DEFORMED HEISENBERG ALGEBRA, FRACTIONAL SPIN FIELDS AND SUPERSYMMETRY WITHOUT FERMIONS

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Abstract

Within a group-theoretical approach to the description of (2+1)-dimensional anyons, the minimal covariant set of linear differential equations is constructed for the fractional spin fields with the help of the deformed Heisenberg algebra (DHA), $[a^-, a^+] = 1 + \nu K$, involving the Klein operator K, $\{K, a^\pm\} = 0$, $K^2 = 1$. The connection of the minimal set of equations with the earlier proposed 'universal' vector set of anyon equations is established. On the basis of this algebra, a bosonization of supersymmetric quantum mechanics is carried out. The construction comprises the cases of exact and spontaneously broken $N = 2$ supersymmetry allowing us to realize a Bose-Fermi transformation and spin-1/2 representation of SU(2) group in terms of one bosonic oscillator. The construction admits an extension to the case of OSp(2|2) supersymmetry, and, as a consequence, both applications of the DHA turn out to be related. A possibility of 'superimposing' the two applications of the DHA for constructing a supersymmetric (2+1)-dimensional anyon system is discussed. As a consequential result we point out that $osp(2|2)$ superalgebra is realizable as an operator algebra for a quantum mechanical 2-body (nonsupersymmetric) Calogero model.

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1 Introduction

In this work, we consider two related applications of the deformed Heisenberg algebra, which was introduced by Vasiliev in the context of higher spin algebras [1]. Subsequently, the generalized and modified (extended) version of this algebra was used in refs. [2]–[4] under investigation of the quantum mechanical N-body Calogero model [5], related to the (1+1)-dimensional anyons [6, 7]. Moreover, the extended version turned out to be useful in establishing the links between Knizhnik-Zamolodchikov equations and Calogero model [8] (see the paper [9] for further references on this model being an interesting example of one-dimensional quantum integrable systems). The applications of the deformed Heisenberg algebra to be considered here are the construction of the minimal set of linear differential equations for (2+1)-dimensional fractional spin fields (anyons) and the bosonization of supersymmetric quantum mechanics. We also present some speculation on the possibility to 'superimpose' these two applications for constructing a supersymmetric (2+1)-dimensional anyon system.

The considerable interest to the (2+1)-dimensional anyons, i.e. particles with fractional spin and statistics [10], is conditioned nowadays by their applications to the theory of planar physical phenomena: fractional quantum Hall effect and high-T_c superconductivity [11]. Anyons also attract a great attention due to their relationship to the different theoretical fields of research such as conformal field theories and braid groups (see, e.g., refs. [12]–[14]).

From the field-theoretical point of view, such particles can be described in two, possibly related, ways. The first way consists in organizing a statistical interaction of the scalar or fermionic field with the Chern-Simons U(1) gauge field, that changes spin and statistics of the matter field [15]. In this approach, manifestly gauge-invariant nonlocal field operators, constructed from the initial gauge noninvariant matter fields following the line integral prescription of Schwinger, carry fractional spin and satisfy anyonic permutation relations [16]. Such redefined matter fields are given on the one-dimensional path — unobservable 'string' going to the space infinity and attached to the point in which the initial matter field is given. Therefore in this approach the anyonic field operators are path-dependent and multivalued [16] (see also ref. [13]). The initial Lagrangian for the statistically charged matter field can be rewritten in the decoupled form in terms of the anyonic gauge-invariant matter fields and Chern-Simons gauge field. However, such a formal transition to the free nonlocal anyonic fields within a path integral approach is accompanied by the appearance of the complicated Jacobian of the transformation [17], and this means a nontrivial relic of the statistical gauge field in the theory. Therefore, the Chern-Simons gauge field approach does not give a minimal description of (2+1)-dimensional anyons.

Another, less developed way consists in attempting to describe anyons within the group-theoretical approach analogously to the case of integer and half-integer spin fields, without using Chern-Simons U(1) gauge field constructions. The program of this approach [18]–[27] consists in constructing equations for (2+1)-dimensional fractional spin field, subsequent identifying corresponding field action and, finally, in realizing a quantization of the theory to reveal a fractional statistics. Within this approach, there are, in turn, two related possibilities: to use many-valued representations of the (2+1)-dimensional Lorentz group SO(2,1), or to work with the infinite-dimensional unitary representations of its universal covering group, $\overline{\mathrm{SO}(2,1)}$ (or $\overline{\mathrm{SL}(2,\mathrm{R})}$, isomorphic to it). Up to now, it is not clear how to construct the action functionals corresponding to the equations for the fractional spin field carrying many-valued representations of SO(2,1) [18, 19, 23]. On the other hand, different variants of the equations and some corresponding field actions were constructed with the use of the unitary infinite-dimensional representations of $\overline{\mathrm{SL}(2,\mathrm{R})}$ [21]–[27]. Nevertheless, the problem of quantizing the theory is still open here. This is connected with the search for the most appropriate set of initial equations for subsequent constructing the action as a basic ingredient for quantization of the theory. Besides, there is a difficulty in quantization of such a theory related to the infinite-component nature of the fractional spin field which is used to describe in a covariant way one-dimensional irreducible representations of the (2+1)-dimensional quantum mechanical Poincaré group $\overline{\mathrm{ISO}(2,1)}$, specified by the values of mass and arbitrary (fixed) spin. Due to this fact, an infinite set of the corresponding Hamiltonian constraints must be present in the theory to exclude an infinite number of the 'auxiliary' field degrees of freedom and leave in the theory only one physical field degree of freedom. This infinite set of constraints should appropriately be taken into account. But, on the other hand, the infinite-component nature of the fractional spin field indicates the possible hidden nonlocal nature of the theory, and, therefore, can be considered in favour of existence of the anyonic spin-statistics relation [13, 28] for the fractional spin fields within the framework of the group-theoretical approach. So, calling fractional spin fields as anyons, we bear in mind such a hypothetical spin-statistics relation.

Here, we shall consider the problem of constructing the minimal covariant set of linear differential equations for (2+1)-dimensional fractional spin fields within a framework of the group-theoretical approach to anyons, developing the investigation initiated in refs. [26, 27]. We shall see that the minimal set of linear differential equations plays for anyons the role analogous to the role played by the Dirac and Jackiw-Templeton-Schonfield equations [29] in the cases of the spinor and topologically massive vector gauge fields, respectively. At the same time, these linear differential equations are similar to the (3+1)-dimensional Dirac positive-energy relativistic wave equations [30]. As it has been declared above, the construction will be realized with the help of the deformed Heisenberg algebra [1]. This algebra involves the so called Klein operator as an essential object, which introduces Z_2-grading structure on the Fock space of the deformed bosonic oscillator. Such a structure, in turn, is an essential ingredient of the supersymmetry considered in ref. [31] as a hidden supersymmetry of the deformed bosonic oscillator. Using this observation, we shall bosonize the supersymmetric quantum mechanics through the realization of $N = 2$ superalgebra on the Fock space of the deformed (or ordinary, undeformed) bosonic oscillator. Moreover, we shall show that the two applications of the deformed Heisenberg algebra turn out to be related through the more general OSp(2|2) superalgebraic structure.

The paper is organized as follows. In section 2, starting from the presentation of the main idea of the group-theoretical approach, we consider a 'universal' (but nonminimal) covariant vector set of linear differential equations for the fractional spin fields [26] and formulate the problem of constructing the minimal spinor set of linear differential equations. Such a construction is realized in section 4 with the help of the deformed Heisenberg algebra, which itself is considered in section 3. In the latter section we shall show, in particular, that the Vasiliev deformed bosonic oscillator, given by this algebra, and the q-deformed Arik-Coon [32] and Macfarlane-Biedenharn oscillators [33] have a general structure of the generalized deformed oscillator considered in ref. [34]. Section 5 is devoted to realization of $N = 2$ supersymmetric quantum mechanics on the Fock space of one (ordinary or deformed) bosonic oscillator. First, we realize the simplest superoscillator in terms of creation and annihilation bosonic operators. Such a construction gives us a possibility to realize a Bose-Fermi transformation in terms of one bosonic oscillator and construct a spin-1/2 representation of the SU(2) group on its Fock space. We shall show that the realization of N=2 supersymmetric quantum mechanics on the Fock space of the bosonic oscillator is achieved due to a specific nonlocal character of the supersymmetry generators. Hence, from the point of view of a nonlocality, the bosonization scheme turns out to be similar to the above mentioned Chern-Simons gauge field constructions for anyons [15, 16]. Then, we shall generalize the constructions to the more complicated $N = 2$ supersymmetric systems, in particular, corresponding to the Witten supersymmetric quantum mechanics with odd superpotential. In conclusion of this section, we shall demonstrate that the construction can be extended to the case of OSp(2|2) supersymmetry. As a consequence, we reveal $osp(2|2)$ superalgebra in the form of the operator superalgebra for the quantum mechanical 2-body (nonsupersymmetric) Calogero model [5].

Section 6 is devoted to the discussion of the results, open problems and possible generalizations of the constructions.

2 'Universal' vector set of equations

Within a group-theoretical approach to anyons, relativistic field with fractional (arbitrary) spin s can be described by the system of the Klein-Gordon equation

$$(P^2 + m^2)\Psi = 0 \tag{2.1}$$

and linear differential equation

$$(PJ - sm)\Psi = 0. \tag{2.2}$$

Here $P_\mu = -i\partial_\mu$, the metric is $\eta_{\mu\nu} = diag(-1, 1, 1)$, and operators J^μ, $\mu = 0, 1, 2$, being the generators of the group $\overline{\mathrm{SL}(2,\mathrm{R})}$, form the $sl(2)$ algebra

$$[J_\mu, J_\nu] = -i\epsilon_{\mu\nu\lambda}J^\lambda, \tag{2.3}$$

where $\epsilon_{\mu\nu\lambda}$ is an antisymmetric tensor normalized so as $\epsilon^{012} = 1$. We suppose that field $\Psi = \Psi^n(x)$ is transformed according to the one of the infinite-dimensional unitary irreducible representations (UIRs) of the group $\overline{\mathrm{SL}(2,\mathrm{R})}$: either of the discrete type series $D^\pm_\alpha$, or of the principal or supplementary continuous series C^θ_σ [35]. These representations are characterized by the value of the Casimir operator J^2 and by eigenvalues j_0 of the operator J_0. In the case of the representations $D^\pm_\alpha$, we have $J^2 = -\alpha(\alpha-1)$, $\alpha > 0$, and $j_0 = \pm(\alpha+n)$, $n = 0, 1, 2, \ldots$, and, therefore, these representations are half-bounded. In the case of the representations of the continuous series C^θ_σ, $J^2 = \sigma$ and $j_0 = \theta+n$, $\theta \in [0, 1)$, $n = 0, \pm1, \pm2, \ldots$, where $\sigma \geq 1/4$ for the principal series and $0 < \sigma < 1/4$, $\sigma > \theta(1-\theta)$ for the supplementary series [35].

The system of eqs. (2.1) and (2.2) has nontrivial solutions under the coordinated choice of the representation and parameter s, and as a result, these equations fix the values of the $\overline{\mathrm{ISO}(2,1)}$ Casimir operators, which are the operators of squared mass, $M^2 = -P^2$, and spin, $S = PJ/M$. In particular, choosing a representation of the discrete series $D^\pm_\alpha$ and parameter $s = \varepsilon\alpha$, $\varepsilon = \pm 1$, we find that these equations have nontrivial solutions describing the states with spin $s = \varepsilon\alpha$, mass $M = m$ and energy sign $\epsilon^0 = \pm\varepsilon$. In this case eq. (2.2) itself is the (2+1)-dimensional analog of the Majorana equation [36] giving the spectrum of the quantized model of relativistic particle with torsion [21]. The spectrum of this equation contains an infinite number of massive states with spin-mass dependence

$$M_n = m\frac{\alpha}{|S_n|}, \quad S_n = \epsilon(\alpha + n), \tag{2.4}$$

and, moreover, comprises massless and tachyonic states. Therefore, from the point of view of the Majorana equation spectrum, the role of the Klein-Gordon equation consists in singling out only one state corresponding to $n = 0$ from infinite spectrum (2.4) and in getting rid off the massless and tachyonic states.

The system of equations (2.1) and (2.2) has an essential shortcoming: they are completely independent, unlike, e.g., the systems of Dirac and Klein-Gordon equations for the case of spinor field or of Jackiw-Templeton-Schonfeld and Klein-Gordon equations for a topologically massive vector U(1) gauge field [29]. Therefore, they are not very suitable for constructing the action and quantum theory of the fractional spin fields, and it is necessary to find out more convenient set of linear differential equations, which would be a starting point for a subsequent realization of the program described above.

In the recent paper [26], the following covariant vector set of linear differential equations for a field with arbitrary fractional spin has been constructed by the author and J.L. Cortés:

$$V_\mu \Psi = 0, \tag{2.5}$$

$$V_\mu = \alpha P_\mu - i\epsilon_{\mu\nu\lambda}P^\nu J^\lambda + \varepsilon m J_\mu, \tag{2.6}$$

where $\varepsilon = \pm 1$, and we suppose that α is an arbitrary dimensionless parameter. This set of three equations has the following remarkable property. From the very beginning one can suppose only that J_μ are the generators of the (2+1)-dimensional Lorentz group satisfying commutation relations (2.3), not fixing at all the choice of the concrete (reducible or irreducible) representation. Then, multiplying eqs. (2.5) by the operators J_μ, P_μ and $\epsilon_{\mu\nu\lambda}P^\nu J^\lambda$, we arrive at the Klein-Gordon equation (2.1) and Majorana-type equation (2.2) with $s = \varepsilon\alpha$ as a consequence of the initial system of equations (2.5), and moreover, we also find that the equation

$$(J^2 + \alpha(\alpha - 1))\Psi = 0 \tag{2.7}$$

is a consequence of eqs. (2.5). Eq. (2.7) is nothing else as the condition of irreducibility of corresponding representation. When $\alpha > 0$, the system of eqs. (2.1), (2.2) and (2.7) has nontrivial solutions only for the choice of the unitary infinite-dimensional representations of the discrete type series $D^{\pm}_{\alpha}$, whereas for $\alpha = -j < 0$, j being integer or half-integer, nontrivial solutions take place only under the choice of the $(2j+1)$-dimensional nonunitary representations $\tilde{D}_j$ of the group $\overline{\mathrm{SL}(2,\mathrm{R})}$. Such $(2j+1)$-dimensional nonunitary representations are related to the corresponding unitary representations D_j of the group $SU(2)$, and, in particular, in the simplest cases $j = 1/2$ and $j = 1$ they describe the spinor and vector representations of the (2+1)-dimensional Lorentz group [26]. In these two cases eqs. (2.5) can be reduced to the Dirac and to the Jackiw-Temleton-Schonfeld equations, respectively. As soon as the representation is chosen (and, so, eq. (2.7) is satisfied identically), the following identity takes place for the vector differential operatorR RV_μ:

$$R^\mu V_\mu \equiv 0,$$
$$R_\mu = \left((\alpha-1)^2\eta_{\mu\nu} - i(\alpha-1)\epsilon_{\mu\nu\lambda}J^\lambda + J_\nu J_\mu\right) P^\nu.$$

This identity means that after fixing a choice of an admissible representation, three equations (2.5) become to be dependent ones, and in general case any two of them are independent (in the above mentioned cases of the finite dimensional nonunitary spinor and vector representations, eqs. (2.5) contain only one independent equation). Therefore, the total set of three equations is necessary only to have an explicitly covariant set of linear differential equations.

Thus, eq. (2.5) is the set of linear differential equations which itself, unlike the set of initial equations (2.1) and (2.2), fixes the choice of the representations $D^\pm_\alpha$ for the description of fractional spin fields in 2+1 dimensions, and, on the other hand, establishes some links between the description of ordinary bosonic integer and fermionic half-integer spin fields, and the fields with arbitrary spin. At the same time we conclude that the minimal number of independent linear differential equations for fractional spin fields is equal to two. Therefore, the corresponding minimal covariant set of such equations must have a spinor form.

To construct such a covariant minimal spinor set of equations, we turn to the deformed Heisenberg algebra [1]. This algebra permits to realize the generators J_μ for the representations $D^\pm_\alpha$ in the form of operators bilinear in the creation-annihilation deformed oscillator operators $a^\pm$. On the other hand, as we shall see, operators $a^\pm$ together with operators J_μ will form $osp(1|2)$ superalgebra, and, as a consequence, two linear combinations of $a^\pm$ will form a two-component self-conjugate object which exactly is a (2+1)-dimensional spinor. It is these two fundamental properties supplied by the deformed bosonic oscillator [1] that will help us to 'extract the square root' from the basic equations (2.1) and (2.2), and to construct a minimal covariant set of linear differential equations for fractional spin fields.

3 Deformed Heisenberg algebra

So, let us consider the algebra [1]

$$[a^-, a^+] = 1 + \nu K, \tag{3.1}$$
$$K^2 = 1, \quad K a^\pm + a^\pm K = 0, \tag{3.2}$$

with the mutually conjugate operators a^- and a^+, $(a^-)^\dagger = a^+$, self-conjugate Klein operator K and real deformation parameter ν. In the undeformed case ($\nu = 0$), eq. (3.2) can be considered simply as a relation defining the Klein operator K.

Under application of algebra (3.1), (3.2) to the quantum mechanical 2-body Calogero model, the Klein operator K is considered as an independent operator being an operator of the permutation of two identical particles, whereas the creation-annihilation operators $a^\pm$ are constructed from the relative coordinate and momentum operators of the particles,

$$a^\pm = \frac{q \mp ip}{\sqrt{2}}, \tag{3.3}$$

and in representation where the operator of the relative coordinate q to be diagonal, the momentum operator is realized in the form [2]–[4]:

$$p = -i\left(\frac{d}{dq} - \frac{\nu}{2q}K\right). \tag{3.4}$$

Therefore, in such a case algebra (3.1), (3.2) is an extension of the Heisenberg algebra. On the other hand, one can realize operator K in terms of the creation-annihilation operators themselves [27, 31], and in this case algebra (3.1), (3.2) gives a specific 'ν-deformed' bosonic oscillator. We shall understand algebra (3.1), (3.2) in this latter sense, i.e. as a deformed Heisenberg algebra.

Let us introduce the Fock-type vacuum,

$$a^-|0>= 0, \quad <0|0>= 1, \quad K|0>= \kappa|0>,$$

where $\kappa = +1$ or -1. Without loss of generality, we put here $\kappa = +1$. Then we get the action of the operator a^+a^- on the states $(a^+)^n|0>$, $n = 0, 1, \ldots$,

$$a^+a^-(a^+)^n|0>= [n]_\nu(a^+)^n|0>, \tag{3.5}$$

where

$$[n]_\nu = n + \frac{\nu}{2}\left(1 + (-1)^{n+1}\right).$$

From here we conclude that in the case when

$$\nu > -1, \tag{3.6}$$

the space of unitary representation of algebra (3.1), (3.2) is given by the complete set of the normalized vectors

$$|n>= \frac{1}{\sqrt{[n]_\nu!}}(a^+)^n|0>, \quad <n|n'>= \delta_{nn'},$$

where

$$[n]_\nu! = \prod_{k=1}^{n}[k]_\nu.$$

In correspondence with eqs. (3.2), the Klein operator K separates the complete set of states $|n>$ into even and odd subspaces:

$$K|n>= (-1)^n|n>, \tag{3.7}$$

and, so, introduces Z_2-grading structure on the Fock space of the deformed ($\nu \neq 0$) or ordinary ($\nu = 0$) bosonic oscillator. Let us introduce the operators Π_+ and Π_-,

$$\Pi_\pm = \frac{1}{2}(1 \pm K), \tag{3.8}$$

which satisfy the equalities $\Pi_\pm^2 = \Pi_\pm$, $\Pi_+\Pi_- = 0$, $\Pi_+ + \Pi_- = 1$, and are the projector operators on the even and odd subspaces of the total Fock space, respectively. Then, from eqs. (3.5) and (3.7) we get the equality:

$$a^+a^- = N + \nu\Pi_-, \tag{3.9}$$

where the number operator N, by definition, satisfies the commutation relations

$$[a^-, N] = a^-, \quad [a^+, N] = -a^+,$$

and equality $N|0 >= 0$, and, as a consequence,

$$N|n >= n|n > .$$

Using eqs. (3.9) and (3.1), we get

$$a^- a^+ = N + 1 + \nu \Pi_+,$$

and, as a result, we arrive at the following expression for the number operator in terms of the operators $a^\pm$:

$$N = \frac{1}{2}\{a^-, a^+\} - \frac{1}{2}(\nu + 1). \tag{3.10}$$

Therefore, due to the completeness of the set of basis states $|n >$, we can realize the Klein operator K in terms of the operators $a^\pm$ by means of equality (3.10): $K = \exp i\pi N$, or, in the explicitly hermitian form,

$$K = \cos \pi N. \tag{3.11}$$

Hence, we have expressed the Klein operator in terms of the creation and annihilation operators, and, as a result, realized algebra (3.1), (3.2) as a deformation of the Heisenberg algebra.

To conclude this section, let us make the following remark. Using eqs. (3.9) and (3.11), one can write the relation

$$N = a^+ a^- + \frac{\nu}{2}(\cos \pi N - 1). \tag{3.12}$$

This relation is a transcendent equation for the number operator given as a function of the normal product $a^+ a^-$. As a result, the defining relations of the ν-deformed oscillator (3.1), (3.2) can be reduced to the form

$$a^- a^+ = g(a^+ a^-), \tag{3.13}$$

where $g(x) = x + f(x)$, and $f(x) = \nu \cos \pi N(x)$ is the function of $x = a^+ a^-$. The generalized deformed oscillator algebra (3.13), containing as other particular cases the algebras of the q-deformed Arik-Coon [32] and Macfarlane-Biedenharn [33] oscillators, was considered by Dascaloyannis [34], and, so, all these deformed bosonic oscillators have a general structure (see also ref. [37]).

4 Spinor system of anyon equations

Let us consider now the following set of three operators J_μ, bilinear in the creation-annihilation operators of the ν-deformed oscillator:

$$J_0 = \frac{1}{4}\{a^+, a^-\}, \quad J_\pm = J_1 \pm iJ_2 = \frac{1}{2}(a^\pm)^2. \tag{4.1}$$

These operators satisfy the algebra (2.3) of the generators of $\overline{\mathrm{SL}(2,\mathrm{R})}$ group for any value of the deformation parameter ν given by eq. (3.6), whereas the value of the Casimir operator $J_\mu J^\mu$ is given here by the relation

$$J^2 = -\hat{\alpha}(\hat{\alpha}-1),$$

where

$$\hat{\alpha} = \frac{1}{4}(1+\nu K). \tag{4.2}$$

So, realization (4.1) gives a reducible representation of the group $\overline{\mathrm{SL}(2,\mathrm{R})}$ on the Fock space of the ν-deformed bosonic oscillator. As we have pointed out, the total Fock space can be separated into two subspaces, spanned by even, $|k>_+$, and odd, $|k>_-$, states being distinguished by the Klein operator:

$$K|k>_\pm = \pm|k>_\pm,$$

$$|k>_+ \equiv |2k>, \quad |k>_- \equiv |2k+1>, \quad k=0,1,\ldots.$$

These subspaces are invariant with respect to the action of operators (4.1):

$$\begin{aligned} J_0|k>_\pm &= (\alpha_\pm + k)|k>_\pm, \quad k=0,1,\ldots, \\ J_-|0>_\pm &= 0, \quad J_\pm|k>_\pm \propto |k\pm 1>_\pm,\ k=1,\ldots, \end{aligned} \tag{4.3}$$

and Casimir operator takes on them the constant values,

$$J_\mu J^\mu |k>_\pm = -\alpha_\pm(\alpha_\pm - 1)|k>_\pm, \tag{4.4}$$

characterized by the parameters

$$\alpha_+ = \frac{1}{4}(1+\nu) > 0, \quad \alpha_- = \alpha_+ + \frac{1}{2} > \frac{1}{2}. \tag{4.5}$$

Relations (4.3) and (4.4) mean that we have realized the UIRs of the discrete series $D^+_{\alpha_+}$ and $D^+_{\alpha_-}$ of the group $\overline{\mathrm{SL}(2,\mathrm{R})}$ on the even and odd subspaces of the total Fock space, spanned by the states $|k>_+$ and $|k>_-$, respectively. The UIRs of the discrete series $D^-_{\alpha_\pm}$ can be obtained from realization (4.1) by means of the obvious substitution

$$J_0 \to -J_0, \quad J_\pm \to -J_\mp.$$

Further on, for the sake of simplicity we shall consider only the case of representations D^+_α.

It is necessary to note here, that the described realization of UIRs $D^\pm_{\alpha_+}$ and $D^\pm_{\alpha_-}$ generalizes the well known realization of the representations $D^\pm_{1/4}$ and $D^\pm_{3/4}$ in terms of the ordinary bosonic oscillator ($\nu=0$) [38].

For the construction of the minimal spinor set of linear differential equations sought for, we introduce the γ-matrices in the Majorana representation,

$$(\gamma^0)_\alpha{}^\beta = -(\sigma^2)_\alpha{}^\beta, \quad (\gamma^1)_\alpha{}^\beta = i(\sigma^1)_\alpha{}^\beta, \quad (\gamma^2)_\alpha{}^\beta = i(\sigma^3)_\alpha{}^\beta,$$

which satisfy the relation $\gamma^\mu\gamma^\nu = -\eta^{\mu\nu} + i\epsilon^{\mu\nu\lambda}\gamma_\lambda$. Here σ^i, $i = 1, 2, 3$, are the Pauli matrices, and raising and lowering the spinor indices is realized by the antisymmetric tensor $\epsilon_{\alpha\beta}$, $\epsilon_{12} = \epsilon^{12} = 1$: $f_\alpha = f^\beta\epsilon_{\beta\alpha}$, $f^\alpha = \epsilon^{\alpha\beta}f_\beta$.

Consider the spinor type operator

$$L_\alpha = \begin{pmatrix} q \\ p \end{pmatrix}, \tag{4.6}$$

constructed from the operators q and p, which, in turn, are defined by the relation of the form of eq. (3.3) as linear hermitian combinations of $a^\pm$. In terms of the operators L_α, $\alpha = 1, 2$, the corresponding part of the deformed Heisenberg algebra (3.1) and (3.2) is presented as

$$[L_\alpha, L_\beta] = i\epsilon_{\alpha\beta}\cdot(1+\nu K), \quad \{K, L_\alpha\} = 0.$$

At the same time, these operators satisfy the following anticommutation relations:

$$\{L_\alpha, L_\beta\} = 4i(J_\mu\gamma^\mu)_{\alpha\beta}. \tag{4.7}$$

Hence, operator (4.6) is the 'square root' operator of the $\overline{\mathrm{SL}(2,\mathrm{R})}$ generators (4.1). Calculating the commutators of L_α with J_μ being the generators of the (2+1)-dimensional Lorentz group, we get the relation:

$$[J_\mu, L_\alpha] = \frac{1}{2}(\gamma_\mu)_\alpha{}^\beta L_\beta, \tag{4.8}$$

which means that the introduced operators (4.6) indeed form the (2+1)-dimensional spinor. Moreover, taking into account (anti)commutation relations (2.3), (4.7) and (4.8), we conclude that the operators J_μ and L_α form the $osp(1|2)$ superalgebra [39] with the Casimir operator

$$C = J^\mu J_\mu - \frac{i}{8}L^\alpha L_\alpha = \frac{1}{16}(1-\nu^2). \tag{4.9}$$

This, in turn, means that we have constructed the generalization of the well known representation of $osp(1|2)$ superalgebra with $C = 1/16$, realized by the ordinary bosonic oscillator ($\nu = 0$) [40, 41], to the case of $C < 1/16$ ($\nu \neq 0$) (see also ref. [31]).

Having the spinor object L_α, which is a 'square root' operator of J_μ, we can construct a covariant spinor linear differential operator with dimensionality of mass of the following most general form not containing a dependence on the Klein operator K:

$$S_\alpha = L^\beta\left((P\gamma)_{\beta\alpha} + \varepsilon m\epsilon_{\beta\alpha}\right), \tag{4.10}$$

where, again, $P_\mu = -i\partial_\mu$ and $\varepsilon = \pm 1$. Therefore, one can consider the spinor set of linear differential equations

$$S_\alpha\Psi = 0. \tag{4.11}$$

Here we suppose that the field $\Psi = \Psi^n(x)$ is an infinite-component function given on the Fock space of the ν-deformed bosonic oscillator, i.e. eq. (4.11) is a symbolic presentation of the set of two infinite-component equations $S_\alpha^{nn'}\Psi^{n'} = 0$ with $S_\alpha^{nn'} = < n|S_\alpha|n' >$.

Operator (4.10) satisfies the relation

$$S^\alpha S_\alpha = L^\alpha L_\alpha(P^2 + m^2).$$

Since $L^\alpha L_\alpha = -i(1+\nu K) \neq 0$ due to restriction (3.6), we conclude that the Klein-Gordon equation (2.1) is the consequence of eqs. (4.11). Moreover, we have the relation

$$L^\alpha S_\alpha = -4i\,(PJ - \varepsilon m\hat{\alpha})\,,$$

where $\hat{\alpha}$ is given by eq. (4.2). Decomposing the field Ψ as $\Psi = \Psi_+ + \Psi_-$,

$$\Psi_\pm = \Pi_\pm \Psi, \tag{4.12}$$

and, so, $\Psi_+ = \Psi^{2k}$, $\Psi_- = \Psi^{2k+1}$, $k = 0, 1, \ldots$, we conclude that as a consequence of basic eqs. (4.11), the fields $\Psi_\pm$ satisfy, respectively, the equations

$$(PJ - \varepsilon\alpha_+ m)\Psi_+ = 0, \tag{4.13}$$

$$\left(PJ - \varepsilon\left(\alpha_- - \frac{1}{2}(1+\nu)\right) m\right)\Psi_- = 0, \tag{4.14}$$

where α_+ and α_- are given by eq. (4.5). Note, that eq. (4.13) is exactly the (2+1)-dimensional analogue of the Majorana equation. Taking into account eq. (2.1), and passing over to the rest frame $\mathbf{P} = \mathbf{0}$ in the momentum representation, we find with the help of eq. (4.1) that eq. (4.13) has the solution of the form $\Psi_+ \propto \delta(P^0 - \varepsilon m)\delta(\mathbf{P})\Psi^0$, whereas eq. (4.14) has no nontrivial solution. Hence, the pair of equations (4.11) has nontrivial solutions only in the case $\Psi = \Psi_+$, describing the field with spin $s = \varepsilon\alpha_+$ and mass m. At $\nu = 0$, eqs. (4.11) turn into Volkov-Sorokin-Tkach equations for a field with spin $s = 1/4 \cdot \varepsilon$ [24, 25] being (2+1)-dimensional analogues of the Dirac (3+1)-dimensional positive-energy relativistic wave equations [30].

Thus, we have constructed the minimal covariant system of linear differential equations (4.11) for the field with arbitrary fractional spin, whose value is defined by the deformation parameter: $s = \varepsilon \cdot \frac{1}{4}(1+\nu) \neq 0$.

There is the following connection between the spinor operator (4.10) and the vector operator (2.6), which is valid on the even subspace of the Fock space of the ν-deformed bosonic oscillator:

$$(\gamma_\mu)^{\alpha\beta} L_\alpha S_\beta \Psi_+ = V_\mu \Psi_+, \tag{4.15}$$

where we suppose that V_μ is given by eq. (2.6) with the operators J_μ being realized in the form (4.1). From here we conclude that the field Ψ_+ satisfying spinor system of independent eqs. (4.11), satisfies also the vector system of dependent equations (2.5), and, therefore, the constructed system of equations (4.11) is a fundamental system of linear differential equations for the fractional spin fields. We shall return to the discussion of these equations and connection (4.15) in last section.

5 Bosonization of supersymmetric quantum mechanics

As it was pointed out in the original papers [1], the deformed Heisenberg algebra (3.1) and (3.2) leads to different higher spin superalgebras. We have shown in the previous section that the ν-deformed bosonic oscillator permits to generalize the well known representation of the $osp(1|2)$ superalgebra, realized by the ordinary bosonic oscillator ($\nu = 0$) and characterized by the value of the Casimir operator $C = 1/16$, to the representations with $C = \frac{1}{16}(1-\nu^2)$. Further, we have seen that the Klein operator introduces Z_2-grading structure on the Fock space of the ordinary or ν-deformed bosonic oscillator. Such a structure is an essential ingredient of supersymmetry. Therefore, we arrive at the natural question: whether it is possible to use this Z_2-grading structure for realizing representations of $N = 2$ supersymmetry in terms of ν-deformed or ordinary ($\nu = 0$) bosonic oscillator, and, therefore, for bosonizing supersymmetric quantum mechanics.

In the present section we shall investigate the problem of bosonization of supersymmetric quantum mechanics [42, 43] in the systematic way. We shall consider the ν-deformed bosonic oscillator, but it is necessary to stress that every time one can put $\nu = 0$, and therefore, all the constructions will also be valid for the case of the ordinary bosonic oscillator.

We begin with the construction of the mutually conjugate nilpotent supercharge operators Q^+ and Q^-, $Q^+ = (Q^-)^\dagger$, in the simplest possible form, linear in the bosonic operators $a^\pm$, but also containing a dependence on the Klein operator K:

$$Q^+ = \frac{1}{2}a^+(\alpha + \beta K) + \frac{1}{2}a^-(\gamma + \delta K).$$

The nilpotency condition $Q^{\pm 2} = 0$ leads to the restriction on the complex number coefficients: $\beta = \epsilon\alpha$, $\delta = \epsilon\gamma$. Therefore, we find that there are two possibilities for choosing operator Q^+:

$$Q^+_\epsilon = (\alpha a^+ + \gamma a^-)\Pi_\epsilon, \quad \epsilon = \pm,$$

where $\Pi_\pm$ are the projector operators (3.8). The anticommutator $\{Q^+_\epsilon, Q^-_\epsilon\}$ has here the form:

$$\{Q^+_\epsilon, Q^-_\epsilon\} = a^{+2}\alpha\gamma^* + a^{-2}\alpha^*\gamma + \frac{1}{2}\{a^+, a^-\}(\gamma\gamma^* + \alpha\alpha^*) - \frac{1}{2}\epsilon K[a^-, a^+](\gamma\gamma^* - \alpha\alpha^*).$$

Whence we conclude that if we choose the parameters in such a way that $\alpha\gamma^* = 0$, the anticommutator will commute with the number operator N, and, as a consequence, the spectra of the corresponding Hamiltonians, $H_\epsilon = \{Q^+_\epsilon, Q^-_\epsilon\}$, $\epsilon = \pm$, will have the simplest possible form. Let us put $\alpha = 0$ and normalize the second parameter as $\gamma = e^{i\varphi}$. Since we can remove this phase factor by the unitary transformation of the operators $a^\pm$, we arrive at the nilpotent operators in the very compact form:

$$Q^+_\epsilon = a^-\Pi_\epsilon, \quad Q^-_\epsilon = a^+\Pi_{-\epsilon}. \tag{5.1}$$

They together with the operator

$$H_\epsilon = \frac{1}{2}\{a^+, a^-\} - \frac{1}{2}\epsilon K[a^-, a^+] \tag{5.2}$$

form the $N = 2$ superalgebra, which we shall also denote, according to ref. [40], as $s(2)$:

$$Q^{\pm 2}_\epsilon = 0, \quad \{Q^+_\epsilon, Q^-_\epsilon\} = H_\epsilon, \quad [Q^\pm_\epsilon, H_\epsilon] = 0. \tag{5.3}$$

Note that the hermitian supercharge operators $Q^{1,2}_\epsilon$,

$$Q^\pm_\epsilon = \frac{1}{2}(Q^1_\epsilon \pm iQ^2_\epsilon), \quad \{Q^i_\epsilon, Q^j_\epsilon\} = 2\delta^{ij}H_\epsilon,$$

have the following compact form:

$$Q^1_\epsilon = \frac{1}{\sqrt{2}}(q + i\epsilon pK), \quad Q^2_\epsilon = \frac{1}{\sqrt{2}}(p - i\epsilon qK) = -i\epsilon Q^1_\epsilon K \tag{5.4}$$

in terms of the coordinate q and momentum p operators of the deformed bosonic oscillator introduced by eq. (3.3).

Consider the spectrum of the supersymmetric Hamiltonian (5.2). Using the expression for the number operator given by eq. (3.10), we find that in the case when $\epsilon = -$, the states $|n>$ are the eigenstates of the operator H_- with the eigenvalues

$$E^-_n = 2[n/2] + 1 + \nu,$$

where $[n/2]$ means the integer part of $n/2$. Therefore, for $\epsilon = -$ we have the case of spontaneously broken supersymmetry, with $E^-_n > 0$ for all n due to restriction (3.6). All the states $|n>$ and $|n+1>$, $n = 2k$, $k = 0, 1, \ldots$, are paired here in supermultiplets. For $\epsilon = +$, we have the case of exact supersymmetry characterized by the spectrum

$$E^+_n = 2[(n + 1)/2],$$

i.e. here the vacuum state with $E^+_0 = 0$ is a supersymmetry singlet, whereas $E^+_n = E^+_{n+1} > 0$ for $n = 2k + 1, k = 0, 1, \ldots$. Hence, we have demonstrated that one can realize both cases of the spontaneously broken and exact $N = 2$ supersymmetries with the help of only one bosonic oscillator, and in the former case the scale of supersymmetry breaking is defined by the deformation parameter ν.

Due to the property $E^-_n > 0$ taking place for $\epsilon = -$, we can construct the Fermi oscillator operators:

$$\begin{aligned} f^\pm &= \frac{Q^\mp_-}{\sqrt{H_-}} \\ &= a^\pm \cdot \frac{\Pi_\pm}{\sqrt{N + \Pi_+}}, \end{aligned} \tag{5.5}$$

$$\{f^+, f^-\} = 1, \quad f^{\pm 2} = 0,$$

i.e. one can realize a Bose-Fermi transformation in terms of one bosonic oscillator. Let us note that though operators $a^\pm$ do not commute with $f^\pm$,

$$[a^\pm, f^\pm] \neq 0,$$

nevertheless, the operator H_ϵ can be written in the form of the simplest supersymmetric Hamiltonian of the superoscillator [42]:

$$H_\epsilon = \frac{1}{2}\{a^+, a^-\} + \epsilon\frac{1}{2}[f^+, f^-].$$

Having fermionic oscillator variables (5.5), we can construct the hermitian operators:

$$S_1 = \frac{1}{2}(f^+ + f^-), \quad S_2 = -\frac{i}{2}(f^+ - f^-), \quad S_3 = f^+f^- - 1/2.$$

They act in an irreducible way on every 2-dimensional subspace of states ($|2k>, |2k+1>$), $k = 0, 1, \ldots$, and form $su(2)$ algebra:

$$[S_i, S_j] = i\epsilon_{ijk} S_k.$$

Due to the relation $S_i S_i = 3/4$, this means that we have realized spin-1/2 representation of the $su(2)$ algebra in terms of one bosonic oscillator in contrast to the well known Schwinger realization of Lie algebra $su(2)$ in terms of two bosonic oscillators [44]. Therefore, one can realize unitary spin-1/2 representation of SU(2) group on the Fock space of one bosonic oscillator.

Before passing over to the generalization of the constructions to the case of more complicated quantum mechanical supersymmetric systems, corresponding to the systems with boson-fermion interaction [43, 45], let us make the following remark on the realized bosonization scheme.

We have pointed out in the first section that the 'anyonization' scheme involving the Chern-Simons gauge field has an essentially nonlocal nature. Our constructions include the Klein operator as a fundamental object, which itself is a nonlocal operator, and, as a result, all the bosonization scheme presented above has an essentially nonlocal character. Indeed, in the simplest case of the ordinary bosonic oscillator ($\nu = 0$), in the Schrödinger representation the Klein operator (3.11) is presented in the nonlocal form

$$K = \sin(\pi H_0)$$

through the Hamiltonian of the linear harmonic oscillator

$$H_0 = \frac{1}{2}\left(-\frac{d^2}{dq^2} + q^2\right).$$

As a consequence, the supersymmetric Hamiltonian (5.2) and hermitian supercharge operators (5.4) also have a nonlocal form:

$$H_\epsilon = H_0 - \frac{1}{2}\epsilon \sin(\pi H_0), \tag{5.6}$$

$$Q_\epsilon^1 = \frac{1}{\sqrt{2}}\left(q + \epsilon\frac{d}{dq}\sin(\pi H_0)\right), \quad Q_\epsilon^2 = -\frac{i}{\sqrt{2}}\left(\frac{d}{dq} + \epsilon q \sin(\pi H_0)\right), \tag{5.7}$$

and, therefore, in this sense, our bosonization constructions turn out to be similar to the Chern-Simons gauge field constructions for (2+1)-dimensional anyons [15, 16].

Let us show how the constructions can be generalized to the case corresponding to the more complicated quantum mechanical supersymmetric systems [43, 45, 46]. To this end, consider the operators

$$\tilde{Q}_\epsilon^\pm = A^\mp \Pi_{\pm\epsilon} \tag{5.8}$$

with odd mutually conjugate operators $A^\pm = A^\pm(a^+, a^-)$, $A^- = (A^+)^\dagger$, $KA^\pm = -A^\pm K$. These properties of $A^\pm$ guarantee that the operators $\tilde{Q}_\epsilon^\pm$ are, in turn, mutually conjugate, $\tilde{Q}_\epsilon^- = (\tilde{Q}_\epsilon^+)^\dagger$, and nilpotent:

$$(\tilde{Q}_\epsilon^\pm)^2 = 0. \tag{5.9}$$

One can take the anticommutator

$$\begin{aligned}\tilde{H}_\epsilon &= \{\tilde{Q}^+_\epsilon, \tilde{Q}^-_\epsilon\} \qquad (5.10)\\ &= \frac{1}{2}\{A^+, A^-\} - \frac{1}{2}\epsilon K[A^-, A^+]. \qquad (5.11)\end{aligned}$$

as the Hamiltonian, and get the $s(2)$ superalgebra given by relations (5.9), (5.10) and by the commutator

$$[\tilde{H}_\epsilon, \tilde{Q}^\pm_\epsilon] = 0. \qquad (5.12)$$

Such a generalized construction given by eqs. (5.8) and (5.11) corresponds to $N = 2$ supersymmetric systems realized in ref. [46] on the Fock space of independent bosonic and fermionic oscillators. In particular, choosing the operators $A^\pm$ as

$$A^\pm = \frac{1}{\sqrt{2}}(\mp ip + W(q))$$

with $W(-q) = -W(q)$, in the case of Heisenberg algebra ($\nu = 0$) we get for supersymmetric Hamiltonian (5.11) the form

$$\tilde{H}_\epsilon = \frac{1}{2}\left(-\frac{d^2}{dq^2} + W^2 - \epsilon K \frac{dW}{dq}\right)$$

corresponding to the Witten supersymmetric quantum mechanics [43] with odd superpotential W.

Concluding the bosonization constructions, we note that the case of the S(2) supersymmetry is contained in the more broad OSp(2|2) supersymmetry, whose superalgebra, as well as $osp(1|2)$ superalgebra considered in the previous section, contains $sl(2)$ algebra (2.3) as a subalgebra [39]. Taking into account this observation, let us show that this more broad OSp(2|2) supersymmetry also can be bosonized. Indeed, one can check that the even operators

$$T_3 = 2J_0, \quad T_\pm = J_\pm, \quad J = -\frac{1}{2}\epsilon K[a^-, a^+], \qquad (5.13)$$

together with the odd operators

$$Q^\pm = Q^\mp_\epsilon, \quad S^\pm = Q^\mp_{-\epsilon}, \qquad (5.14)$$

form the $osp(2|2)$ superalgebra given by the nontrivial (anti)commutators

$$\begin{gathered}[T_3, T_\pm] = \pm 2T_\pm, \quad [T_-, T_+] = T_3,\\ \{S^+, Q^+\} = T_+, \quad \{Q^+, Q^-\} = T_3 + J, \quad \{S^+, S^-\} = T_3 - J,\\ [T_+, Q^-] = -S^+, \quad [T_+, S^-] = -Q^+, \quad [T_3, Q^+] = Q^+,\\ [T_3, S^-] = -S^-, \quad [J, S^-] = -S^-, \quad [J, Q^+] = -Q^+, \qquad (5.15)\end{gathered}$$

and corresponding other nontrivial (anti)commutators which can be obtained from eqs. (5.15) by hermitian conjugation. Therefore, even operators (5.13) form a subalgebra $sl(2) \times u(1)$, whereas $s(2)$ superalgebra (5.3), as a subalgebra, is given by the sets of generators $Q^\pm$ and $T_3 + J$, or $S^\pm$ and $T_3 - J$. Hence, the both cases of the exact and spontaneously broken $N = 2$ supersymmetry are contained in the extended supersymmetry Osp(2|2). In eqs. (5.13) and (5.14) we suppose that operators J_μ and $Q^\pm_\epsilon$ are realized by ν-deformed oscillator operators $a^\pm$ in the form of eqs. (4.1) and (5.1), respectively, whereas the Klein operator K is realized with the help of eqs. (3.11) and (3.10).

Superalgebra (5.15), obviously, also takes place in the case discussed at the beginning of section 3, when algebra (3.1), (3.2) is considered as an extended Heisenberg algebra with the Klein operator being the permutation operator, independent from $a^\pm$, as it happens under consideration of the quantum mechanical Calogero model [2]–[4]. Therefore, we conclude that $osp(2|2)$ superalgebra (as well as smaller $s(2)$ superalgebra) can be realized as an operator superalgebra for the 2-body (nonsupersymmetric) Calogero model. Note here that the supersymmetric extension of the N-body Calogero model, possessing OSp(2|2) supersymmetry as a dynamical symmetry, was constructed by Freedman and Mende [47]. The generators of this dynamical symmetry of the supersymmetric Calogero model [47] were realized by Brink, Hansson and Vasiliev in terms of bilinears of the modified bosonic creation and annihilation operators (of the form given by eqs. (3.3) and (3.4) in the case of 2-body model) and of independent fermionic operators, and our notation for the generators in eqs. (5.13) and (5.14) have been chosen in correspondence with that in ref. [4].

6 Discussion and concluding remarks

We have constructed the minimal covariant set of linear differential equations (4.11) for the fractional spin fields, which is related to the vector set of equations (2.5) via eq. (4.15). The latter set of equations possesses a property of 'universality': it describes ordinary integer or half-integer spin fields under the choice of the $(2j + 1)$-dimensional nonunitary representations of the group $\overline{\mathrm{SL}(2,\mathrm{R})}$ instead of the infinite-dimensional unitary representations $D^\pm_\alpha$, necessary for the description of the fractional spin. Therefore, it would be interesting to investigate the minimal spinor set of equations from the point of view of the possible analogous properties of universality. We hope to consider this problem elsewhere.

The minimal set of equations (4.11) represents by itself two independent infinite sets of equations for one infinite-component field. Therefore, it is necessary to introduce some auxiliary fields for the construction of the corresponding field action and subsequent quantization of the theory. Investigation of the previous problem as well as the search for a possible hidden relation of the theory to the approach involving Chern-Simons U(1) gauge field constructions [15, 16] could be helpful for constructing a 'minimal' field action leading to eqs. (4.11) and comprising the minimal number of auxiliary fields. Since both sets of linear differential equations (4.11) and (2.5), as well as two other known sets of linear differential equations considered in refs. [20, 22], use the half-bounded infinite dimensional representations of the $\overline{\mathrm{SL}(2,\mathrm{R})}$ for the description of fractional spin fields, one can consider this fact as an indication on the possible hidden connection of a group-theoretical approach with the approach involving Chern-Simons gauge field constructions. Indeed, the half-bounded nature of these representations ($n = 0, 1, 2, \ldots$) could be associated with the half-infinite nonobservable 'string' of the nonlocal anyonic field operators within the framework of the latter approach. This is, of course, so far pure speculation.

As it has been pointed out in section 3, under application of algebra (3.1), (3.2) to the quantum mechanical 2-body Calogero model, the Klein operator is considered as the operator of permutation for identical particles on the line, being independent from the operators $a^\pm$ in the sense of its realization. In this case algebra (3.1), (3.2) is the extension of the Heisenberg algebra, and operators $a^\pm$ can be realized with the help of relation (3.3) through the relative coordinate and momentum operators of the particles, $q = q_1 - q_2$, $p = p_1 - p_2$: in representation with the operator q to be diagonal, the momentum operator is given by eq. (3.4). Obviously, our constructions from section 4 will also be valid under consideration of algebra (3.1), (3.2) as the extended one. In this case the field Ψ will depend on two arguments, $\Psi = \Psi(x, q)$, and nontrivial solutions of eqs. (4.11) will be described by even fields $\Psi(x, q) = \Psi(x, -q)$. On the other hand, we have pointed out in section 1 on the relationship of the Calogero model to the systems of (1+1)-dimensional anyons [6, 7]. In particular, as it was shown by Hansson, Leinaas and Myrheim [7], the system of 2-dimensional anyons in the lowest Landau level is effectively the system of 1-dimensional anyons, described by means of the Calogero model. Therefore, the described possibility for the reinterpretation of our constructions in terms of extended Heisenberg algebra means that the fields $\Psi(x, q)$ can be understood as the fractional spin fields in (2+1)-dimensional space time, and, at the same time, as the fields describing the system of two 1-dimensional anyons, related to the 2-body Calogero model. Such a reinterpretation seems to be very attractive from the point of view of possible revealing spin-statistics relation for (2+1)-dimensional fractional spin fields within a framework of the approach under consideration.

We have shown that $osp(2|2)$ superalgebra can be revealed in the form of an operator (spectrum generating) algebra for the 2-body (nonsupersymmetric) Calogero model. It seems that with the help of generalization of algebra (3.1), (3.2) given in refs. [2]–[4], one could also reveal this superalgebra in the form of the spectrum generating algebra for the general case of N-body Calogero model. Note that here we have an analogy with the case of the ordinary (nonsupersymmetric) harmonic oscillator, for which the superalgebra $osp(1|2)$ is the spectrum generating superalgebra [40].

As a further generalization of the bosonization constructions given in section 5, one could investigate a possibility to bosonize $N > 2$ supersymmetric [40] and parasupersymmetric [48] quantum mechanical systems. For the former case, one could try to use the generalizations of the Klein operator of the form: $\tilde{K}^l = 1$, $l > 2$. Another interesting problem is the construction of the classical Lagrangians, which would lead after quantization to the supersymmetric systems described in previous section, in particular, to the simplest system described by Hamiltonian (5.6). The knowledge of the form of such Lagrangians and corresponding actions ((0+1)-dimensional from the field-theoretical point of view) could be helpful in possible generalizing the constructions to the case of the quantum field systems, (1+1)-dimensional in the simplest case. In connection with such hypothetical possible generalization it is necessary to point out that earlier some different problem was investigated by Aratyn and Damgaard [49]. They started from the (1+1)-dimensional supersymmetric field systems, bosonized them with the help of the Mandelstam nonlocal constructions [50], and, as a result, arrived at the pure bosonic quantum field systems, described by the local action functionals. On the other hand, an essentially nonlocal form of just the simplest Hamiltonian (5.6) is an indication that the corresponding generalization of the constructions presented here to the case of quantum field theory would lead to the nonlocal bosonic quantum field systems.

To conclude, let us point out the possibility for a 'superposition' of the two applications of the deformed Heisenberg algebra, presented here.

The essential object for the construction of the minimal set of linear differential equations for fractional spin fields is the $osp(1|2)$ superalgebra realized in terms of the ν-deformed bosonic oscillator. This superalgebra is the superalgebra of automorphisms of the fundamental (anti)commutation relations for the system containing one bosonic (described by mutually conjugate coordinate and momentum operators) and one fermionic (given by the selfconjugate generator of the Clifford algebra C_1) degrees of freedom. At the same time, it is a subalgebra of the superalgebra $osp(2|2)$ being a superalgebra of automorphisms for the system with one bosonic and two fermionic (given by the two selfconjugate generators of the Clifford algebra C_2) degrees of freedom [40]. So, the both our constructions turn out to be related through the $osp(2|2)$ superalgebra (5.15). Note, that the operators $a^{\pm}$ (or their hermitian linear combinations L_α), being the odd generators of $osp(1|2)$, are presented in terms of odd generators (5.14) of $osp(2|2)$ in the form:

$$a^{\pm} = Q^{\pm} + S^{\pm}.$$

The even subalgebra of $osp(1|2)$ is $sl(2)$ algebra with generators (4.1). We have seen that the deformed (or extended) Heisenberg algebra permits to realize irreducible representations $D^{\pm}_{\alpha_+}$ and $D^{\pm}_{\alpha_-}$ of the corresponding group $\overline{\mathrm{SL}(2,\mathrm{R})}$, characterized by the parameters (4.5), shifted in 1/2 and contained in one irreducible representation of $osp(1|2)$ superalgebra, characterized, in turn, by the Casimir operator (4.9). The constructed system of eqs. (4.11) has nontrivial solutions only on the even subspace of the total Fock space, described by functions Ψ_+ (or on even functions $\Psi(x,q) = \Psi(x,-q)$ in the case of extended Heisenberg algebra), i.e. only on the space of the $D^{\pm}_{\alpha_+}$ series of representations. But if we could 'enliven' the second series of representation through the construction of the system of equations having solutions for both series, we would have the system of two states with spins shifted in one-half. If, moreover, these two states will have equal masses, we would get a (2+1)-dimensional supersymmetric system of fractional spin fields. So, it would be very interesting to investigate the possibility to supersymmetrize the system of eqs. (4.11) by using odd operators (5.14) (being spinor operators from the point of view of $sl(2)$ generators (4.1)) instead of the standard introduction of additional Grassmann spinor variables [25], and therefore, to realize the superposition of the both constructions presented here. The work in this direction is in progress.

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