A Toy Model Approach to the Canonical Non-perturbative Quantization of the Spatially Flat Robertson-Walker Spacetimes with Cosmological Constant

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Abstract

We present a toy model approach to the canonical non-perturbative quantization of the spatially-flat Robertson-Walker Universes with cosmological constant, based on the fact that such models are exactly solvable within the framework of a simple Lagrangian formulation. The essential quantum dynamical metric-field and the corresponding Hamiltonian, explicitly derived in terms of annihilation and creation operators, point out that the Wheeler - DeWitt equation is a natural (quantum) generalization of the G_{44} - Einstein equation for the classical De Sitter spacetime and selects the physical states of the quantum De Sitter Universe. As a result of the exponential universal expansion, the usual Fock states (defined as the eigenstates of the number-operator) are no longer invariant under the derived Hamiltonian. They exhibit quantum fluctuation of the energy and of the metric field which lead to a (geometrical) volume quantization.

1 General

The basic idea of this paper is to look for one of the most simple, but physically interesting, spacetime structures with a minimum number of essential dynamical variables, to write down the action functional for the gravitational field in terms of these variables and choosing the Lagrangian whose Euler-Lagrange equations have to be compatible with the Einstein's equations for that spacetime, to perform a canonical non-perturbative quantization. Is almost obvious that such a spacetime might be the well known Friedmann-Robertson-Walker Universe with cosmological constant and flat t - leveled hypersurfaces, described by the metric

$$ds^2 = \eta_{ab}\omega^a\omega^b = e^{2f(t)}\delta_{\alpha\beta}dx^\alpha dx^\beta - (dt)^2 \tag{1}$$

where

$$\begin{array}{lll} a) & (\eta_{ab}) & = diag(1,\, 1,\, 1,\, -1) \\ b) & \omega^\alpha & = e^{f(t)}dx^\alpha \\ c) & \omega^4 & = dt \end{array} \tag{2}$$

and $f : R \to R$ is the primitive of the Hubble's function $h = f,_4$, where $(\cdot),_4 = \frac{\partial \cdot}{\partial t}$, since the dual tetrads ω^α are entirely defined by $f(t)$ and the connection 1-forms (without torsion)

$$\Gamma_{ab} \wedge \omega^b = -\eta_{ab}d\omega^b \tag{3}$$

which, as can be easily checked, concretely read

$$\left.\begin{array}{ll} a) & \Gamma_{\alpha 4} = f,_4\,\omega^\alpha \\ b) & \Gamma_{\alpha\beta} = 0 \end{array}\right. ,\ \alpha,\, \beta = \overline{1,\, 3} \tag{4}$$

are completely defined by the Hubble's function $f,_4$. Due to the general simplicity of the analysed system there is no need to engage in full the "3 + 1 decomposition" - formalism of quantization [1, 2, 3] because the metric (1), possesing the scalar curvature

$$R = 6\left[f,_{44} + 2(f,_4)^2\right], \tag{5}$$

already leads to the following easily handling expression

$$S[f] = \frac{3}{\kappa_0} \int e^{3f} \left[f,_{44} + 2(f,_4)^2 - \frac{\Lambda}{3} \right] d^3x \, dt \quad , \; \kappa_0 = 8\pi G \tag{6}$$

for the action functional of the gravitational field in the spatially flat FRW Universe with the cosmological constant Λ. Before proceeding to quantization it is worthy to have a good image of the classical dynamics of the system as it is described by Einstein's equations. Sure, at this level (classically) nothing is new but when the quantization is engaged it will turn out that the G_{44} - Einstein equation not only heavily lies on the Hamiltonian associated to the Lagrangian derived from (6) but also directly expresses the well known Wheeler-DeWitt equation [4]

$$\hat{H} \mid \psi\rangle = 0 \tag{7}$$

obeyed by the physical states $\mid \psi\rangle$.

2 Geometry and Exact Solutions

The algebraically essential components of the Einstein tensor G_{ab} with respect to the dual pseudo-orthonormal basis (2) can be worked out as usual, starting with the 2-nd Cartan structure equation

$$\Omega_{ab} = d\Gamma_{ab} + \Gamma_{ac} \wedge \Gamma^{c}_{\cdot b} \tag{8}$$

which leads to the curvature 2-forms

$$\begin{aligned} a) \quad & \Omega_{\alpha\beta} = (f,_4)^2 \omega^\alpha \wedge \omega^\beta \\ b) \quad & \Omega_{\alpha 4} = -\left[f,_{44} + (f,_4)^2 \right] \omega^\alpha \wedge \omega^4 \end{aligned} \tag{9}$$

from which the curvature components read

$$\begin{aligned} a) \quad & R_{\alpha\beta\alpha\beta} = (f,_4)^2 \\ b) \quad & R_{\alpha 4 \alpha 4} = -\left[f,_{44} + (f,_4)^2 \right] \end{aligned} \tag{10}$$

(without summation after the repeated indices) and so, the Ricci tensor components

$$R_{ab} = \eta^{cd} R_{cadb} \tag{11}$$

get the expressions

$$\begin{aligned} a) \quad & R_{\alpha\beta} = \left[f,_{44} + 3(f,_4)^2 \right] \delta_{\alpha\beta} \\ b) \quad & R_{44} = -3 \left[f,_{44} + (f,_4)^2 \right] \end{aligned} \tag{12}$$

and the scalar curvature

$$R = \eta^{ab} R_{ab} \tag{13}$$

is given by (5) and the only non-vanishing components of Einstein tensor concretely read

$$\begin{aligned} a) \quad & G_{\alpha\beta} = - \left[2f,_{44} + 3(f,_4)^2 \right] \delta_{\alpha\beta} \\ b) \quad & G_{44} = 3(f,_4)^2 \end{aligned} \tag{14}$$

Thus, the Einstein's equations with cosmological constant

$$G_{ab} + \eta_{ab} \Lambda = 0 \tag{15}$$

(in the empty spacetime case) reduce to the following overdetermined system of two ordinary differential equations for the metric function f:

$$\begin{aligned} a) \quad & 2f,_{44} + 3(f,_4)^2 = \Lambda \\ b) \quad & 3(f,_4)^2 = \Lambda \end{aligned} \tag{16}$$

which are known respectively as the "cosmological pressure" - equation and the "energy" - equation. As Λ is constant, is clear from the "energy" - equation that

$$f,_{44} = 0 \tag{17}$$

and this assures the compatibility of (16), the "cosmological pressure" - equation being actually the same with (16.b). Therefore, classically, the dynamics of the spatially flat FRW Universe with cosmological constant and no (ordinary) matter-content is entirely encoded in the "energy" - equation (16.b) which posses the general solutions

$$
\begin{aligned}
&a)\quad f_+(t) = f_+^0 + \sqrt{\frac{\Lambda}{3}}\,t \\
&b)\quad f_-(t) = f_-^0 - \sqrt{\frac{\Lambda}{3}}\,t
\end{aligned}
\tag{18}
$$

with $t \in (-\infty, \infty)$ and $f_+^0,\, f_-^0 \in R$ being the scale factors. Setting $f_\pm^0 = 0$ i.e. at $t = 0$ the analyzed spacetime is Minkowskian, the metric (1) becomes

$$
ds^2 = e^{\pm 2\sqrt{\frac{\Lambda}{3}}t}\,\delta_{\alpha\beta}dx^\alpha dx^\beta - (dt)^2 \tag{19}
$$

and these two "disjoint" cosmological vacuum-type exact solutions of Einstein's field equations are (very well) known as describing the so called "red-shifted" and "blue-shifted" De Sitter Universes (respectively) [5]. Both of them are possesing a 10-dimensional group of motion (the De Sitter group) with a globally defined timelike Killing vector, which is a very important feature for defining an universal cosmic time, fulfiling this way the Perfect Cosmological Principle [6], and moreover the "red-shifted" one contains also an event horizon [5, 7]. At the present observational level it can easily be shown that the "blue-shifted" De Sitter solution is physically unacceptable [5] and thus

$$
f_+(t) = \sqrt{\frac{\Lambda}{3}}\,t\,,\ t \in (-\infty, \infty) \tag{20}
$$

is the only physically significant solution of the "energy" - equation (16.b). For what it follows a short comment on the exact dynamical role of each of the field equations (16) is welcome. This regards the question "what it would be if the Universe were driven only by the "cosmological pressure" - equation ? " Mathematically the answer can be given immediately since, performing in (16.a) the transformation

$$
f = \alpha \ln u,\ \alpha \in R \tag{21}
$$

one gets the equation

$$
u\left[u_{,44} - \frac{\Lambda}{2\alpha}u\right] + \left(\frac{3}{2}\alpha - 1\right)(u_{,4})^2 = 0 \tag{22}
$$

and setting $\alpha = 2/3$ it results, omitting the trivial solution $u = 0$,

$$u,_{44} - \frac{3\Lambda}{4} u = 0. \tag{23}$$

The general solution of (23) reads

$$u(t) = u_+^0 \exp\left(\frac{\sqrt{3\Lambda}}{2} t\right) + u_-^0 \exp\left(-\frac{\sqrt{3\Lambda}}{2} t\right), \tag{24}$$

that means

$$f(t) = \frac{2}{3} \ln\left[u_+^0 \exp\left(\frac{\sqrt{3\Lambda}}{2} t\right) + u_-^0 \exp\left(-\frac{\sqrt{3\Lambda}}{2} t\right)\right] \tag{25}$$

and it represents a 2 - parameter family of spatially flat FRW Universes driven by the "cosmological pressure" alone. Physically, the result (25) combined with the "energy" - equation (16.b) points out two important features:

1. for $t \to \pm\infty$ the scale function behaves as $f_\pm$ i.e. in the "very distant" past the 2 - parameter family was degenerated in a "blue-shifted" De Sitter Universe and will be degenerated again, this time in a "red - shifted" one, in the "very distant" future. Thus, for arbitrary u_+^0, u_-^0 the pressure driven FRW - Universes "have started" and "end" as De Sitter spacetimes.

2. The general solution (25) of (16.a) naturally contains the De Sitter solutions (18) of (16.b) for $u_-^0 \equiv 0$ and respectively $u_+^0 \equiv 0$. Thus, putting it in other words, the dynamics encoded in the "cosmological pressure" - equation is more general than that controled by the "energy" - equation but it might fail to be a physical one since (in general) it violates the energy-spacetime geometry balance stated by (16.b).

Generalizing this comment to the case when somehow the quantization has been engaged it becomes clear that working with a field operator related to f by a suitable construction, one might solve the quantum analogous of the "cosmological pressure" - equation, expresses the Hamiltonian with respect to the derived solution for the field operator and imposing at the end the quantum version of the "energy" - equation as a constraint on the formally possible states of the system associated to its quantum dynamics. That's exactly what we are going to do in the followings. Therefore, let's come back to the action functional (6) and perform a partial integration of the term containing the second derivative of the metric function f in order to get a "surface" - term and a suitable form for the Lagrange function $L\,(f; f,_4)$:

$$S[f] = \frac{3}{\kappa_0}\left[\int d^3x\right]\left[e^{3f} f,_4\right]_{t_-}^{t_+} - \frac{3}{\kappa_0}\left[\int d^3x\right] \cdot \int e^{3f}\left[(f,_4)^2 + \frac{\Lambda}{3}\right] dt \qquad (26)$$

Considering the spatial integration performed over an arbitrary spacelike compact region of Euclidian volume V, (26) becomes

$$S[f] = \frac{V}{\kappa_0}\left[(e^{3f}),_4\right]_{t_-}^{t_+} + \int L\,(f; f,_4)\,dt \qquad (27)$$

where

$$L(f; f,_4) = -\frac{3V}{\kappa_0} e^{3f}\left[(f,_4)^2 + \frac{\Lambda}{3}\right] \qquad (28)$$

is the Lagrangian of the spatially flat FRW Universe characterized by the metric function f that becomes a canonical variable. In terms of $(f; f,_4)$ the Hamiltonian

$$H[f] = \frac{\partial L}{\partial f,_4}(f,_4) - L$$

reads

$$H[f] = -\frac{3V}{\kappa_0} e^{3f}\left[(f,_4)^2 - \frac{\Lambda}{3}\right]$$

and is quite obvious that the "energy" - equation (16.b) gets the physically meaningfull expression

$$H[f] = 0$$

that represents the constraint imposed on the general solution of the Euler-Lagrange equation (16.a) in order to be a physically acceptable one from a classical viewpoint.

3 Quantizing the Model

3.1 Standard Procedures

Proceeding to quantization we notice that the substitution

$$\phi = \sqrt{\frac{8V}{3\kappa_0}} \exp\left[\frac{3}{2} f\right] \qquad (29)$$

casts the Lagrangian (28) into the form

$$L[\phi] = -\frac{1}{2}\left[(\phi_{,4})^2 + \frac{3\Lambda}{4}\phi^2\right] \tag{30}$$

which, except the sign of the first term, is practically the same with the Lagrangian density of a real scalar field of mass $(3\Lambda/4)^{1/2}$. This leads straigheforwardly to the non-perturbative canonical quantization of the field ϕ since, treating ϕ as a coordinate-like operator and

$$\pi = \frac{\partial L}{\partial \phi_{,4}} = -\phi_{,4} \tag{31}$$

as its canonically conjugated momentum-operator, it is natural to impose the Bohr-Heisenberg commutation relation

$$[\pi, \phi] = -i. \tag{32}$$

With (30) and(31), the associated Hamiltonian reads

$$H[\pi, \phi] = -\frac{1}{2}[\pi^2 - \mu^2\phi^2], \tag{33}$$

where

$$\mu = \frac{\sqrt{3\Lambda}}{2}, \tag{34}$$

Coming back to (30), we get the Euler-Lagrange equation as

$$\phi_{,44} - \mu^2\phi = 0 \tag{35}$$

and it yields for the field operator the hyperbolic expression

$$\phi = a_+ e^{\mu t} + a_- e^{-\mu t} \tag{36}$$

where, in order to do not violate at the classical level the reality of ϕ , the operators $a_\pm$ have to be hermitians. With (36), the commutation relation becomes

$$[a_+ e^{\mu t} - a_- e^{-\mu t},\, a_+ e^{\mu t} + a_- e^{-\mu t}] = \frac{i}{\mu}$$

i.e. the $a_\pm$ - hermitic operators must satisfy

$$[a_+,\, a_-] = \frac{i}{2\mu} \tag{37}$$

For later use, the Hamilton operator (33) can be worked out by (36) in terms of $a_\pm$:

$$H[\phi] = \mu^2[a_+a_- + a_-a_+] \tag{38}$$

Inspired by (37), (38) and their analogous formulas for the case of the quantum harmonic oscillator, let us express the operators $a_\pm$ as a linear combination with respect to a linear operator "C" and its adjoint "C^+" :

$$\begin{aligned} a)\quad a_+ &= \frac{\rho}{\sqrt{2\mu}}[e^{i\alpha}C + e^{-i\alpha}C^+] \\ b)\quad a_- &= \frac{\rho}{\sqrt{2\mu}}[e^{i\beta}C + e^{-i\beta}C^+] \end{aligned} \tag{39}$$

where $\rho, \alpha, \beta \in R$. With (39) the commutation relation (37) becomes

$$\rho^2[e^{i\alpha}C + e^{-i\alpha}C^+,\, e^{i\beta}C + e^{-i\beta}C^+] = i$$

i.e. for

$$2\rho^2\sin(\alpha-\beta) = 1,\ \textit{with the constraint}\ \alpha-\beta \in \bigcup_n (2n\pi, (2n+1)\pi), n = 0, 1, 2, ... \tag{40}$$

it reads

$$[C,\, C^+] = 1 \tag{41}$$

and it can be considered as the commutator between the annihilation and creation operators C, C^+ respectively, which act on a Fock-state of n - quanta according to the formulas

$$\begin{aligned} C \mid n\rangle &= \sqrt{n} \mid n-1\rangle \quad ; \quad \langle n \mid C^+ = \sqrt{n}\langle n-1 \mid \\ C^+ \mid n\rangle &= \sqrt{n+1} \mid n+1\rangle \quad ; \quad \langle n \mid C = \sqrt{n+1}\langle n+1 \mid \end{aligned} \tag{42}$$

with

$$\begin{aligned} C \mid 0\rangle &= 0 \\ \langle 0 \mid C^+ &= 0 \end{aligned}$$

In terms of C, C^+, using the linear decomposition (39), the Hamiltonian (38) gets the expression

$$H[\phi] = \mu\rho^2\left[e^{i(\alpha+\beta)}(C)^2 + \cos(\alpha-\beta)(CC^+ + C^+C) + e^{-i(\alpha+\beta)}(C^+)^2\right] \tag{43}$$

which leads by the natural condition

$$\langle 0 \mid H[\phi] \mid 0 \rangle = 0, \tag{44}$$

using (42), to the equation

$$\cos(\alpha - \beta) = 0 \tag{45}$$

having, because of the constraint mentioned in (40), the solution

$$\alpha - \beta = (4k+1)\frac{\pi}{2},\ k = 0, 1, 2, ... \tag{46}$$

Thus,

$$\sin(\alpha - \beta) = 1,\ \rho^2 = \frac{1}{2},\ \alpha = \beta + \frac{\pi}{2} + 2k\pi$$

and, without restraining the generality, we choose

$$\begin{aligned} a)\quad & \rho = \frac{1}{\sqrt{2}} \\ b)\quad & \alpha = \beta + \frac{\pi}{2} \end{aligned} \tag{47}$$

that casts (39) into the form

$$\begin{aligned} a)\quad & a_+ = \frac{i/2}{\sqrt{\mu}}[e^{i\beta}C - e^{-i\beta}C^+] \\ b)\quad & a_- = \frac{1/2}{\sqrt{\mu}}[e^{i\beta}C + e^{-i\beta}C^+] \end{aligned} \tag{48}$$

and expresses the Hamiltonian (43) as:

$$H[\phi] = \frac{i\mu}{2}\left[e^{2i\beta}(C)^2 - e^{-2i\beta}(C^+)^2\right] \tag{49}$$

With respect to the orthonormal basis $\{\mid n\rangle; n = 0, 1, 2, ...\}$ formed by the Fock-states of n - quanta, the Hamiltonian (49) is an infinite-dimensional extra-bidiagonal Hermitian matrix with the elements

$$H_{n'n} = \frac{i\mu}{2}\left[e^{2i\beta}\sqrt{n(n-1)}\,\delta_{n',n-2} - e^{-2i\beta}\sqrt{(n+1)(n+2)}\,\delta_{n',n+2}\right] \tag{50}$$

and its eigenvalues are given by the "secular" - equation

$$det\left[\lambda\delta_{n'n} - \frac{i\mu}{2}\left(e^{2i\beta}\sqrt{n(n-1)}\delta_{n',n-2} - e^{-2i\beta}\sqrt{(n+1)(n+2)}\delta_{n',n+2}\right)\right] = 0 \tag{51}$$

As it can be easily seen, because of the Hermitian character of (50), the phase β doesn't affect the eigenvalues λ (obtained from (51)) and so it can be fixed arbitrarily. As an example, in the case of a 9-th order truncation of the Hamiltonian matrix (50), its eigenvalues λ, obtained by solving the corresponding 9-th degree algebraic equation (51), are:

$$\begin{aligned}
a)\quad \lambda_{-4} &= -\frac{\mu}{\sqrt{2}}\{22 + \frac{3}{\sqrt{2}}[1 + (99 - 4\sqrt{2})^{1/2}]\}^{1/2} \\
b)\quad \lambda_{-3} &= -\frac{\mu}{\sqrt{2}}\{22 + \frac{3}{\sqrt{2}}[1 - (99 - 4\sqrt{2})^{1/2}]\}^{1/2} \\
c)\quad \lambda_{-2} &= -\frac{\mu}{\sqrt{2}}[17 + \sqrt{226}]^{1/2} \\
d)\quad \lambda_{-1} &= -\frac{\mu}{\sqrt{2}}[17 - \sqrt{226}]^{1/2} \\
e)\quad \lambda_{0} &= 0 \\
f)\quad \lambda_{1} &= \frac{\mu}{\sqrt{2}}[17 - \sqrt{226}]^{1/2} \\
g)\quad \lambda_{2} &= \frac{\mu}{\sqrt{2}}[17 + \sqrt{226}]^{1/2} \\
h)\quad \lambda_{3} &= \frac{\mu}{\sqrt{2}}\{22 + \frac{3}{\sqrt{2}}[1 - (99 - 4\sqrt{2})^{1/2}]\}^{1/2} \\
i)\quad \lambda_{4} &= \frac{\mu}{\sqrt{2}}\{22 + \frac{3}{\sqrt{2}}[1 + (99 - 4\sqrt{2})^{1/2}]\}^{1/2}
\end{aligned} \tag{52}$$

It can be noticed that the negative - and positive - energy eigenstates of H appear in pairs, but this is not a problem, in what it concerns the negative - energy eigenstates, for the physical states $\mid \psi\rangle$ have to safisfy the "energy-geometry" - constraint, i.e. Wheeler-DeWitt equation

$$H[\phi] \mid \psi\rangle = 0 \tag{53}$$

which, for the analyzed system, is the quantum analogous of the "energy" - equation (16.b).

3.2 Some Quantum Properties

Before dealing with (53) in order to compute the physical solution(s) $\mid \psi\rangle$, it is useful to derive some of the quantum properties "generated" by the field operator (36), with the linear decomposition (48), and by the Hamilton operator (49).

- Starting with the field operator ϕ casted into the form

$$\phi = \frac{1}{\sqrt{\mu}} \left\{ \frac{i}{2}[e^{i\beta} C - e^{-i\beta} C^{+}] e^{\mu t} + \frac{1}{2}[e^{i\beta} C + e^{-i\beta} C^{+}] e^{-\mu t} \right\} \tag{54}$$

is clear first that, as it should be, its mean value in a given Fock-state of n - quanta is

$$\langle n \mid \phi \mid n\rangle \equiv 0, \ \forall\, n = 0, 1, 2, ... \tag{55}$$

For the Hermitic operator ϕ^2 the situation is different and we get

$$\begin{aligned} \phi^2 = & - \frac{1}{4\mu}[e^{2i\beta}(C)^2 - (CC^{+} + C^{+}C) + e^{-2i\beta}(C^{+})^2] e^{2\mu t} + \\ & + \frac{1}{4\mu}[e^{2i\beta}(C)^2 + (CC^{+} + C^{+}C) + e^{-2i\beta}(C^{+})^2] e^{-2\mu t} + \\ & + \frac{i}{4\mu}[e^{2i\beta}(C)^2 - e^{-2i\beta}(C^{+})^2] \end{aligned}$$

i.e.

$$\langle n \mid \phi^2 \mid n\rangle = \frac{n + 1/2}{\mu} \cosh(2\mu t) \tag{56}$$

which means that the mean-deviation of the field ϕ, expressing its quantum fluctuation at any given moment t in a state with n - quanta, is given by the formula

$$(\triangle\phi)_n(t) = \left[\frac{n + 1/2}{\mu} \cosh(2\mu t)\right]^{1/2} \tag{57}$$

Having a look at (29), which states the classical relation between ϕ and the scale-function f, we also derive from (56) that

$$\langle n \mid e^{3f} \mid n\rangle = \frac{\kappa_0}{4V}\sqrt{\frac{3}{\Lambda}}(n + 1/2)\cosh(\sqrt{3\Lambda}t) \tag{58}$$

As V is the Euclidian volume of an (arbitrary) t - leveled compact region, it results that

$$= \int_{(V)} \sqrt{-g}\, d^3x = V e^{3f} \tag{59}$$

represents its "universal" - volume and so (58) it yields

$${}_n(t) = \frac{\kappa_0}{4}\sqrt{\frac{3}{\Lambda}}(n + 1/2)\cosh(\sqrt{3\Lambda}t) \tag{60}$$

pointing out a very interesting volume-quantization at any instant t in an n - quanta state. Probably the most puzzling feature outlined by (60) is that in comparison to the classical case when at any instant t, can be arbitrary large (because of V), in the quantum case the accesible universal-volume in a state of n - quanta (at the instant t) *is finite*. As a consequence, at the Minkowskian epoch $t = 0$ in the Fock state of 0 - quanta, it exists an elementary 0-universal-volume

$${}_0 = \frac{\kappa_0}{8}\sqrt{\frac{3}{\Lambda}}$$

which, considering Λ as a fundamental constant, is expressed only with respect to the universal constants $c, \hbar, G, \Lambda$:

$${}_0 = \frac{\pi G \hbar}{c^3}\sqrt{\frac{3}{\Lambda}} \tag{61}$$

- Passing to $H[\phi]$ given by (49) it results, according to (50) also, that its mean-value in any n - quanta state is zero,

$$\langle n \mid H[\phi] \mid n \rangle \equiv 0, \tag{62}$$

and for the square-mean-deviation it yields

$$(\triangle H)^2_n = \langle n \mid H^2[\phi] \mid n \rangle = \frac{\mu^2}{4}\langle n \mid (C)^2(C^+)^2 + (C^+)^2(C)^2 \mid n \rangle,$$

i.e.

$$\langle n \mid H^2[\phi] \mid n \rangle = \frac{\mu^2}{2}(n^2 + n + 1) \tag{63}$$

and so, the mean-fluctuation of H in the Fock-state $\mid n\rangle$ reads

$$(\triangle H)_n = \frac{\mu}{\sqrt{2}}(n^2+n+1)^{1/2}, \tag{64}$$

increasing linearly with n at large values. Therefore the "energy per quanta"- mean-fluctuation is approximately constant

$$\frac{(\triangle H)_n}{n} \cong \frac{\mu}{\sqrt{2}} \tag{65}$$

being equal with the energy-mean-fluctuation in the "no-quanta" - state, $\mid 0\rangle$. This might be an expression of the "time-energy" Heisenberg relation because in a continuesly evolving spatially flat FRW Universe (with cosmological constant) the Fock-states $\{\mid n\rangle\}_{n=0,1,2,...}$ are no longer invariant under the action of H,

$$H \mid n\rangle = \frac{i\mu}{2}\{e^{2i\beta}\sqrt{n(n-1)} \mid n-2\rangle - e^{-2i\beta}\sqrt{(n+1)(n+2)} \mid n+2\rangle\}, \tag{66}$$

hence the number of "FRW - quanta" is no longer conserved, and so, the mean-life-time τ of a "FRW - quanta" can be estimated from

$$\frac{(\triangle H)_n}{n}\cdot\tau \approx 1 \;\;\Leftrightarrow\;\; \frac{\mu}{\sqrt{2}}\tau \approx 1 \tag{67}$$

i.e.

$$\tau = \sqrt{\frac{8/3}{\Lambda}} \tag{68}$$

By the way, in what it concerns the quanta-number non-conservation, let us derive the differential equation satisfied by the number-operator

$$N = C^+C \tag{69}$$

and work out the "phenomenological" solution $n(t)$. First, we have

$$\frac{dN}{dt} = i[H,N] = -\frac{\mu}{2}\{e^{2i\beta}[C^2, C^+C] - e^{-2i\beta}[(C^+)^2, C^+C]\}$$

i.e.

$$\frac{dN}{dt} = -\mu\{e^{2i\beta}(C)^2 + e^{2i\beta}(C^+)^2\} \tag{70}$$

For the second-order derivative of the number-operator it results

$$\frac{d^2N}{dt^2} = i\left[H, \frac{dN}{dt}\right] = \mu^2[(C)^2, (C^+)^2],$$

such that, the differential equation obeyed by N reads:

$$\frac{d^2N}{dt^2} - 4\mu^2 N = 2\mu^2 \tag{71}$$

Its averaged-version,

$$\frac{d^2n}{dt^2} - 4\mu^2 n = 2\mu^2\ , \tag{72}$$

rules the quanta-number dynamics, $n(t)$, possessing the general solution

$$n(t) = n_1^0 e^{2\mu t} + n_2^0 e^{-2\mu t} - 1/2 \tag{73}$$

and the phenomenological "annihilation - creation rate" of FRW-quanta reads

$$S(t) = \frac{dn}{dt} = 2\mu[n_1^0 e^{2\mu t} - n_2^0 e^{-2\mu t}] \tag{74}$$

Imposing $n_1^0, n_2^0 > 0$ as a necessary condition for $n(t) \geq 0, \forall t \in (-\infty, \infty)$ the relations (73), (74) point out physically that in the "very distant" past, $t \to -\infty$, the Universe was a De Sitter - "blue-shifted" one filled with a very large number of quanta

$$n_{-\infty}(t) \cong n_2^0 e^{-2\mu t}$$

possesing a huge phenomenological absorbtion rate

$$S_{-\infty}(t) \cong -2\mu n_2^0 e^{-2\mu t}$$

As the time passes everything slows down but the phenomenological creation rate is getting bigger and bigger and after some stationary epoch t_0 (depending on n_1^0, n_2^0) the Universe becomes a De Sitter - "red-shifted" one with a very large number of quanta

$$n_{+\infty}(t) \cong n_1^0 e^{2\mu t}$$

and a huge pnenomenological creation rate

$$S_{+\infty}(t) \cong 2\mu n_1^0 e^{2\mu t}$$

for compensating the exponential rate $\sim e^{2\mu t}$ of the universal-volume expantion.

3.3 The Physical Quantum States

Finally, let us come back to the Wheeler-DeWitt equation (53) which, inserting (49), concretely reads:

$$\left[e^{i(2\beta+\frac{\pi}{2})}(C)^2 + e^{-i(2\beta+\frac{\pi}{2})}(C^+)^2\right] \mid \psi >= 0 \tag{75}$$

As the phase β is dynamically irrelevant, let us set

$$\beta = -\frac{\pi}{4} \tag{76}$$

dealing this way with the "very symmetric" equation:

$$[(C^+)^2 + (C)^2] \mid \psi >= 0. \tag{77}$$

Expressing $\mid \psi\rangle$ as a mixed quantum state with respect to the orthonormal Fock-basis $\{\mid n\rangle\}_{n=0,1,2,...}$ i.e.

$$\mid \psi\rangle = \sum_{n=0}^{\infty} a_n \mid n\rangle \tag{78}$$

we get from (78) the state equation

$$\begin{aligned} &\sqrt{2}a_2 \mid 0\rangle + \sqrt{2\cdot 3}a_3 \mid 1\rangle + \\ &+ \sum_{n=2}^{\infty}\left[\sqrt{(n+1)(n+2)}a_{n+2} + \sqrt{n(n-1)}a_{n-2}\right] \mid n\rangle = 0 \end{aligned} \tag{79}$$

that demands

$$a_2 = a_3 \equiv 0 \tag{80}$$

and transforms into the difference-equation

$$\sqrt{(n+1)(n+2)}\,a_{n+2} + \sqrt{n(n-1)}\,a_{n-2} = 0,\ n = 2, 3, ... \tag{81}$$

that must be safisfied by the amplitudes $\{a_n\}_{n=0,1,2,...}$ of the physical quantum state(s) of spatially-flat FRW-Universe with cosmological constant, Λ. A very interesting feature of (81) with the necessary and sufficient condition (80) (for admitting a non-trivial solution) is that there are two fundamental decoupled states, i.e. orthonormal, $\mid \psi_+\rangle$, $\mid \psi_-\rangle$ possesing the general forms

$$\begin{aligned} a) \quad \mid \psi_+\rangle &= \sum_{k=0}^{\infty} a_{4k} \mid 4k\rangle \\ b) \quad \mid \psi_-\rangle &= \sum_{k=0}^{\infty} a_{4k+1} \mid 4k+1\rangle \end{aligned} \tag{82}$$

such that "ψ_+" - , "ψ_-" - difference equations obtained from (81) explicitly read

$$\begin{aligned} a) \quad & [(4k+1)(4k+2)]^{1/2} a_{4k+2} + [4k(4k-1)]^{1/2} a_{4k-2} = 0 \\ b) \quad & [(4k+2)(4k+3)]^{1/2} a_{4k+3} + [4k(4k+1)]^{1/2} a_{4k-1} = 0 \end{aligned} \tag{83}$$

the non-trivial solutions being obtained, because of (80), for $k = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$

In order not to deal with semi-integer values for k and also in order to work directly with the "index-structure" of (82) the rescaling $4k \to 4k+2$ in both (83.a) and (83.b) is extremely convenient because it separates at once the recurrent-equations

$$\begin{aligned} a) \quad & a_{4k+4} = -\left\{\frac{(4k+1)(4k+2)}{(4k+3)(4k+4)}\right\}^{1/2} a_{4k} \\ & \qquad ; k = 0, 1, 2, \ldots \\ b) \quad & a_{4k+5} = -\left\{\frac{(4k+2)(4k+3)}{(4k+4)(4k+5)}\right\}^{1/2} a_{4k+1} \end{aligned} \tag{84}$$

for the physical amplitudes $\{a_{4k}\}_{k\in N}, \{a_{4k+1}\}_{k\in N}$ of the states $\mid \psi_+\rangle, \mid \psi_-\rangle$ given by (82). Hence, the solutions of (84.a), (84.b) read respectively

$$\begin{aligned} a) \quad & a_{4n} = (-1)^n \prod_{k=0}^{n-1} \left\{\left[\frac{(4k+1)(4k+2)}{(4k+3)(4k+4)}\right]^{1/2}\right\} a_0 \\ & \qquad ; \; n = 1, 2, \ldots \\ b) \quad & a_{4n+1} = (-1)^n \prod_{k=0}^{n-1} \left\{\left[\frac{(4k+2)(4k+3)}{(4k+4)(4k+5)}\right]^{1/2}\right\} a_1 \end{aligned} \tag{85}$$

and the amplitudes a_0, a_1 are fixed up to a phase factor by the normalization

$$a)\quad \mid a_0 \mid = \left\{1 + \sum_{n=1}^{\infty} \mid \frac{a_{4n}}{a_0} \mid^2 \right\}^{-1/2} = \left\{\sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2} \frac{\Gamma(3/4)}{\Gamma(1/4)} \frac{\Gamma(n+1/4)}{\Gamma(n+3/4)}\right\}^{-1/2}$$

$$b)\quad \mid a_1 \mid = \left\{1 + \sum_{n=1}^{\infty} \mid \frac{a_{4n+1}}{a_1} \mid^2 \right\}^{-1/2} = \left\{\sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2} \frac{\Gamma(5/4)}{\Gamma(3/4)} \frac{\Gamma(n+3/4)}{\Gamma(n+5/4)}\right\}^{-1/2} \tag{86}$$

In the end, as $\mid \psi_+\rangle, \mid \psi_-\rangle$ given by (82), with (85), (86), are two physical quantum states of the system, let us compute, at least formally, the effective propagators

$$a)\quad G_+(t,t_0) = \langle \psi_+ \mid T[\phi(t)\phi(t_0)] \mid \psi_+\rangle$$

$$b)\quad G_-(t,t_0) = \langle \psi_- \mid T[\phi(t)\phi(t_0)] \mid \psi_-\rangle, \tag{87}$$

where the T - operation stands for the usual time-ordered product. So, using the field operator expression (54), we get

$$T[\phi(t)\phi(t_0)] = \frac{1/2}{\mu}\{[C^+C + CC^+]\cosh[\mu(t+t_0)] + i\sinh[\mu(t-t_0)] + \\ +i[e^{2i\beta}(C)^2 - e^{-2i\beta}(C^+)^2]\cosh[\mu(t-t_0)] - \\ -[e^{2i\beta}(C)^2 + e^{-2i\beta}(C^+)^2]\sinh[\mu(t+t_0)]\}, \tag{88}$$

with $t \geq t_0$, and because

$$\langle \psi_+ \mid (C)^2 \mid \psi_+\rangle = 0 \quad , \quad \langle \psi_+ \mid (C^+)^2 \mid \psi_+\rangle = 0$$

$$\langle \psi_- \mid (C)^2 \mid \psi_-\rangle = 0 \quad , \quad \langle \psi_- \mid (C^+)^2 \mid \psi_-\rangle = 0 \tag{89}$$

(as it can be easily seen in the light of (80)) it results from (88) after some simple calculations :

$$
\begin{aligned}
a)\quad G_+(t,t_0) = &\ \frac{1/2}{\mu}\{\cosh[\mu(t+t_0)] + i\sinh[\mu(t-t_0)] + \\
&+ 8\,\frac{\sum_{n=1}^{\infty} n\frac{(2n)!}{2^{2n}(n!)^2}\frac{\Gamma(n+1/4)}{\Gamma(n+3/4)}}{\sum_{n=0}^{\infty}\frac{(2n)!}{2^{2n}(n!)^2}\frac{\Gamma(n+1/4)}{\Gamma(n+3/4)}}\cosh[\mu(t+t_0)]\} \\
b)\quad G_-(t,t_0) = &\ \frac{1/2}{\mu}\{3\cosh[\mu(t+t_0)] + i\sinh[\mu(t-t_0)] + \\
&+ 8\,\frac{\sum_{n=1}^{\infty} n\frac{(2n)!}{2^{2n}(n!)^2}\frac{\Gamma(n+3/4)}{\Gamma(n+5/4)}}{\sum_{n=0}^{\infty}\frac{(2n)!}{2^{2n}(n!)^2}\frac{\Gamma(n+3/4)}{\Gamma(n+5/4)}}\cosh[\mu(t+t_0)]\}
\end{aligned} \tag{90}
$$

4 Summary and Discussion

We have noticed that, with respect to the modified coherent metric field $\phi(t)$, the spatially-flat Robertson-Walker Universe with cosmological constant is an exactly solvable Hamiltonian system, with the Lagrangian (30) of the "reversed oscillator" - type. This feature allows a canonical non-perturbative quantization under the Wheeler - DeWitt constraint (53), which arises naturally as the quantum analogous of the Einsteinian "energy" - equation (16.b) and filters the two orthogonal physical states (82) of the (quantum) system, exactly as Einstein equation did at the classical level, with the disjoint "blue" and "red - shifted" De Sitter (exact) solutions. The derived expression of the field operator (54) casting the Hamiltonian into the form (43) leads to the non-conservation of the number of "FRW - quanta", mathematically viewed as the eigenvalues of the number operator. Therefore, each n - quanta Fock state is non-trivially evolving, exhibiting energy fluctuations and a finite "universal" - volume quantization, (60), induced by the modified coherent metric field fluctuations. Spontaneously, at $t_0 = t$, the effective field propagators (90) are real and point out that the large scale infinity of the (modeled) Universe is due to the quantum - geometrical contributions of <u>all</u> $\mid 4n\rangle$ or $\mid 4n+1\rangle$ "FRW - quanta" that build up its physical states $\mid \psi_\pm\rangle$.

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