

The Probability for Primordial Black Holes

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Abstract

We consider two quantum cosmological models with a massive scalar field: an ordinary Friedmann universe and a universe containing primordial black holes. For both models we discuss the complex solutions to the Euclidean Einstein equations. Using the probability measure obtained from the Hartle-Hawking no-boundary proposal, we find that the only unsuppressed black holes start at the Planck size but can grow with the horizon scale during the roll down of the scalar field to the minimum.

1 Introduction

In this paper we ask how likely it is for the universe to have contained primordial black holes. We investigate universes which undergo a period of inflation in their earliest stage. In our models the inflation is driven by a minimally coupled scalar field ϕ of mass m . It starts out at a large initial value ϕ_0 and acts as a cosmological constant for some time until it reaches the minimum of its potential and inflation ends. We consider two different types of spacetimes: in the first, the spacelike sections are simply 3-spheres and no black holes are present; in the second, they have the topology $S^1 \times S^2$, which is the topology of the spatial section of the Schwarzschild-de Sitter solution. Thus these spaces can be interpreted as inflationary universes with a pair of black holes. In the inflationary period, the first type will be similar to a de Sitter universe, the second to a Nariai universe [1]. To find the likelihood for primordial black holes, we assign probabilities to both types of spacetimes using the Hartle-Hawking no-boundary proposal (NBP) [2].

The NBP framework is summarized in Section 2. In Sections 3 and 4 we review its implementation for cases with a fixed cosmological constant. In Section 5 we introduce a massive scalar field and discuss the solutions of the Euclidean Einstein equations for the S^3 case. They will be slightly complex due to the time dependence of the effective cosmological constant $(m\phi)^{-2}$. We obtain the Euclidean action for those solutions. In Section 6 we go through a similar procedure for the $S^1 \times S^2$ case. We find that the black hole grows during the inflationary period, a noteworthy difference to the Nariai case with a fixed cosmological constant. In Section 7 we use the action to estimate the relative probability of the two types of universes. We find that black holes are suppressed for all but very large initial values of ϕ_0 .

2 The Wave Function of the Universe

The Hartle-Hawking no-boundary proposal states that the wave function of the universe is given by

$$\Psi_0[h_{ij}, \Phi_{\partial M}] = \int D(g_{\mu\nu}, \Phi) \exp[-I(g_{\mu\nu}, \Phi)], \quad (2.1)$$

where $(h_{ij}, \Phi_{\partial M})$ are the 3-metric and matter field on a spacelike boundary ∂M and the path integral is taken over all compact Euclidean four geometries

$g_{\mu\nu}$ that have ∂M as their only boundary and matter field configurations Φ that are regular on them; $I(g_{\mu\nu}, \Phi)$ is their action.

The gravitational part of the action is given by

$$I_E = -\frac{1}{16\pi} \int_M d^4x g^{1/2} (R - 2\Lambda) - \frac{1}{8\pi} \int_{\partial M} d^3x h^{1/2} K, \quad (2.2)$$

where R is the Ricci-scalar, Λ is the cosmological constant, and K is the trace of K_{ij} , the second fundamental form of the boundary ∂M in the metric g . For the origin of the boundary term, see, e. g., ref. [3].

In the standard 3+1 decomposition [4], the metric is written as

$$ds^2 = N^2 d\tau^2 + h_{ij} (dx^i + N^i d\tau)(dx^j + N^j d\tau). \quad (2.3)$$

Assuming that the NBP is satisfied at $\tau = 0$, the Euclidean action then takes the form

$$I_E = -\frac{1}{16\pi} \int_{\tau=0}^{\tau_{\partial M}} N d\tau \int d^3x h^{1/2} (-K_{ij} K^{ij} + K^2 + {}^3R - 2\Lambda) + \frac{1}{8\pi} \int_{\tau=0} d^3x h^{1/2} K. \quad (2.4)$$

Here 3R is the scalar curvature of the surface, and tensor operations are carried out with respect to the surface metric h_{ij} . In the first term the boundary terms are implicitly subtracted out at $\tau = 0$ and $\tau = \tau_{\partial M}$. But it is an essential prescription of the NBP that there *is no* boundary at $\tau = 0$. So the second term explicitly adds the contribution from $\tau = 0$ back in. It vanishes for universes with spacelike section of topology S^3 , but can be non-zero for the topology $S^1 \times S^2$.

There are unresolved questions on how to choose the integration contour and make the integral converge [5], but we shall not discuss them here. Instead, we will use the semiclassical approximation

$$\Psi_0[h_{ij}, \Phi_{\partial M}] \approx \sum_n A_n e^{-I_n}, \quad (2.5)$$

where the sum is over the saddlepoints of the path integral, i. e. the solutions of the Euclidean Einstein equations. In this paper, we neglect the prefactors A_n and take only one saddlepoint into account for a given argument of the wave function. So the probability measure will be

$$|\Psi_0[h_{ij}, \Phi_{\partial M}]|^2 = \left| e^{-I} \right|^2 = e^{-2I^{Re}}, \quad (2.6)$$

where I^{Re} is the real part of the Euclidean saddlepoint action.

By considering only spaces of high symmetry (homogeneous S^3 or $S^1 \times S^2$ spacelike sections) we restrict the degrees of freedom in the metric to a finite number q^α . The Euclidean action for such a minisuperspace model with bosonic matter will typically have the form

$$I = - \int N d\tau \left[\frac{1}{2} f_{\alpha\beta} \frac{dq^\alpha}{d\tau} \frac{dq^\beta}{d\tau} + U(q^\alpha) \right]. \quad (2.7)$$

The saddlepoints will in general be complex solutions $q^\alpha(\tau)$ in the τ -plane. In the semiclassical approximation the following relations for the real and imaginary parts of the saddlepoint actions hold:

$$-\frac{1}{2} (\nabla I^{Re})^2 + \frac{1}{2} (\nabla I^{Im})^2 + U(q^\alpha) = 0 \quad (2.8)$$

$$\nabla I^{Re} \cdot \nabla I^{Im} = 0, \quad (2.9)$$

where the gradient and the dot product are both with respect to $f^{\alpha\beta}$. Therefore I^{Im} will be a solution of the Lorentzian Hamilton-Jacobi equation in regions of minisuperspace where Ψ has the property that

$$(\nabla I^{Re})^2 \ll (\nabla I^{Im})^2, \quad (2.10)$$

This allows us to reintroduce a concept of Lorentzian time in such regions: We find the integral curves of ∇I^{Im} in minisuperspace and define the Lorentzian time t as the parameter naturally associated with them. Reversely, if we demand that the NBP should predict classical Lorentzian universes at sufficiently late Lorentzian time, condition (2.10) must be satisfied. This means that there must be saddlepoint solutions for which the path in the τ -plane can be deformed such that it is eventually almost parallel to the imaginary τ axis and that all the q^α should be virtually real at late Lorentzian times. In summary, the following conditions must be met:

- i. The NBP must be satisfied at $\tau = 0$.
- ii. At the endpoint $\tau_{\partial M}$ of the path, the q^α must take on the real values $q_{\partial M}^\alpha$ of the arguments of the wavefunction:

$$q^\alpha(\tau_{\partial M}) = q_{\partial M}^\alpha. \quad (2.11)$$

iii. The q^α must remain nearly real in the Lorentzian vicinity of the endpoint:

$$\text{Re} \left(\left. \frac{dq^\alpha}{d\tau} \right|_{\tau_{\partial M}} \right) \approx 0. \quad (2.12)$$

3 The de Sitter Spacetime

In this and the next section we review vacuum solutions of the Euclidean Einstein equations with a cosmological constant Λ . First we look for a solution with spacelike sections S^3 . Therefore we choose the metric ansatz

$$ds^2 = N(\tau)^2 d\tau^2 + a(\tau)^2 d\Omega_3^2. \quad (3.1)$$

The Euclidean action is

$$I = -\frac{3\pi}{4} \int N d\tau a \left(\frac{\dot{a}^2}{N^2} + 1 - \frac{\Lambda}{3} a^2 \right), \quad (3.2)$$

A dot ($\dot{}$) denotes differentiation with respect to τ . We define

$$H = \sqrt{\frac{\Lambda}{3}}. \quad (3.3)$$

Variation of a and N yields the equation of motion

$$\frac{\ddot{a}}{a} + H^2 = 0 \quad (3.4)$$

and the Hamiltonian constraint

$$\frac{\dot{a}^2}{a^2} - \frac{1}{a^2} + H^2 = 0 \quad (3.5)$$

in the gauge $N = 1$. A solution of equations (3.4) and (3.5) is given by

$$a(\tau) = H^{-1} \sin H\tau. \quad (3.6)$$

It is called the de Sitter spacetime. The NBP is satisfied at $\tau = 0$, where $a = 0$ and $\frac{da}{d\tau} = 1$. If we choose a path along the τ^{Re} -axis to $\tau = \frac{\pi}{2H}$, the solution will describe half of the Euclidean de Sitter instanton S^4 . Choosing

the path to continue parallel to the τ^{I^m} -axis, $a(\tau)$ remains real and the conditions *i* to *iii* of the previous section will be fulfilled:

$$a(\tau^{I^m})\Big|_{\tau^{Re}=\frac{\pi}{2H}} = H^{-1} \cosh H\tau^{I^m}. \quad (3.7)$$

This describes half of an ordinary Lorentzian de Sitter universe.

So with the above choice of path, equation (3.6) corresponds to half of a real Euclidean 4-sphere joined to a real Lorentzian hyperboloid of topology $R^1 \times S^3$. It can be matched to any $a_{\partial M} > 0$ by choosing the endpoint appropriately, and for $a_{\partial M} > H^{-1}$ the wavefunction oscillates and a classical Lorentzian universe is predicted.

The real part of the action for this saddlepoint is

$$I_{deSitter}^{Re} = \frac{3\pi}{2} \int_0^{\frac{\pi}{2H}} d\tau^{Re} a (H^2 a^2 - 1) = -\frac{3\pi}{2\Lambda}. \quad (3.8)$$

The Lorentzian segment of the path only contributes to I^{Im} .

4 The Nariai Spacetime

We still consider vacuum solutions of the Euclidean Einstein equations with a cosmological constant, but we now look for solutions with spacelike sections $S^1 \times S^2$. The corresponding ansatz is the Kantowski-Sachs metric

$$ds^2 = N(\tau)^2 d\tau^2 + a(\tau)^2 dx^2 + b(\tau)^2 d\Omega_2^2. \quad (4.1)$$

The Euclidean action is

$$I = -\pi \int N d\tau a \left(\frac{\dot{b}^2}{N^2} + 2\frac{b}{a} \frac{\dot{a}\dot{b}}{N^2} + 1 - \Lambda b^2 \right) + \pi \left[-\dot{a}b^2 - 2abb \right]_{\tau=0}, \quad (4.2)$$

where the second term is the surface term of equation (2.4). We define

$$H = \sqrt{\Lambda}. \quad (4.3)$$

Variation of a , b and N gives the equations of motion and the Hamiltonian constraint:

$$\frac{\ddot{b}}{b} - \frac{\dot{a}\dot{b}}{ab} = 0 \quad (4.4)$$

$$\frac{\ddot{b}}{b} + \frac{\dot{a}\dot{b}}{ab} + \frac{\ddot{a}}{a} + H^2 = 0 \quad (4.5)$$

$$2\frac{\dot{a}\dot{b}}{ab} + \frac{\dot{b}^2}{b^2} - \frac{1}{b^2} + H^2 = 0. \quad (4.6)$$

A solution is given by

$$a(\tau) = H^{-1} \sin H\tau, \quad b(\tau) = H^{-1} = \text{const}. \quad (4.7)$$

It is called the Nariai spacetime. The NBP is satisfied at $\tau = 0$, where

$$a = 0, \quad \dot{a} = 1, \quad b = b_0 \text{ and } \dot{b} = 0. \quad (4.8)$$

(There is a second way of satisfying the NBP for the Kantowski-Sachs metric [6], but it will not lead to a universe containing black holes.) The path along the τ^{Re} -axis describes half of the Euclidean Nariai instanton $S^2 \times S^2$. Both 2-spheres have the radius H^{-1} . Continuing parallel to the τ^{Im} -axis, the solution remains real:

$$a(\tau^{Im}) \Big|_{\tau^{Re}=\frac{\pi}{2H}} = H^{-1} \cosh H\tau^{Im}, \quad b(\tau^{Im}) \Big|_{\tau^{Re}=\frac{\pi}{2H}} = H^{-1}. \quad (4.9)$$

This describes half of a Lorentzian Nariai universe. Its spacelike sections can be visualized as 3-spheres of radius a with a “hole” of radius b punched through the North and South pole. This gives them the topology of $S^1 \times S^2$. Their physical interpretation is that of 3-spheres containing two black holes at opposite ends. The black holes have the radius b and accelerate away from each other as a grows. The Nariai universe is a degenerate case of the Schwarzschild-de Sitter spacetime, with the black hole horizon and the cosmological horizon having equal radius [7].

The above path corresponds to half of a 2-sphere joined to a two-dimensional hyperboloid at its minimum radius H^{-1} , cross a 2-sphere of constant radius H^{-1} . It can be matched to any $a_{\partial M} > 0$ but only to $b_{\partial M} = H^{-1}$ so the wavefunction will be highly peaked around that value of b .

The first term of equation (4.2) vanishes and so the real part of the action for the Nariai solution comes entirely from the second term:

$$I_{Nariai}^{Re} = -\pi b_0^2 = -\frac{\pi}{\Lambda}. \quad (4.10)$$

Now we compare the probability measures corresponding to the de Sitter and Nariai solutions. We find that in these models with a fixed cosmological constant primordial black holes are strongly suppressed, unless Λ is at least of order 1 in Planck units:

$$\exp\left(-2I_{Nariai}^{Re}\right) = \exp\left(\frac{2\pi}{\Lambda}\right) \ll \exp\left(\frac{3\pi}{\Lambda}\right) = \exp\left(-2I_{deSitter}^{Re}\right). \quad (4.11)$$

5 An Inflationary Model Without Black Holes

Of course, we know that $\Lambda \approx 0$, and therefore the models of the previous section are rather unrealistic. However, in inflationary cosmology it is assumed that the very early universe underwent a period of exponential expansion. It has proven very successful to model this behaviour by introducing a massive scalar field Φ with a potential $\frac{1}{2}m^2\Phi^2$. If this field is sufficiently far from equilibrium at the beginning of the universe, the corresponding energy density acts like a cosmological constant until the field has reached its minimum and starts oscillating. During this time the universe behaves much like the Lorentzian de Sitter or Nariai universes described above.

But there are two important differences due to the time dependence of the effective cosmological constant Λ_{eff} : Firstly, for the solutions of the Euclidean Einstein equations in the complex τ -plane one can no longer find a path on which the minisuperspace variables are always real. However, we shall see that it is possible to satisfy conditions *i* to *iii* of Section 2 by choosing appropriate complex initial values. Secondly, it will be found in the next section that the black hole radius b is no longer constant during inflation.

In this section, we introduce the massive scalar field for the model corresponding to de Sitter spacetime, where the spacelike slices are 3-spheres containing no black holes. This model was first put forward by Hawking [8]. From the fluctuations in the cosmic microwave background as measured by COBE [9] it follows that m is small compared to the Planck mass [10]:

$$m \approx O\left(10^{-5}\right). \quad (5.1)$$

We will find complex solutions and the complex initial value of the scalar field, and we calculate the real part of the action. This has been done before

by Lyons [11], but his paper contains a logical error to which we will come back later.

The ansatz for the Euclidean metric is again

$$ds^2 = N(\tau)^2 d\tau^2 + a(\tau)^2 d\Omega_3^2. \quad (5.2)$$

Using the rescaled field

$$\phi^2 = 4\pi\Phi^2 \quad (5.3)$$

we obtain the Euclidean action

$$I = -\frac{3\pi}{4} \int N d\tau a \left(\frac{\dot{a}^2}{N^2} + 1 - \frac{1}{3} a^2 \frac{\dot{\phi}^2}{N^2} - \frac{1}{3} a^2 m^2 \phi^2 \right), \quad (5.4)$$

so that the effective cosmological constant is

$$\Lambda_{eff}(\tau) = m^2 \phi(\tau)^2. \quad (5.5)$$

In analogy to equation (3.3) we define

$$H(\tau) = \sqrt{\frac{\Lambda_{eff}(\tau)}{3}} = \frac{m\phi(\tau)}{\sqrt{3}}. \quad (5.6)$$

Variation with respect to a , ϕ and N gives the Euclidean equations of motion and the Hamiltonian constraint:

$$\frac{\ddot{a}}{a} + \frac{2}{3} \dot{\phi}^2 + \frac{1}{3} m^2 \phi^2 = 0 \quad (5.7)$$

$$\ddot{\phi} + 3 \frac{\dot{a}}{a} \dot{\phi} - m^2 \phi = 0 \quad (5.8)$$

$$\frac{\dot{a}^2}{a^2} - \frac{1}{a^2} - \frac{1}{3} \dot{\phi}^2 + \frac{1}{3} m^2 \phi^2 = 0. \quad (5.9)$$

To evaluate $\Psi_0(a_{\partial M}, \phi_{\partial M})$ using a semiclassical approximation we must find solutions in the complex τ -plane that meet conditions *i* to *iii* of Section 2. In particular, the NBP must be satisfied:

$$a = 0, \dot{a} = 1, \phi = \phi_0 \text{ and } \dot{\phi} = 0 \text{ for } \tau = 0. \quad (5.10)$$

Assume that the initial value of the scalar field is large and nearly real:

$$\phi_0^{Re} \gg 1 \gg \phi_0^{Im}. \quad (5.11)$$

An approximate solution near the origin is given by

$$a_{\mathcal{I}}(\tau) = \frac{1}{H_0^{Re}} \sin H_0^{Re} \tau \quad (5.12)$$

$$\phi_{\mathcal{I}}(\tau) = \phi_0 + \sum_{n=1}^{\infty} \frac{1}{n!} \gamma_n \tau^n \quad (5.13)$$

$$\text{for } |\tau| < O(1/H_0^{Re}),$$

where the Taylor series is obtained by solving equation (5.8) iteratively for $\ddot{\phi}$, using the NBP conditions (5.10) and the approximation (5.12) for a . It has the property that

$$\gamma_{2n+1} = 0 \text{ for all } n. \quad (5.14)$$

We call equations (5.12) and (5.13) the *inner approximation*. Writing down the Taylor expansion explicitly to lowest non-trivial order

$$\phi(\tau) = \phi_0 \left[1 + \frac{3}{8\phi_0^2} (H_0\tau)^2 \right] + O(\tau^4) \quad (5.15)$$

shows that ϕ is almost constant near the origin.

As an *outer approximation* we use:

$$\phi_{\mathcal{O}}(\tau) = \psi_0 + \frac{im}{\sqrt{3}}\tau + \chi_0 \exp(3iH_0\tau) \quad (5.16)$$

$$a_{\mathcal{O}}(\tau) = a_0 \exp \left[-\frac{im}{\sqrt{3}} \int_0^\tau \phi(\tau') d\tau' \right] + c_0 \exp \left[\frac{im}{\sqrt{3}} \int_0^\tau \phi(\tau') d\tau' \right] \quad (5.17)$$

$$\text{for } 0 < \tau^{Im} \ll \frac{\sqrt{3}\phi_0^{Re}}{m}.$$

While this solution does not satisfy the NBP, it will be good outside the validity of the inner approximation. Both the χ_0 -term and the c_0 -term can be neglected for $\tau^{Im} \gg 1/H_0^{Re}$, but they are useful for matching $a_{\mathcal{O}}$ and $\phi_{\mathcal{O}}$ to $a_{\mathcal{I}}$ and $\phi_{\mathcal{I}}$ at some $|\tau| \approx O(1/H_0^{Re})$. Comparison with equation (5.15) shows that

$$\chi_0 \approx O\left(\frac{\sqrt{3}}{\phi_0^{Re}}\right), \quad Im(\chi_0) \approx 0. \quad (5.18)$$

In the region of the inner approximation, a will be nearly real on the Lorentzian line $\tau^{Re} = \frac{\pi}{2H_0^{Re}}$. Matching $a_{\mathcal{O}}$ to $a_{\mathcal{I}}$ fixes

$$a_0 \approx \frac{i}{2H_0^{Re}}, \quad c_0 \approx \frac{-i}{2H_0^{Re}} \quad (5.19)$$

and ensures that a will remain nearly real on this line. To make $\phi(\tau)$ roughly real on the same line, by equations (5.16) and (5.18) we have to choose

$$\psi_0^{Im} = -\frac{\pi}{2\phi_0^{Re}} \quad (5.20)$$

in the outer approximation.

ϕ_0^{Im} in turn is fixed by matching ϕ_I to ϕ_O . Since it is very small, this requires evaluation of equation (5.13) to a very high order n . However, we need not calculate any coefficients since, by equation (5.14), ϕ_I^{Im} is constant along the imaginary axis to any order n :

$$\phi_I^{Im}(\tau^{Im})\Big|_{\tau^{Re}=0} = \phi_0^{Im}. \quad (5.21)$$

Therefore it is convenient to choose a matching point τ_M on the imaginary axis:

$$\tau_M^{Re} = 0, \quad \tau_M^{Im} = O\left(1/H_0^{Re}\right). \quad (5.22)$$

By equations (5.16) and (5.18) ϕ_O^{Im} is also constant along this axis:

$$\phi_O^{Im}(\tau^{Im})\Big|_{\tau^{Re}=0} = \psi_0^{Im}, \quad (5.23)$$

so the result of the matching analysis will be independent of the precise choice of τ_M on the axis, as it should be. The matching condition is

$$\phi_I^{Im}(\tau_M) = \phi_O^{Im}(\tau_M) \quad (5.24)$$

and by equations (5.20), (5.21) and (5.23) we obtain

$$\phi_0^{Im} = \psi_0^{Im} = -\frac{\pi}{2\phi_0^{Re}}. \quad (5.25)$$

This result is non-trivial (e. g. $\phi_0^{Re} \neq \psi_0^{Re}$). We now see why the correct value for ϕ_0^{Im} is obtained in ref. [11], although actually only ψ_0^{Im} is calculated there.

We have thus satisfied condition *ii* of Section 2. By the continuity of the outer approximation, condition *iii* can be satisfied by fine-tuning ϕ_0^{Im} . Condition *i* is satisfied by the construction of the inner approximation. The only freedom left is the choice of ϕ_0^{Re} . This variable parametrizes the set of solutions.

To calculate the Euclidean action for the solutions given above, we consider a path going along the real τ -axis from the origin to $\tau^{Re} = \frac{\pi}{2H_0^{Re}}$ and then parallel to the imaginary τ -axis to $\tau_{\partial M}$. Both a and ϕ are nearly real on the Lorentzian segment of this path, so the real part of the action can be approximated by an integral only over the first segment, using the inner approximation [11]:

$$I_{S^3}^{Re} \approx \frac{3\pi}{2} \int_0^{\pi/2H_0^{Re}} d\tau^{Re} a_I \left(\frac{1}{3} a_I^2 m^2 \phi_I^2 - 1 \right) \approx -\frac{3\pi}{2m^2(\phi_0^{Re})^2}. \quad (5.26)$$

The outer approximation is not valid after inflation ends, when $\phi \approx 0$. However, at this point we are already well inside the classical regime. A dust phase will ensue where ϕ oscillates; a and ϕ will both remain real. Approximate solutions for this regime have been given by Hawking and Page [12].

6 An Inflationary Model With Black Holes

We now introduce a massive scalar field on a universe with spacelike sections $S^1 \times S^2$. Thus we will obtain a cosmological model similar to the Nariai universe of Section 4. We find the complex solutions, initial conditions and the action in analogy to the previous section, but point out a few differences.

Again we use the Kantowski-Sachs metric

$$ds^2 = N(\tau)^2 d\tau^2 + a(\tau)^2 dx^2 + b(\tau)^2 d\Omega_2^2 \quad (6.1)$$

and the rescaled field

$$\phi^2 = 4\pi\Phi^2. \quad (6.2)$$

The Euclidean action is

$$I = -\pi \int N d\tau a \left(\frac{\dot{b}^2}{N^2} + 2\frac{b}{a} \frac{\dot{a}\dot{b}}{N^2} + 1 - b^2 \frac{\dot{\phi}^2}{N^2} - b^2 m^2 \phi^2 \right) + \pi \left[-\dot{a}b^2 - 2ab\dot{b} \right]_{\tau=0}, \quad (6.3)$$

and like in the previous section the effective cosmological constant is given by

$$\Lambda_{eff}(\tau) = m^2 \phi(\tau)^2. \quad (6.4)$$

In analogy to equation (4.3) we define

$$H(\tau) = \sqrt{\Lambda_{eff}(\tau)} = m\phi(\tau). \quad (6.5)$$

Variation with respect to a , b , ϕ and N gives the Euclidean equations of motion and the Hamiltonian constraint:

$$\frac{\ddot{b}}{b} - \frac{\dot{a}\dot{b}}{ab} + \dot{\phi}^2 = 0 \quad (6.6)$$

$$\frac{\ddot{b}}{b} + \frac{\dot{a}\dot{b}}{ab} + \frac{\ddot{a}}{a} + \dot{\phi}^2 + m^2\phi^2 = 0 \quad (6.7)$$

$$\ddot{\phi} + \left(\frac{\dot{a}}{a} + 2\frac{\dot{b}}{b}\right)\dot{\phi} - m^2\phi = 0 \quad (6.8)$$

$$2\frac{\dot{a}\dot{b}}{ab} + \frac{\dot{b}^2}{b^2} - \frac{1}{b^2} - \dot{\phi}^2 + m^2\phi^2 = 0. \quad (6.9)$$

The NBP conditions corresponding to an instanton of topology $S^2 \times S^2$ are:

$$a = 0, \dot{a} = 1, b = b_0, \dot{b} = 0, \phi = \phi_0 \text{ and } \dot{\phi} = 0 \text{ for } \tau = 0. \quad (6.10)$$

With the new definition (6.5) of H the *inner approximation* is given by:

$$a_{\mathcal{I}}(\tau) = \frac{1}{H_0^{Re}} \sin H_0^{Re} \tau \quad (6.11)$$

$$\phi_{\mathcal{I}}(\tau) = \phi_0 + \sum_{n=1}^{\infty} \frac{1}{n!} \gamma_n \tau^n \quad (6.12)$$

$$b_{\mathcal{I}}(\tau) = \frac{1}{m\phi_{\mathcal{I}}(\tau)} \quad (6.13)$$

$$\text{for } |\tau| < O(1/H_0^{Re}).$$

The *outer approximation* is:

$$\phi_{\mathcal{O}}(\tau) = \psi_0 + im\tau + \chi_0 \exp(iH_0\tau) \quad (6.14)$$

$$a_{\mathcal{O}}(\tau) = a_0 \exp\left[-im \int_0^\tau \phi(\tau') d\tau'\right] + c_0 \exp\left[im \int_0^\tau \phi(\tau') d\tau'\right] \quad (6.15)$$

$$b_{\mathcal{O}}(\tau) = \frac{1}{m\phi_{\mathcal{O}}(\tau)} \quad (6.16)$$

$$\text{for } 0 < \tau^{Im} \ll \frac{\phi_0^{Re}}{m}.$$

A matching analysis completely analogous to that of the previous section shows that a , b and ϕ will be nearly real on the Lorentzian line $\tau^{Re} = \frac{\pi}{2H_0^{Re}}$, if we choose the following initial values:

$$\phi_0^{Im} = -\frac{\pi}{2\phi_0^{Re}}, \quad b_0 = \frac{1}{m\phi_0}; \quad (6.17)$$

ϕ_0^{Re} is a free parameter.

An interesting feature of the outer approximation is that the black hole radius grows with the horizon scale during inflation. On the Lorentzian line $\tau^{Re} = \frac{\pi}{2H_0^{Re}}$ the field decreases linearly with time until it reaches zero and inflation ends. By equations (6.14) and (6.16) b becomes very large on the timescale

$$\Delta\tau_{growth} = \frac{\phi_0^{Re}}{m}. \quad (6.18)$$

Again the inner approximation is used to calculate the real part of the Euclidean action. As in Section 4 it comes entirely from the $\tau = 0$ term:

$$I_{S^1 \times S^2}^{Re} \approx -\pi \left(b_0^{Re}\right)^2 \approx -\frac{\pi}{m^2 (\phi_0^{Re})^2}. \quad (6.19)$$

7 The Probability for Primordial Black Holes

In the previous two sections we have calculated the action for two inflationary universes. We now compare the corresponding probability measures

$$P_{S^3}(\phi_0^{Re}) = \exp\left(\frac{3\pi}{m^2 (\phi_0^{Re})^2}\right) \text{ and } P_{S^1 \times S^2}(\phi_0^{Re}) = \exp\left(\frac{2\pi}{m^2 (\phi_0^{Re})^2}\right). \quad (7.1)$$

For a given ϕ_0^{Re} , the universe containing black holes is heavily suppressed, unless the field is large enough to make the effective cosmological constant at least equal to the Planck value. That will be the case if

$$\phi_0^{Re} > O(10^5). \quad (7.2)$$

The corresponding black hole mass is of the order of the Planck mass or less:

$$b_0^{Re} < O(1). \quad (7.3)$$

The Nariai solution is unstable to quantum fluctuations [7]. At the beginning of inflation it becomes a non-degenerate Schwarzschild-de Sitter space-time. Once the black hole horizon is inside the cosmological horizon the black hole will start to lose mass due to Hawking radiation. If the black hole horizon is somewhat smaller than the cosmological horizon, the black hole will evaporate and disappear. However, there is a significant probability that the areas of the two horizons will be nearly enough equal for them to increase together. The consequences of this result for the global structure of the universe will be presented in a forthcoming paper.

References

- [1] H. Nariai, *Science Reports of the Tohoku Univ.* **35**, 62 (1951).
- [2] J. B. Hartle and S. W. Hawking, *Phys. Rev. D* **28**, 2960 (1983).
- [3] S. W. Hawking, *The Path Integral Approach to Quantum Gravity*, in *General Relativity: An Einstein Centenary Survey*, eds. S. W. Hawking and W. Israel (Cambridge University Press, 1979).
- [4] A. Arnowitt, S. Deser and C. Misner in *Gravitation*, ed. M. J. Bowick and F. Gürsey (World Scientific, 1986).
- [5] J. J. Halliwell and J. Luoko, *Phys. Rev. D* **42**, 3997 (1990).
- [6] R. LaFlamme, *Ph.D. thesis*, 1988.
- [7] P. Ginsparg and M. J. Perry, *Nucl. Phys.* **B222**, 245 (1983).
- [8] S. W. Hawking, *Nucl. Phys.* **B239**, 257 (1984).
- [9] G. Smoot et al., *Ap. J. Lett.* **396**, L1 (1992).
- [10] J. J. Halliwell and S. W. Hawking, *Phys. Rev. D* **31**, 1777 (1985).
- [11] G. W. Lyons, *Phys. Rev. D* **46**, 1546 (1992).
- [12] S. W. Hawking and D. N. Page, *Nucl. Phys.* **B298**, 789 (1988).