsilon$. Then $C \approx \epsilon/2M$, and

$$T_{\mu\nu} u^\mu u^\nu \approx -\frac{k^2}{48\pi \epsilon^2}. \quad (39)$$

Note that as r increases from $2M + \epsilon$ to $2M + 2\epsilon$, $T_{\mu\nu} u^\mu u^\nu$ falls off by a factor of 4, so we can take $\delta r = \epsilon$ and $\delta\tau = \epsilon/k$. Then

$$\begin{aligned} \int_0^\infty T_{\mu\nu} u^\mu u^\nu d\tau &\approx \left(T_{\mu\nu} u^\mu u^\nu \right)_{r=2M+\epsilon} \delta\tau \\ &\approx -\frac{1}{48\pi \delta\tau}. \end{aligned} \quad (40)$$

Note that Eqs. (35) and (40), the results for the Unruh and Boulware vacua, respectively, are identical. This indicates that the dominant contribution to the integral is the static vacuum energy rather than the Hawking radiation.

The inequality, Eq. (36), represents a bound on the degree of AWEC violation seen by timelike geodesic observers who start out very close to the horizon and are shot outward at very high speed. In this limit, the longer the (proper) timescale over which the observer sees a significant change in the energy density, the smaller is the magnitude of the integrated negative energy density seen by that observer. Since the negative energy density drops off rapidly with increasing r , to remain in the negative energy density the observer must stay close to the horizon. The closer the observer is to the horizon, the larger is the magnitude of the negative energy density. However, in order to remain close to the horizon for a long time as seen by a distant observer, the observer's trajectory must be nearly lightlike. Therefore, although the observer spends a long time in the negative energy region as seen by the distant observer, the *proper* time spent in the region of appreciable negative energy (as measured by $\delta\tau$) decreases with the observer's proximity to the horizon.

The QI, Eq. (36), for outgoing observers in 2D spacetime who are shot out from near $r = 2M$, is very similar to that of the flux constraint for these observers found in Ref. [29], i.e., $|F|(\Delta\tau)^2 \lesssim 1$. Since

$$\int_0^\infty T_{\mu\nu} u^\mu u^\nu d\tau \approx \left(T_{\mu\nu} u^\mu u^\nu \right)_0 \delta\tau, \quad (41)$$

we have

$$\left| T_{\mu\nu} u^\mu u^\nu \right|_0 \delta\tau \lesssim \frac{1}{\delta\tau}. \quad (42)$$

Hence the expression above is analogous to $|F|(\Delta\tau) \lesssim 1/(\Delta\tau)$. This similarity is not surprising, since for these 2D ultrarelativistic observers $|T_{\mu\nu} u^\mu u^\nu| \approx |F|$.

2.2 Null Geodesics in 2D: Off the Horizon

We now wish to examine ANEC for null geodesics in 2D Schwarzschild spacetime. From Eq. (14), and from the equation for null geodesics, we have

$$\frac{dt}{d\lambda} = \pm C^{-1} E, \quad \frac{dr}{d\lambda} = \pm E, \quad (43)$$

where E is an arbitrary positive constant, whose value fixes the scale of the affine parameter, λ . We want to examine ANEC along both ingoing and outgoing future-directed (i.e., $d\lambda > 0$) null geodesics, inside and outside the horizon. There are four cases (See Fig. 1):

$$1) \quad \frac{dt}{d\lambda} > 0, \quad \frac{dr}{d\lambda} = E > 0, \quad \text{for } r > 2M \text{ (outgoing rays)}, \quad (44)$$

$$2) \quad \frac{dt}{d\lambda} > 0, \quad \frac{dr}{d\lambda} = -E < 0, \quad \text{for } r > 2M \text{ (ingoing rays)}, \quad (45)$$

$$3) \quad \frac{dt}{d\lambda} > 0, \quad \frac{dr}{d\lambda} = -E < 0, \quad \text{for } r < 2M \text{ (outgoing rays)}, \quad (46)$$

$$4) \quad \frac{dt}{d\lambda} < 0, \quad \frac{dr}{d\lambda} = -E < 0, \quad \text{for } r < 2M \text{ (ingoing rays)}. \quad (47)$$

Note that $C = (1 - 2M/r)$ changes sign inside the horizon, while $d\lambda$ remains positive for future-directed null rays.

If we use Eqs. (44) - (47), and Eqs. (16) - (18), we obtain for the Unruh vacuum state

$$T_{\mu\nu} K^\mu K^\nu = \frac{E^2}{24\pi} \left(1 - \frac{2M}{r}\right)^{-2} \left[\frac{6M^2}{r^4} - \frac{4M}{r^3} \right] \text{ (outgoing)}, \quad (48)$$

and

$$T_{\mu\nu} K^\mu K^\nu = \frac{E^2}{24\pi} \left(1 - \frac{2M}{r}\right)^{-2} \left[\frac{6M^2}{r^4} - \frac{4M}{r^3} + \frac{1}{8M^2} \right] \text{ (ingoing)}. \quad (49)$$

To obtain Eq. (49), we have used the fact that the sign of T_{tr} changes inside the horizon, as can be seen from Eq. (17), together with Eqs. (45) and (47). Note that for Eq. (49), the term in square brackets vanishes as $r \rightarrow 2M$. A Taylor expansion shows that, to leading order, it vanishes as $(r - 2M)^2$, so it will cancel the $(r - 2M)^2$ divergence in the denominator. Therefore Eq. (49) is finite on the horizon, as it should be. However, there appears to be a discontinuity in Eq. (48) for an outgoing null ray on the horizon, $r = 2M$. The situation for null rays on the horizon is sufficiently subtle as to warrant a separate discussion. This is provided in the next subsection.

From Eq. (48), $T_{\mu\nu} K^\mu K^\nu < 0$ for outgoing null geodesics when $r > 1.5M$. Therefore the local null energy condition, Eq. (2) is violated along outgoing null vectors slightly inside the horizon, as well as along all outgoing rays outside (and on) the horizon. For $r > 2M$, consider an outgoing null geodesic starting at

$r = 2M + \epsilon$, with $\epsilon \ll M$. Examine ANEC along this ray to obtain

$$I_1 \equiv \int_0^\infty T_{\mu\nu} K^\mu K^\nu d\lambda = \frac{E}{24\pi} \int_{2M+\epsilon}^\infty \frac{1}{(r-2M)^2} \left[\frac{6M^2}{r^2} - \frac{4M}{r} \right] dr, \quad (50)$$

where we have used Eq. (44). Here $\lambda = 0$ is the value of the affine parameter at $r = 2M + \epsilon$. For $r < 2M$, along an outgoing null geodesic starting just inside the horizon at $r = 2M - \epsilon$:

$$I_2 \equiv \int_0^{\lambda_f} T_{\mu\nu} K^\mu K^\nu d\lambda = \frac{E}{24\pi} \int_{r_{min}}^{2M-\epsilon} \frac{1}{(r-2M)^2} \left[\frac{6M^2}{r^2} - \frac{4M}{r} \right] dr, \quad (51)$$

where we have used Eq. (46), and λ_f is the value of the affine parameter at $r = r_{min}$, the minimum value of r attained by the null geodesics. From Eqs. (45), (47), and (49), for ingoing null geodesics starting at $r = r_0$ (either inside or outside the horizon) we find

$$I_3 \equiv \int_0^{\lambda_f} T_{\mu\nu} K^\mu K^\nu d\lambda = \frac{E}{24\pi} \int_{r_{min}}^{r_0} \frac{1}{(r-2M)^2} \left[\frac{6M^2}{r^2} - \frac{4M}{r} + \frac{r^2}{8M^2} \right] dr, \quad (52)$$

Notice that the integrand in Eqs. (51) and (52) will be dominated by the positive $6M^2/r^2$ term which diverges for small r . Thus we immediately see that ANEC is satisfied for these two sets of null geodesics.

In the limit of small ϵ , we may expand the integrand in Eq. (50) around $r = 2M$ and then perform the integration to find

$$\lim_{\epsilon \rightarrow 0} I_1 \sim -\frac{E}{48\pi \epsilon} < 0. \quad (53)$$

Similarly, for Eqs. (51) and (52), we may find the forms for small r_{min} by expanding the integrands around $r = 0$. The results are

$$\lim_{r_{min} \rightarrow 0} I_2 \sim \frac{E}{16\pi r_{min}} > 0 \text{ for fixed } \epsilon > 0, \quad (54)$$

and

$$\lim_{r_{min} \rightarrow 0} I_3 \sim \frac{E}{16\pi r_{min}} > 0 \text{ for fixed } r_0. \quad (55)$$

Thus we see that ANEC is violated for outgoing null rays just outside the horizon, but it is satisfied for outgoing null rays just inside the horizon. (The divergence in Eq. (53) as $\epsilon \rightarrow 0$ may be circumvented by an appropriate choice of scaling of the affine parameter, i.e., by an appropriate choice of E . This and related issues will be discussed in the next sub-section.) Even though outgoing rays which originate in the region $1.5M < r < 2M$ initially encounter negative energy, the sign of the integral of $T_{\mu\nu} K^\mu K^\nu$ is determined by a positive vacuum polarization term which

dominates near the singularity, $r = 0$. We also see that ANEC holds for ingoing null geodesics, whether they originate outside or inside the horizon, and that once again the dominance of the positive vacuum polarization term near the singularity determines the sign of the integral.

One might at first sight conclude that the reasoning here is a bit circular. It could be argued that it is no surprise that evaluation of the ANEC integral for null geodesics inside the horizon yields a divergent positive result. After all, we placed our quantized field on a background which already had a singularity in it. Thus one could argue that divergent values of our ANEC integrals for these geodesics are not unexpected, because the behavior of the fields inside the horizon is dominated by the singularity. However, it is not obvious, a priori, what the *sign* of these integrals should be. Since these are test fields on a given background, as opposed to fields which generate the background spacetime (which we cannot have in any case in 2D), it is possible that the integrals might have turned out to diverge with either sign. Similar ANEC integrals for quantized massless fermion fields in 2D have the same signs as for the massless scalar field. This follows from the fact that the renormalized fermion stress tensor is identical to that in the scalar case [31], even though the formally divergent tensors have opposite signs in the two cases.

Let us now take the “null limit” of our integrals for outgoing timelike observers. We can think of this as the limit of a series of outgoing timelike geodesic observers who are shot outward from near the horizon with ever increasing speeds. Define

$$\lambda = \alpha k \tau, \quad \lambda_0 = \alpha k \tau_0, \quad \delta \lambda = \alpha k \delta \tau, \quad (56)$$

where α is an arbitrary constant, and as before, $\delta \tau = \epsilon/k$. In the null limit, $k \rightarrow \infty$ as $\tau \rightarrow 0$, such that the product $k\tau$ remains finite [32]. Therefore the null limit of Eq. (26), representing outgoing null geodesics just outside the horizon in the Unruh vacuum state, is

$$\frac{2\lambda_0}{\pi} \int_0^\infty \frac{T_{\mu\nu} K^\mu K^\nu d\lambda}{\lambda^2 + \lambda_0^2} \approx -\frac{1}{24\pi^2 \lambda_0 \delta \lambda}. \quad (57)$$

If we set the width of our sampling function, λ_0 , equal to $\delta \lambda$, we get

$$\frac{2\lambda_0}{\pi} \int_0^\infty \frac{T_{\mu\nu} K^\mu K^\nu d\lambda}{\lambda^2 + \lambda_0^2} \approx -\frac{1}{24\pi^2 \lambda_0^2} \quad (58)$$

which has the form of a QI-type bound [19],

$$\frac{2\lambda_0}{\pi} \int_0^\infty \frac{T_{\mu\nu} K^\mu K^\nu d\lambda}{\lambda^2 + \lambda_0^2} \gtrsim -\frac{1}{\lambda_0^2}. \quad (59)$$

If instead we take the $\lambda_0 \rightarrow \infty$ limit of Eq. (57), we “sample” the entire (i.e., half-infinite) null geodesic and find

$$\int_0^\infty T_{\mu\nu} K^\mu K^\nu d\lambda \approx -\frac{1}{48\pi \delta \lambda}, \quad (60)$$

which has the form

$$\int_0^\infty T_{\mu\nu} K^\mu K^\nu d\lambda \gtrsim -\frac{1}{\delta\lambda}. \quad (61)$$

As a check, note that with $\delta\lambda = \epsilon/E$, Eq. (60) agrees with Eq. (53), which we obtained by initially starting with the expressions for null geodesics. Here we can think of $\delta\lambda$ as a characteristic lapse of affine parameter over which the energy density changes significantly along the null geodesic. Therefore, although ANEC is violated along outgoing null geodesics outside the horizon, there appears to be a QI-type bound on the degree of the violation, at least in our simple 2D model. We find the same null limit for high k observers, i.e., Eq. (61), in the Boulware vacuum as well. Note that Eqs. (57)- (61) are all invariant under rescaling of the affine parameter [19].

2.3 Null Geodesics in 2D: On the Horizon

Recall that our results for outgoing null geodesics outside the horizon are formulated in terms of the null tangent vector

$$K^\mu = E (C^{-1}, 1). \quad (62)$$

where the components are given in terms of Schwarzschild t, r coordinates. Since these coordinates are badly-behaved on the horizon, let us switch to Kruskal null coordinates, given by

$$U = -e^{-\kappa u}, \quad V = e^{\kappa v}, \quad (63)$$

where $u = t - r^*$, $v = t + r^*$, with r^* the usual tortoise coordinate and $\kappa = 1/(4M)$. In these coordinates, our null vector has the form

$$K^{\bar{\mu}} = \left(\frac{dU}{d\lambda}, \frac{dV}{d\lambda} \right) = \left(0, \frac{2E}{C} \kappa e^{\kappa v} \right). \quad (64)$$

If E has the same value for all outgoing null geodesics, then $K^{\bar{\mu}}$ would not be defined in the limit where the geodesics approach the horizon, since $C \rightarrow 0$. However, E need only be constant along each null geodesic, but not the same constant for different null geodesics.

Consider a sequence of outgoing null geodesics which start at different values of r , e.g., the histories of a sequence of photons emitted by an infalling observer. Let r_0 be the value of r at which the null geodesic begins (which will be different for each null geodesic). We might choose the various values of E for different null geodesics by the following prescription. Let the observer emit photons of fixed frequency ω_e in his rest frame. The time-component of the observer's two-velocity (in Schwarzschild coordinates) is

$$u^t = \frac{dt}{d\tau} = \left(1 - \frac{2M}{r_0} \right)^{-1} \tilde{E}, \quad (65)$$

where \tilde{E} is the observer's energy per unit rest mass, measured at infinity. This is just the rate of a clock at infinity as compared to the rate of the clock carried by the

infalling observer. The frequency of a photon at infinity is ω_∞ , which is related to ω_e by

$$\frac{\omega_\infty}{\omega_e} = \frac{d\tau}{dt} = \left(1 - \frac{2M}{r_0}\right) \tilde{E}^{-1}. \quad (66)$$

Note that the expression on the right-hand-side includes both the Doppler effect and the gravitational redshift. The factor of \tilde{E}^{-1} reflects the fact that the faster the observer is shot toward the black hole (i.e., the larger \tilde{E} is), the greater will be the Doppler shift. If we choose a scaling for the affine parameter such that $E = \omega_\infty$, for fixed ω_e , then

$$E = \left(1 - \frac{2M}{r_0}\right) \omega_e \tilde{E}^{-1} \propto C(r_0). \quad (67)$$

Now $E/C(r_0)$ is finite in the limit $r_0 \rightarrow 2M$, so that the components of K^μ are finite on the horizon, in either Schwarzschild or Kruskal coordinates.

However, $K^{\bar{\mu}}$ is still not the same as $k^{\bar{\mu}}$, the *affinely* parameterized null tangent vector on the horizon. In Kruskal null coordinates the vector $k^{\bar{\mu}}$ has components

$$k^{\bar{\mu}} = (0, \kappa). \quad (68)$$

(See the discussion on p. 331 of Ref. [33].) On the horizon,

$$K^{\bar{\mu}} = \left(\frac{2E}{C}\right)_{r=2M} e^{\kappa v} (0, \kappa), \quad (69)$$

so these vectors still differ by a factor of $\exp(\kappa v)$. Thus, a null vector such as K^μ , which is an affinely parameterized null tangent vector off the horizon, does not necessarily remain affinely parameterized *on* the horizon.

With these lessons in hand, let us now consider the integral $\int T_{\bar{\mu}\bar{\nu}} k^{\bar{\mu}} k^{\bar{\nu}} dV$ along a portion of the future event horizon. Here $T_{\bar{\mu}\bar{\nu}}$ is the vacuum expectation value of the stress-tensor in the Unruh vacuum state expressed in Kruskal null coordinates, and $k^{\bar{\mu}}$ is an affinely parameterized null tangent vector on the horizon. The Kruskal advanced time coordinate V , is an affine parameter on the horizon (see for example, p. 331 of Ref. [33] or p. 122 of Ref. [34]). From Eq. (68), and a straightforward coordinate transformation of Eqs. (16)- (18), we obtain

$$\int_{V_0}^{\infty} T_{\bar{\mu}\bar{\nu}} k^{\bar{\mu}} k^{\bar{\nu}} dV = -\frac{1}{768 \pi M^2 V_0}. \quad (70)$$

This integral is taken over a portion of the future horizon; V_0 is the value of the $V = \text{const}$ line which intersects the future horizon at the (arbitrary) event where we start our null geodesic.

The right-hand-side of Eq. (70) goes to $-\infty$ as $V_0 \rightarrow 0$. However, this would correspond to integrating along the full future horizon of an eternal black hole spacetime. The Unruh vacuum state is singular on the past horizon (i.e., $V = 0$) of such a spacetime, so it is not surprising that we get a divergence in this case. Physically,

this would correspond to a black hole that Hawking-radiates for an infinite time. For the Unruh vacuum state to be realizable, we must consider only physically realistic collapse spacetimes in which the black hole forms at a finite time in the past. Therefore, the smallest allowed value of $V_0 = V_{min}$, which is the value of the $V = const$ line at which the the worldline of the surface of the collapsing star intersects the future event horizon, as shown in Fig. (2). For such spacetimes, the ANEC integral along a portion of the future horizon is negative but finite. This result seems to be in agreement with the results of Wald and Yurtsever [13]. They find that ANEC is satisfied along the complete null geodesic comprised of the future horizon plus the null line which would have been the past horizon in an eternal black hole spacetime. In a collapse spacetime, this complete null geodesic originates at past null infinity, propagates through the collapsing star, “reflects off” $r = 0$, and then ultimately joins on to the future horizon at $V_0 = V_{min}$ (see Fig. (2)). Evidently there is a positive contribution to the ANEC integral from the part of the geodesic prior to its exit from the collapsing star [35]. This contribution must be greater than or equal to $(768 \pi M^2 V_{min})^{-1}$. (Note that the source of the positive contribution we refer to here is the quantized field, not the classical collapsing matter. The latter would presumably make the ANEC integral even more positive.)

The dimensions of $k^{\bar{\mu}}$ are $(length)^{-1}$, whereas $T_{\bar{\mu}\bar{\nu}}$, V , and V_0 are dimensionless. Let us now rescale the coordinates to get a more familiar choice of dimensions. Let $x^{\hat{\mu}} = \kappa^{-1} x^{\mu}$, i.e., $\hat{U} = \kappa^{-1} U$, $\hat{V} = \kappa^{-1} V$, so $\hat{V}_0 = \kappa^{-1} V_0$. Then $T_{\hat{\mu}\hat{\nu}} = \kappa^2 T_{\bar{\mu}\bar{\nu}}$, and $k^{\hat{\mu}} = \kappa^{-1} k^{\bar{\mu}}$. If we rewrite Eq. (70) in terms of the rescaled coordinates, we obtain

$$\begin{aligned} \int_{\hat{V}_0}^{\infty} T_{\hat{\mu}\hat{\nu}} k^{\hat{\mu}} k^{\hat{\nu}} d\hat{V} &= -\frac{1}{768 \pi \kappa^2 M^2 \hat{V}_0} \\ &= -\frac{1}{48\pi \hat{V}_0}. \end{aligned} \quad (71)$$

Compare this result with our result for null rays outside the horizon, i.e.,

$$\int_0^{\infty} T_{\mu\nu} K^{\mu} K^{\nu} d\lambda \gtrsim -\frac{1}{\delta\lambda}, \quad (72)$$

where $\delta\lambda$ was interpreted as the characteristic affine parameter distance along the null geodesic over which the negative energy density falls off. Let us now show that a similar interpretation holds *on* the horizon. The integrand of Eq. (71) is equal to $-1/(48\pi \hat{V}^2)$, so

$$\begin{aligned} \int_{\hat{V}_0}^{\infty} T_{\hat{\mu}\hat{\nu}} k^{\hat{\mu}} k^{\hat{\nu}} d\hat{V} &= -\frac{1}{48\pi} \int_{\hat{V}_0}^{\infty} \left(\frac{1}{\hat{V}^2} \right) d\hat{V} \\ &= -\frac{1}{48\pi \hat{V}_0}. \end{aligned} \quad (73)$$

Note that \hat{V}_0 can be interpreted as the characteristic affine parameter distance over which the integrand falls off, i.e., as \hat{V} increases from $\hat{V} = \hat{V}_0$ to $\hat{V} = 2\hat{V}_0$, $T_{\hat{\mu}\hat{\nu}} k^{\hat{\mu}} k^{\hat{\nu}}$ falls off to 1/4 of its initial value. Thus the interpretation of the QI for null geodesics on the horizon is the same as that for Eq. (72).

3 4D Evaporating Black Holes

In this section, we examine AWEC and ANEC in four-dimensional evaporating black hole spacetimes. In these spacetimes, the components of the renormalized expectation value of the stress-energy tensor are only known numerically. Accordingly, we make use of the numerical data of Elster [36] for the conformally-coupled massless scalar field and that of Jensen, McLaughlin, and Ottewill [37] for the electromagnetic field, evaluated in the Unruh vacuum state. We examine the cases of outgoing radial timelike observers, outgoing radial null geodesics, and orbiting null geodesics. We find that although AWEC and ANEC are violated in all three cases, as in two-dimensional spacetime, there appear to be QI-type bounds in each case which constrain the degree of the violations.

3.1 Outgoing Radial Timelike Observers

The four-velocity of an outgoing radial timelike observer in 4D Schwarzschild spacetime is given by

$$u^\mu = \left(\frac{k}{C}, \sqrt{k^2 - C}, 0, 0 \right), \quad (74)$$

where again $C = (1 - 2M/r)$. Let us adopt the notation of Elster [36] so that: $\mu(r) = -T^t_t$, $p_r(r) = T^r_r$, $L = -4\pi r^2 C T_{tr}$, where the components of the stress-tensor here represent renormalized expectation values in the Unruh vacuum state. Therefore, we may write

$$T_{\mu\nu} u^\mu u^\nu = \frac{k^2}{C} (\mu + p_r) - p_r - \frac{kL}{2\pi r^2 C^2} \sqrt{k^2 - C}. \quad (75)$$

If a QI-type bound of the kind found in 2D, i.e., Eq. (36), also holds in 4D, then we might expect it [38] to have the form

$$\int_0^\infty T_{\mu\nu} u^\mu u^\nu d\tau \gtrsim -\frac{1}{M^2 \delta\tau}, \quad (76)$$

where again $\delta\tau$ is the characteristic proper time which the observer spends in the negative energy. (Note that since the fields we will be considering are massless, the mass M of the black hole is only natural length scale which appears in the problem.) We want to numerically compare the left and right-hand-sides of Eq. (76), in order to see if the inequality is satisfied.

The integrand in Eq. (76) drops off rapidly with increasing r near the horizon, hence the dominant contribution to the AWEC integral comes from the initial part of the trajectory. We will therefore approximate the AWEC integral as

$$\int_0^\infty T_{\mu\nu} u^\mu u^\nu d\tau \approx T(r)|_{r_0} \delta\tau, \quad (77)$$

where $T(r) \equiv T_{\mu\nu} u^\mu u^\nu$ is understood to be evaluated at the starting point r_0 (minimum r) of the trajectory. Since the magnitude of $T(r)$ rapidly decreases near the

r_0	$\mu(r_0)M^4$	$p_r(r_0)M^4$	$\delta r/M$	k	$\delta \tau/M$	$T \delta \tau$	$-(M^2 \delta \tau)^{-1}$
2.1	-2.7×10^{-3}	1.9×10^{-3}	0.5	0.5	1.77	-1.1×10^{-2}	-0.56
				1.0	0.54	-1.0×10^{-2}	-1.85
				5.0	0.10	-4.4×10^{-2}	-10
				10.0	0.05	-8.5×10^{-2}	-20
2.2	-1.8×10^{-3}	1.5×10^{-3}	1.0	1	1.16	-5.7×10^{-3}	-0.86
				5	0.20	-1.7×10^{-2}	-5
				10	0.10	-3.5×10^{-2}	-10
2.5	-1.1×10^{-3}	6.8×10^{-4}	1.0	1	1.22	-3.3×10^{-3}	-0.82
				5	0.20	-1.0×10^{-2}	-5
				10	0.10	-2.1×10^{-2}	-10
3	-5.4×10^{-4}	3.2×10^{-4}	2	1	2.82	-2.8×10^{-3}	-0.45
				5	0.40	-6.8×10^{-3}	-2.5
				10	0.20	-1.3×10^{-2}	-5

Table 1: Outgoing timelike geodesics moving through the quantized electromagnetic field.

horizon, our estimate of using $T(r_0)$ gives a result which is somewhat more negative than the true result. Therefore, correction of this estimate only goes in the direction of strengthening the inequality.

Our results for the electromagnetic field are displayed in Table 1. The values of μ and p_r for the electromagnetic field are from the numerical data given in Ref. [37], and the value of $L = 3.4 \times 10^{-4}/M^2$ used in this case is the photon luminosity given by Page [39]. In the fourth column of Table 1, δr is chosen such that the magnitude of $T(r)$ decreases by at least a factor of two from r_0 to $r_0 + \delta r$. The characteristic proper time, $\delta \tau$, is computed from

$$\delta \tau = \int_{r_0}^{r_0 + \delta r} \frac{dr}{\sqrt{k^2 - C}}. \quad (78)$$

The next to last column in the table is our approximation of the AWEC integral given in Eq. (77), while the last column represents the right-hand-side of our QI given by Eq. (76). By examining the last two columns of Table 1, we see that the QI, Eq. (76), is easily satisfied. A similar analysis for the conformally-coupled massless scalar field shows that the QI is easily satisfied in that case as well. Numerical values of μ , p_r , and $L = 7.44 \times 10^{-5}/M^2$ for the scalar field were obtained from the numerical data given in Ref. [36].

3.2 Radial Null Rays

The components of the tangent vector to an affinely parameterized outgoing radial null geodesic (outside the horizon) are given by

$$K^\mu = E(C^{-1}, 1, 0, 0). \quad (79)$$

Thus we have

$$T_{\mu\nu}K^\mu K^\nu = \frac{E^2}{2\pi(r-2M)^2} \left\{ 2\pi r(r-2M)[\mu(r) + p_r(r)] - L \right\}. \quad (80)$$

The 4D analog of the QI for null geodesics in 2D given by Eq. (61) is

$$\int_0^\infty T_{\mu\nu}K^\mu K^\nu d\lambda \gtrsim -\frac{1}{M^2 \delta\lambda}, \quad (81)$$

where $\delta\lambda$ is again interpreted as the characteristic affine parameter distance along the null geodesic over which the negative energy density falls off. If we define

$$T \equiv \frac{1}{E^2} (T_{\mu\nu}K^\mu K^\nu), \quad (82)$$

and use the fact that $dr/d\lambda = E$ and $\delta\lambda = \delta r/E$, then it is easily seen that the QI, Eq. (81) is equivalent to

$$\int_{r_0}^\infty T dr \gtrsim -\frac{1}{M^2 \delta r}, \quad (83)$$

where $r_0 > 2M$ is the initial starting value of r for the outgoing null geodesic and δr is the characteristic interval in r over which the negative energy falls off. We numerically evaluate each side of Eq. (83) to see if the inequality is satisfied. Following a similar procedure to that used for outgoing radial timelike observers, we approximate the ANEC integral by

$$\int_{r_0}^\infty T dr \approx T(r_0) \delta r. \quad (84)$$

As discussed in the last section, this approximation gives a result which is somewhat more negative than the true result. Therefore, if the QI holds for our approximation, then the inequality in the actual case is even stronger.

Our results for the scalar field are given in Table 2, and those for the electromagnetic field appear in Table 3, where as before the numerical data of Refs. [36] and [37], were used for the scalar and electromagnetic fields, respectively. The next to last column in each table is our approximation of the ANEC integral given in Eq. (84), while the last column represents the right-hand-side of our QI given by Eq. (83). By examining the last two columns of each table, we see that the QI, Eq. (83), is easily satisfied and hence so is Eq. (81). Note that the inequalities in 4D seem to be satisfied by a wider margin than the corresponding QI's in 2D. This is due to the fact that the numerical factors which appear in the components of the renormalized expectation values of the stress-tensor are characteristically smaller in 4D than in 2D spacetime, at least for radial null geodesics in the scalar field case.

In our calculations, we have ignored the backreaction of the Hawking radiation on the background spacetime. It might be better to repeat our calculations on an evolving background, such as a Vaidya spacetime. However, we can estimate the effects of backreaction on an outgoing null ray just outside the horizon in the following

r_0/M	$\mu(r_0)M^4$	$p_r(r_0)M^4$	$\delta r/M$	$T \delta r M^3$	$-M/\delta r$
2.5	-4×10^{-6}	0	0.5	-3.4×10^{-5}	-2
4	-4×10^{-7}	-3×10^{-7}	3	-4.8×10^{-6}	-0.3
5	-1.5×10^{-6}	-3×10^{-7}	5	-7.8×10^{-6}	-0.2

Table 2: Data is given for outgoing null geodesics, where μ and p_r are the energy density and pressure for a massless scalar field.

r_0/M	$\mu(r_0)M^4$	$p_r(r_0)M^4$	$\delta r/M$	$T \delta r M^3$	$-M/\delta r$
2.1	-2.7×10^{-3}	1.9×10^{-3}	0.5	-0.01	-2
2.2	-1.8×10^{-3}	1.5×10^{-3}	1	-0.004	-1
2.5	-1.1×10^{-3}	6.8×10^{-4}	1	-0.002	-1
3	-5.4×10^{-4}	3.2×10^{-4}	2	-1.4×10^{-3}	-0.5

Table 3: Data is given for outgoing null geodesics, where μ and p_r are the energy density and pressure for the electromagnetic field.

way. Our approximation should hold as long as the time of escape of a null ray from near the horizon is small compared to the evaporation time of the black hole. Here these times are measured by an observer at infinity. For such a null ray we have

$$\int_0^{t_{esc}} dt = \int_{2M+\epsilon}^{xM} (1 - 2M/r)^{-1} dr. \quad (85)$$

Solving for t_{esc} yields

$$t_{esc} = xM + 2M \ln\left(\frac{x}{2} - 1\right) - 2M - \epsilon + 2M \ln\left(\frac{2M}{\epsilon}\right), \quad (86)$$

where t_{esc} is the time for the ray to go from $r = 2M + \epsilon$ to $r = xM$. For $x \gg 2$ and finite, in the limit of small ϵ :

$$t_{esc} \rightarrow 2M \ln\left(\frac{2M}{\epsilon}\right). \quad (87)$$

By contrast, the evaporation time of the hole is $t_{evap} = AM^3$, where $A \approx 10^4/n$, and n is the number of species of particles, each of which is assumed to contribute approximately $10^{-4} M^{-2}$ to the black hole's luminosity. Our approximation should break down only when $\epsilon \lesssim 2M \exp[-AM^2/2]$.

3.3 Orbiting Null Rays

We now examine orbiting null geodesics in Schwarzschild spacetime. The line element is

$$ds^2 = -C dt^2 + C^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (88)$$

where again $C = (1 - 2M/r)$. For an orbiting null ray in Schwarzschild with $\theta = \pi/2 = \text{const}$,

$$\frac{dt}{d\lambda} = \frac{r}{\sqrt{C}} \frac{d\phi}{d\lambda}. \quad (89)$$

Here we choose $d\phi/d\lambda > 0$, for our future-directed ray. From the geodesic equations, we have

$$\dot{t} \equiv \frac{dt}{d\lambda} = \frac{E}{C}, \quad (90)$$

$$\dot{\phi} \equiv \frac{d\phi}{d\lambda} = \frac{L}{r^2}, \quad (91)$$

where E and L are constants. It follows that

$$\frac{L}{E} = r C^{-1/2}. \quad (92)$$

At $r = 3M$, we have

$$\frac{L}{E} \Big|_{r=3M} = 3\sqrt{3} M. \quad (93)$$

From the numerical values in Figs. (1) and (3) of Ref. [37] for the quantized electromagnetic field, and Eq. (93), we find that

$$\begin{aligned} T_{\mu\nu} K^\mu K^\nu \Big|_{r=3M} &= 3 E^2 \left[-T_t^t + T_\phi^\phi \right] \Big|_{r=3M} \\ &\approx -3.1 \times 10^{-4} \frac{E^2}{M^4}, \end{aligned} \quad (94)$$

where $K^\mu = dx^\mu/d\lambda$ is the tangent to the null geodesic. For one orbit, $\Delta\phi = 2\pi = (L/9M^2) \Delta\lambda$, so

$$\frac{E}{M} = \frac{2\sqrt{3}\pi}{\Delta\lambda}. \quad (95)$$

We may therefore rewrite Eq. (94) as

$$T_{\mu\nu} K^\mu K^\nu \Big|_{r=3M} \approx -3.7 \times 10^{-2} \frac{1}{M^2 (\Delta\lambda)^2}. \quad (96)$$

Since $T_{\mu\nu} K^\mu K^\nu \Big|_{r=3M}$ is a constant, we can trivially construct a QI-type integral:

$$T_{\mu\nu} K^\mu K^\nu \Big|_{r=3M} = \frac{2\lambda_0}{\pi} \int_0^\infty \frac{T_{\mu\nu} K^\mu K^\nu d\lambda}{\lambda^2 + \lambda_0^2} \approx -3.7 \times 10^{-2} \frac{1}{M^2 (\Delta\lambda)^2}, \quad (97)$$

where we have used Eq. (25) to creatively multiply by 1. A minor technical point is in order. Strictly speaking, we cannot have $\lambda = -\infty$ as the lower bound on our integral, since that would imply that the black hole was eternal. Here we can imagine creating a photon in the $r = 3M$ orbit (after the black hole has been formed), and

set $\lambda = 0$ at this creation event, so our integral goes from $\lambda = 0$ to $\lambda = \infty$. (Since the $r = 3M$ orbit is unstable, realistically we would expect the null ray to at best only make a few orbits.)

We may also integrate $T_{\mu\nu}K^\mu K^\nu$ over one orbit to get

$$\begin{aligned} \int_0^{\Delta\lambda} T_{\mu\nu}K^\mu K^\nu d\lambda &\approx -3.1 \times 10^{-4} \frac{E^2}{M^4} (\Delta\lambda) \\ &\approx -3.7 \times 10^{-2} \frac{1}{M^2 (\Delta\lambda)}. \end{aligned} \quad (98)$$

Thus we see that for orbiting null geodesics

$$\int_0^{\Delta\lambda} T_{\mu\nu}K^\mu K^\nu d\lambda \gtrsim -\frac{1}{M^2 (\Delta\lambda)}. \quad (99)$$

The expressions given by Eqs. (98) and (99) are of the same form as those for radial null geodesics discussed earlier. The difference here is that we integrate only over one orbit, i.e., we do not consider multiple traversals through the same negative energy region. The inclusion of the contributions of an arbitrary number of orbits would violate the inequality given in Eq. (99). However, it is of interest to note that if we consider the null geodesics which generate the boundary of the future of an event P with $r = 3M$, then the orbiting null geodesics leave this boundary after half an orbit. This is due to the fact that orbiting null geodesics originating at P , which initially set off in opposite spatial directions, encounter a conjugate point (i.e., the geodesics cross) on the far side of the black hole after half an orbit [40], [41]. At this point, the geodesics leave the boundary of the future of P (i.e., an event Q which lies beyond the conjugate point along either null geodesic can be connected to P by a timelike curve), and therefore are not achronal.

This observation may be relevant in the context of proving singularity theorems in the presence of negative energy. For example, in the proof of a singularity theorem such as the Penrose theorem, one uses either the local or the averaged null energy condition to insure the focusing of the null geodesics which generate the boundary of the future of a closed trapped surface. Geodesics of the orbiting variety considered here would not be relevant, at least in the spherically symmetric case, since they leave the boundary of the future of a point in any case, irrespective of energy condition-focusing arguments. We also note in passing that, at least in this simple case, while these null geodesics are in the boundary they obey the inequality, Eq. (99).

Let us compare the previous results with the following situation. Consider a 4D Minkowski spacetime with one spatial dimension compactified, and a quantized, minimally-coupled, massless scalar field in the Casimir vacuum state on this background. Let $K^\mu = E(1, 1, 0, 0)$ be the tangent to a null geodesic in the x -direction. The vacuum expectation value of the stress-tensor in this state is

$$T_{\mu\nu} = \frac{\pi^2}{45 L^4} \times \text{diag} \{-1, -3, 1, 1\}, \quad (100)$$

where we have taken x as the compactified dimension with circumference L . It follows that

$$T_{\mu\nu}K^\mu K^\nu = -\frac{4\pi^2 E^2}{45 L^4}. \quad (101)$$

The affine parameter lapse for one orbit is $\lambda_p = L/E$. If we integrate over one orbit, we find that

$$\begin{aligned} \int_0^{\lambda_p} T_{\mu\nu}K^\mu K^\nu d\lambda &= (T_{\mu\nu}K^\mu K^\nu)(\lambda_p) \\ &= -\frac{4\pi^2}{45 L^2 \lambda_p}. \end{aligned} \quad (102)$$

We see that this result has the same form as Eq. (98) for the black hole case. Both of these expressions are invariant under rescaling of the affine parameter. Here, as in the black hole case, we have ignored the possibility of multiple orbits, which would violate the inequality. However, it is possible that in the 4D compactified case, as was found previously in 2D [19], an ANEC-type bound might hold on the *difference* between the expectation values of $T_{\mu\nu}K^\mu K^\nu$ in an arbitrary quantum state and in the Casimir vacuum state.

4 Conclusions

In this paper, we have examined AWEC and ANEC in both two and four- dimensional evaporating Schwarzschild black hole spacetimes, for quantized massless scalar and electromagnetic fields. It was discovered that AWEC is violated for outgoing radial timelike geodesics which start just outside the horizon and reach infinity. Similarly, ANEC is violated for half-complete outgoing radial null geodesics just outside the horizon, and also on the horizon in 2D. This condition is violated for orbiting null geodesics in 4D as well. In a 2D black hole spacetime, ANEC is satisfied along ingoing null geodesics and along outgoing null geodesics inside the horizon. However, in all cases where the conditions are violated, there appear to be QI-type bounds which limit the degree of the violations. In curved spacetime, let $\delta\tau$ be the characteristic proper time for timelike geodesics and $\delta\lambda$ be the affine parameter length for null geodesics over which the negative energy density changes significantly. Then the QI bounds have the form that the integrated negative energy density along the timelike or null geodesic (i.e., the AWEC or ANEC integral) is greater than or equal to minus the inverse of $\delta\tau$ or $\delta\lambda$, respectively. The application of QI bounds to constrain the dimensions of traversable wormholes is currently under investigation.

Can the semi-classical effects of negative energy, in processes such as black hole evaporation, invalidate the singularity theorems before quantum gravity effects become significant? Although we do not have a definite proof in the general case, our results suggest that the answer may be no. It remains possible that even though ANEC (or even a weaker energy condition [16]) might fail in some regions of a given

spacetime, it may hold in enough other regions for the conclusions of the singularity theorems to be maintained. Our (admittedly) 2D results for ANEC would seem to support this view. In Ref. [8], it was shown that if ANEC is satisfied along the null geodesics which generate the boundary of the future of a trapped surface, then Penrose’s singularity theorem will still hold. There the local null energy condition was replaced by ANEC along half-complete null geodesics. However, two points should be emphasized about that result. First, it is not necessary to assume that the ANEC integral is non-negative to prove the result [8, 16]. In fact, from Eq. (5) of Ref. [8] and the Einstein equations, it is sufficient to have

$$\int_0^\infty T_{\mu\nu} K^\mu K^\nu d\lambda \geq \frac{\theta_0}{2}, \quad (103)$$

where $\theta_0 < 0$ is the initial expansion of the null geodesic congruence at the trapped surface. The ANEC integral may be negative as long as it is not more negative than $\theta_0/2$. Physically this implies that a singularity will still form provided that the defocusing effects due to the presence of any negative energy are more than compensated for by the initial convergence of the null geodesics produced by the trapped surface.

Second, the existence of only *one* such trapped surface whose orthogonal null geodesics obey either ANEC or the weaker inequality, Eq. (103), is required to prove the occurrence of a singularity. There may exist other trapped surfaces in the spacetime whose orthogonal null geodesics do not obey either of the inequalities. However, as long as there exists *at least one* trapped surface with the desired properties, a singularity is inevitable.

Consider an evaporating black hole spacetime, with backreaction included. The Penrose diagram for the standard scenario is shown in Fig. (3). Let us assume that this standard picture is correct. The line H^+ is the event horizon, the dashed curve A is the apparent horizon, and the lines g_1 , g_2 , and g_3 are outgoing null geodesics orthogonal to the trapped surfaces labelled 1, 2, and 3, respectively. The outgoing null rays from 1 do not focus but instead reach future null infinity, so one expects that *any* averaged energy condition which would insure focusing, such as ANEC or Eq. (103), would fail along them. The closest analogs to these rays in our analysis are the outgoing null rays just outside the horizon shown in Fig. (1). It was found that in both two and four-dimensional Schwarzschild spacetime, ANEC was violated along these rays, although in each case a QI limits the degree of the violation. One would expect that if an analogous QI holds in more general cases, such as Fig. (3), it will not be strong enough to guarantee focusing of null rays in the region between H^+ and A in Fig. (3). We could not examine the weaker inequality Eq. (103), since in a static evaporating black hole background there are no trapped surfaces outside the horizon. For the outgoing null rays from 2 in Fig. (3) (a trapped surface inside the collapsing matter), one would suspect that ANEC probably holds. In this case, the focusing effects of the matter of the star most likely overwhelm any defocusing by negative energy. For the outgoing null rays from 3 (a trapped surface “behind” the event horizon, in the vacuum outside the collapsing matter), one might expect

either ANEC or a similar but weaker inequality such as Eq. (103) to hold, since the null rays eventually do run into a singularity. This is assuming, of course, that the “standard” scenario depicted in Fig. (3) is in fact the correct one - a view which could well be wrong. In our 2D analysis, we found that ANEC was satisfied along these rays, and also along ingoing null rays. We showed that our approximation of neglecting backreaction should fail only when the time of escape of a null ray from near the horizon is comparable to the evaporation time of the black hole. This implied that such a ray would have to start out extremely close to the horizon. Nevertheless, it would be useful to attempt a similar analysis to ours for evaporating black hole spacetimes with backreaction taken into account. The situation is complicated by our ignorance of the detailed form of the stress-tensor components in such cases.

Recent work of Kuo and Ford [42, 43] has shown that negative energy densities in flat spacetime are subject to large fluctuations. In these circumstances, one does not expect the semi-classical Einstein equations to be a good approximation. If the same is true of negative energy densities in curved spacetime, and we see no reason to believe otherwise, then the question arises as to whether such fluctuations could cause a gross failure of the semi-classical approximation at scales well above the Planck scale. Large stress-tensor fluctuations would presumably produce large fluctuations in spacetime geometry. Therefore, it is very important to determine whether such fluctuations might render suspect the semi-classical picture of the Hawking process, or the conclusions of the singularity theorems in the presence of negative energy. On the other hand it could turn out, for example in the Hawking process, that if the timescale of the stress-tensor fluctuations is very short compared to other relevant time scales, such as the evaporation time of the hole, then the standard analysis may still be valid. In the proofs of the singularity theorems, one assumes a fixed classical spacetime background which determines the causal structure. If the background geometry fluctuates, it is not clear that the usual global techniques employed in classical general relativity will still be useful [44]. These questions are currently under active investigation.

Acknowledgements

The authors would like to thank Arvind Borde, Matt Visser, Ulvi Yurtsever, and Eanna Flanagan for helpful discussions. TAR would like to thank the members of the Tufts Institute of Cosmology for their kind hospitality and encouragement while this work was being done. This research was supported in part by NSF Grant No. PHY-9208805 (Tufts) and by the U.S. Department of Energy (D.O.E.) under cooperative agreement # DF-FC02-94ER40818 (MIT).

References

- [1] S.W. Hawking and G.F.R. Ellis, *The Large Scale Structure of Spacetime* (Cambridge University Press, London, 1973), pp. 88-96.

- [2] H. Epstein, V. Glaser, and A. Jaffe, *Nuovo Cim.* **36**, 1016 (1965).
- [3] H.B.G. Casimir, *Proc. Kon. Ned. Akad. Wet.* **B51**, 793 (1948).
- [4] F.J. Tipler, *Phys. Rev. D* **17**, 2521 (1978).
- [5] C. Chicone and P. Ehrlich, *Manuscr. Math.* **31**, 297 (1980).
- [6] A. Borde, *Class. Quantum Grav.* **4**, 343 (1987).
- [7] T.A. Roman, *Phys. Rev. D* **33**, 3526 (1986).
- [8] T.A. Roman, *Phys. Rev. D* **37**, 546 (1988).
- [9] J. Friedman, K. Schleich, and D. Witt, *Phys. Rev. Lett.* **71**, 1486 (1993).
- [10] Note that in a general spacetime, unlike the relation between the local null and weak energy conditions, ANEC does not simply follow by continuity from AWEC.
- [11] G. Klinkhammer, *Phys. Rev. D* **43**, 2542 (1991).
- [12] A. Folacci, *Phys. Rev. D* **46**, 2726 (1992).
- [13] R. Wald and U. Yurtsever, *Phys. Rev. D* **44**, 403 (1991).
- [14] M. Visser, *Phys. Lett.* **B349**, 443 (1995).
- [15] U. Yurtsever, *Phys. Rev. D* **51**, 5797 (1995).
- [16] U. Yurtsever, “A Note on the Averaged Null Energy Condition in Quantum Field Theory”, JPL preprint, gr-qc/9501040.
- [17] M. Morris and K. Thorne, *Am. J. Phys.* **56**, 395 (1988).
- [18] M. Morris, K. Thorne, and U. Yurtsever, *Phys. Rev. Lett.* **61**, 1446 (1988).
- [19] L. Ford and T. Roman, *Phys. Rev. D*, **51**, 4277 (1995).
- [20] L.H. Ford, *Proc. Roy. Soc. Lond. A* **364**, 227 (1978).
- [21] L.H. Ford, *Phys. Rev. D* **43**, 3972 (1991).
- [22] This particular choice was initially made in Ref. [21] purely for mathematical convenience. Presumably, one could prove analogous inequalities with other choices of sampling functions.
- [23] L.H. Ford and T.A. Roman, *Phys. Rev. D* **41**, 3662 (1990).
- [24] L.H. Ford and T.A. Roman, *Phys. Rev. D* **46**, 1328 (1992).
- [25] U. Yurtsever, *Class. Quantum Grav.* **7**, L251 (1990).

- [26] R. Penrose, *Phys. Rev. Lett.* **14**, 57 (1965).
- [27] S. W. Hawking, *Commun. Math. Phys.* **43**, 199 (1975).
- [28] A. Borde, “Topology Change in Classical General Relativity”, Tufts Institute of Cosmology preprint TUTP 94-13, gr-qc 9406053.
- [29] L.H. Ford and T.A. Roman, *Phys. Rev. D* **48**, 776 (1993).
- [30] W.G. Unruh, *Phys. Rev. D* **15**, 365 (1977).
- [31] P.C.W. Davies and W.G. Unruh, *Proc. Roy. Soc. Lond. A* **356**, 259 (1977).
- [32] Such a limit works in this case, but a similar limit need not hold in spacetimes with a more complicated causal structure.
- [33] R. M. Wald, *General Relativity* (Univ. of Chicago Press, Chicago, 1984).
- [34] R. M. Wald, *Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics* (Univ. of Chicago Press, Chicago, 1994).
- [35] U. Yurtsever, private communication.
- [36] T. Elster, *Phys. Lett.* **94A**, 205 (1983).
- [37] B.P. Jensen, J.G. McLaughlin, and A.C. Ottewill, *Phys. Rev. D* **43**, 4142 (1991).
- [38] This follows from dimensional considerations and the demand that our inequality be invariant under a rescaling of the proper time.
- [39] D. N. Page, *Phys. Rev.* **D13**, 198 (1976).
- [40] A. Borde, private communication.
- [41] See p. 56, footnote 2 of R. Penrose, *Techniques of Differential Topology in Relativity* (Society for Industrial and Applied Mathematics, Philadelphia, 1972), for an interesting example with timelike geodesics.
- [42] C.-I Kuo and L.H. Ford, *Phys. Rev. D* **47**, 4510 (1993).
- [43] C.-I Kuo, *Quantum Fluctuations and Semiclassical Gravity Theory*, Ph.D. thesis, Tufts University (1994), unpublished.
- [44] The authors are grateful to A. Borde for this remark.

Figure Captions

- [1] Radial null geodesics in the spacetime of a black hole formed by gravitational collapse: an ingoing geodesic (a), an outgoing geodesic inside the horizon (b), and an outgoing geodesic outside of the horizon (c).
- [2] The spacetime of a black hole formed by gravitational collapse. The null ray which lies in the future horizon enters the collapsing body (shaded region), and then re-imerges into the vacuum spacetime at V_{min} .
- [3] The spacetime of an evaporating black hole, including the effects of backre-
action.