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Spontaneous Symmetry Breaking in the Non-Abelian Anyon Fluid

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Abstract

We study the theory of non-relativistic matter coupled to the non-Abelian $U(2)$ Chern-Simons gauge field in (2+1) dimensions. We adopt the mean field approximation in the current-algebra formulation already applied to the Abelian anyons. We first show that this method is able to describe both "boson-based" and "fermion-based" anyons and yields consistent results over the whole range of fractional statistics. In the non-Abelian theory, we find a superfluid (and superconductive) phase, which is smoothly connected with the Abelian superfluid phase originally discovered by Laughlin. The characteristic massless excitation is the Goldstone particle of the specific mechanism of spontaneous symmetry breaking. An additional massive mode is found by diagonalizing the non-local, non-Abelian Hamiltonian in the radial gauge.

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1 Introduction

The dynamics of anyons - particle with fractional statistics in $(2+1)$ dimensions [1] - has been considerably investigated in the past few years. These collective excitations can arise in planar condensed-matter systems like the fractional quantum Hall effect [2] and the high-temperature superconductivity [3]. An effective field theory for anyons is obtained by coupling non-relativistic matter particles - either bosons or fermions - to the Abelian Chern-Simons gauge field, which provides the statistical interaction [4] [5].

The remarkable property of *superfluidity* is exhibited by anyons in the thermodynamic limit at constant density [6]: the anyon *fluid* possesses a ground state with uniform density and a massless longitudinal excitation. This is a Goldstone mode which gives rise to superconductivity by the usual Higgs mechanism when coupled to the physical electromagnetic field. This theory was originally proposed by Laughlin [6] for explaining the high-temperature superconductivity of cuprates [3]. However, the explicit breaking of P and T symmetries by the fractional statistics [1] has not been confirmed by the experiments so far [7].

Independently of its physical application to high-temperature superconductivity, we believe that the anyon superfluid is very interesting and deserves a deeper analysis. Few non-perturbative, semiclassical, ground states are known in field theory, thus any new one is worth understanding for its own sake. This may find wider applications than the original physical problem, as it has occurred to spontaneous symmetry breaking. Actually, the Anyon fluid is closely related to the usual superfluid, because both exhibit the spontaneous breaking of the $U(1)$ *global* symmetry related to particle number conservation.

In this paper, we show that the anyon superfluidity also arises in the presence of a *non-Abelian* Chern-Simons interactions. We consider the simplest case of particles having a *isospin* $1/2$ quantum number with $U(2)$ gauge symmetry. The application of this theory to some $(2+1)$-dimensional physical problems has been discussed in ref. [8]. Here, we solve it in the mean field approximation, describe the *specific* mechanism of spontaneous symmetry breaking and study the low-energy excitations.

In section two, we review the mean field approximation [6] which allows to describe the non-perturbative ground state of the Abelian anyons. One assumes self-consistently that the matter density is spatially uniform and obtains a uniform magnetic field by the Chern-Simons Gauss law, $\langle\rho\rangle = -\langle B\rangle/\kappa$, where κ is the coupling constant. Thus, the particles uniformly fill up the Landau levels determined by this mean magnetic field. The quadratic fluctuations around the mean field can be described [9] by using the Dashen-Sharp current algebra formalism [10]; their diagonalization by a Bogoliubov transformation produces a relativistic longitudinal excitation at low energy, as in the familiar case of the superfluid

[11]. This is by far the simplest method for describing anyon superfluidity [6].

While reviewing this method [9], we clarify one property of the Chern-Simons interaction in the Hamiltonian formulation. This long-range, topological interaction produces non-trivial boundary effects, whose strength depends on the type of mean field ground state. These boundary effects can be removed by normal-ordering the Hamiltonian, but produce an effective *local* interaction, which is ground-state dependent. This property is crucial for describing both "boson-based" and "fermion-based" anyons within the current algebra approach. Actually, the two descriptions of anyons are more accurate for fractional statistics $\theta/\pi \sim 0$ and $\theta/\pi \sim 1$, respectively, and agree at the mid-point of *semions* ($\theta/\pi = 1/2$): their combination yields a consistent approximation for all values of the statistics. In particular, we obtain the approximate second-order ground-state energy.

In section three, we extend this method to the $U(2)$ non-Abelian Chern-Simons interaction with two independent coupling constants, $\kappa_{U(1)} = \kappa$ and $\kappa_{SU(2)} = \widetilde{\kappa}$. The mean field approximation produces two copies of Landau levels for isospin-up and isospin-down matter, which have opposite contributions to the mean iso-magnetic field. For $\kappa\widetilde{\kappa} > 0$, the ground-state configuration corresponds to equal populations of spin-up and spin-down particles, and to a vanishing iso-magnetic field; if $1/\kappa \to 0$, this ground state is P and T invariant because parity-violating effects cancel between the two populations - only excitations can break P and T explicitly. Models of this kind have been discussed in refs. [12].

Here we describe a different phase of the system, which exists for $\kappa\widetilde{\kappa} < 0$ and $1/|\widetilde{\kappa}| < 4/|\kappa|$, and has a ground state with maximally unbalanced populations. This phase is continuously connected to the Abelian theory by tuning $1/|\widetilde{\kappa}| \to 0$. The ground state breaks spontaneously the $U(2)$ global symmetry to a $U(1)$ subgroup, as in the Standard Model of electroweak interactions [13]. We find it interesting that the low-energy dynamics of a non-Abelian gauge theory can be solved in closed form in a toy model for spontaneous symmetry breaking. Moreover, the dynamics of non-Abelian anyons has not been much investigated so far [4][14].

In section four, we discuss the low-energy collective excitations above the mean-field ground state. The quadratic expansion of the Hamiltonian, written in terms of non-Abelian currents, consists of two independent parts, corresponding to matter-density and isospin-density fluctuations, respectively. The former fluctuations are massless and similar to those of the Abelian anyon fluid. The latter have a non-trivial, yet solvable, non-Abelian dynamics; we solve explicitly the Gauss law constraint by using the radial gauge $\sum_{i=1}^{2} x^i A_i^a = 0$, which maintains manifest rotation invariance and breaks translation invariance [15]. The non-Abelian Hamiltonian resembles a non-local deformation of the Landau-level Hamiltonian, because isospin-flip excitations feel the mean iso-magnetic field and, moreover, self-interact.

We obtain the complete spectrum and show that this is gapful and discrete, with Gaussian fall-off of correlations. Special care is paid to the gauge invariance of the set of physical states, which is actually translational invariant. This massive excitation does not spoil the Laughlin superconductivity mechanism, because both low energy excitations become gapful upon coupling to the physical electro-magnetic field.

In section five, the Abelian and non-Abelian anyon superfluidities are explained in terms of the spontaneous breaking of the global gauge symmetries, and the specific mechanisms are compared with those of the Higgs and Standard Models of four dimensional gauge theories [13]. Finally, in the conclusion, we discuss other possible physical applications of the non-Abelian anyon fluid. In the appendix, we collect some additional informations on the eigen-functions of the non-Abelian Hamiltonian.

2 The mean field approximation in terms of currents

2.1 Hamiltonian and Abelian current algebra

In this section, we review the mean field for the Abelian anyon fluid in the current algebra approach of ref.[9]. The Lagrangian for non-relativistic matter coupled to the Abelian Chern-Simons gauge field is [4],

$$\mathcal{L} = i\Psi^\dagger D_0 \Psi - \frac{1}{2m}(D_i\Psi)^\dagger D_i\Psi + \frac{\kappa}{2}\epsilon^{\alpha\beta\gamma} A_\alpha \partial_\beta A_\gamma \ , \qquad (2.1)$$

where $D_\mu = \partial_\mu + iA_\mu$ is the covariant derivative of the gauge field and Ψ is the non-relativistic matter field*. The equation of motion for the matter field can be used to derive the Hamiltonian,

$$\mathcal{H} = \int d^2x \ \frac{1}{2m}(D_i\Psi)^\dagger (D_i\Psi) \ , \qquad (2.2)$$

while the equations for the gauge field are,

$$\begin{aligned} -F_{12} &= B = \epsilon_{ij}\partial_i A_j = -\frac{1}{\kappa}\ \rho \qquad (Gauss\ \ law), \\ F_{0i} &= \partial_0 A_i + \partial_i A_0 = -\frac{1}{\kappa}\ \epsilon_{ij} J^j \ . \end{aligned} \qquad (2.3)$$

The conserved, gauge-invariant matter current $J^\mu = (\rho, J^i = J_i)$ is given by

$$\rho = \Psi^\dagger\Psi \ , \qquad J_i = \frac{1}{2im}(\Psi^\dagger D_i \Psi - (D_i\Psi)^\dagger \Psi) \ . \qquad (2.4)$$

*We choose units $c = \hbar = 1$ and set the electric charge equal to one. We use the metric $\eta_{\mu\nu} = diag(1,-1,-1)$, while for two-dimensional expressions we use δ_{ij}, i.e. $A_\mu = (A_0, -A^i = -A_i)$ and $x^\mu = (x^0, x^i = x_i)$.

The Chern-Simons field has no local physical degrees of freedom, thus it can be solved in terms of the matter field at equal time by choosing a complete gauge fixing. The Coulomb gauge $\partial_i A_i = 0$ has been often used for non-relativistic theories; here, we prefer the (spatial) radial gauge [15],

$$x^i A_i = 0 \ , \tag{2.5}$$

because it is better suited for the non-Abelian theory discussed in the next section. Actually, any choice of gauge is equivalent for the Abelian theory, because it will be described in terms of gauge-invariant quantities. The solutions of gauge field equations (2.3) in the radial gauge are [15],

$$\begin{aligned} A_0 &= -\frac{1}{\kappa}\,\frac{1}{x\cdot\partial}\ \epsilon_{ij} x^i J_j \ , \\ A_i &= \frac{1}{\kappa}\,\frac{1}{1+x\cdot\partial}\ \epsilon_{ij} x^j \rho \ , \end{aligned} \tag{2.6}$$

and do not actually involve time derivatives. A precise meaning of the operator $(x\cdot\partial)^{-1}$ is not important here, and will be discussed in section four. By using (2.6), we can write the Hamiltonian in terms of matter fields only, and quantize it by requiring the *bosonic* commutation relations

$$\left[\Psi(x,t),\Psi^\dagger(y,t)\right] = \delta(x-y) \ . \tag{2.7}$$

Let us now choose the current J_i and the density ρ as basic variables. Their algebra can be computed by using (2.7) and (2.6), and reads,

$$\begin{aligned} [\rho(x),\rho(y)] &= 0 \ , \\ [\rho(x),J_i(y)] &= \frac{1}{im}\frac{\partial}{\partial x^i}\left(\delta(x-y)\rho(x)\right) \ , \\ [J_i(x),J_j(y)] &= \frac{1}{im}\left[\frac{\partial}{\partial x^j}\left(\delta(x-y)J_i(x)\right) - \frac{\partial}{\partial y^i}\left(\delta(x-y)J_j(y)\right)\right] \end{aligned} \tag{2.8}$$

An important property of this algebra is its *independence* of the Chern-Simons coupling constant, which only appears in the representation of the algebra (the states) and in the *normal-ordered* Hamiltonian [9]. The Hamiltonian (2.2) can also be written in terms of currents as follows [10],

$$\mathcal{H} = \int d^2x\ \frac{1}{8m}\left(\partial_i\rho + 2imJ_i\right)^\dagger\ \frac{1}{\rho}\left(\partial_i\rho + 2imJ_i\right) \ . \tag{2.9}$$

2.2 Mean field approximation

Let us assume that the ground state $|\Omega\rangle$ has a spatially uniform density,

$$\langle\Omega|\rho(x)|\Omega\rangle = \rho_0, \qquad \langle\Omega|J_i(x)|\Omega\rangle = 0 \ , \tag{2.10}$$

which corresponds to a uniform magnetic field,

$$\langle\Omega|B(x)|\Omega\rangle = B_0 = -\frac{1}{\kappa}\rho_0 \ , \tag{2.11}$$

by the Gauss law (2.3). Next, we seek for self-consistency of this hypothesis in the approximate quantum theory. By decoupling matter (ρ) and field (B) fluctuations in (2.11), we quantize the field Ψ in the external average magnetic field B_0:

$$\mathcal{H} \to \mathcal{H}^{(0)} = \int \ d^2x \ \frac{1}{2m}\left(D_i^{(0)}\Psi\right)^{\dagger}\left(D_i^{(0)}\Psi\right) \ , \tag{2.12}$$

where $D_i^{(0)} = \partial_i - iA_i^{(0)}$, and $A_i^{(0)} = \epsilon_{ij}x^j \ \rho_0/2\kappa$. This is the well-known Hamiltonian of the Landau levels in the so-called symmetric gauge [16]. The one-particle energy of the n-th Landau level is,

$$\epsilon_n = \frac{B_0}{m}(n+\frac{1}{2}) \ , \tag{2.13}$$

and the eigen-functions of the lowest level are

$$\psi_{0,\ell}(x) = \frac{1}{\lambda\sqrt{\pi\ell!}} \ \left(\frac{z}{\lambda}\right)^{\ell} \ e^{-|z|^2/2\lambda^2} \ , \qquad \left(z = x^1 + ix^2\right) \ , \tag{2.14}$$

where $\lambda = \sqrt{2/eB_0}$ is the magnetic length, and ℓ is the angular momentum. Note that these eigen-functions satisfy

$$(D_1^{(0)} + iD_2^{(0)})\psi_{0,\ell} = 0 \ . \tag{2.15}$$

The angular momentum orbitals (2.14) have degenerate energy, and their number is $(B_0A/2\pi)$ in a finite domain of area A (independent of n).

The mean field hypothesis is self-consistent for all the ground states of $\mathcal{H}^{(0)}$ with N particles which have uniform density. For bosonic matter, these have been found in ref.[9], and correspond to filling each Landau orbital of the lowest level with the same number n of particles. Actually, using (2.14,2.15), one can compute

$$\lim_{N\to\infty}\langle\Omega|\rho(x)|\Omega\rangle = \frac{nB_0}{2\pi} = \rho_0 \ , \qquad \lim_{N\to\infty}\langle\Omega|J_i(x)|\Omega\rangle = 0 \ . \tag{2.16}$$

These values for ρ_0 agree with the Gauss law (2.3), provided that $\kappa = -n/2\pi$. Therefore, the mean field approximation is self-consistent for these integer values of the coupling constant. The ground-state energies, for a system of area A, are obtained from (2.13),

$$E_0^{(0)} = \epsilon_0 \ N = \frac{\pi}{nm} \ \rho_0^2 \ A \ , \qquad \left(boson - based \ anyons, \quad \kappa = -\frac{n}{2\pi}\right) \ . \tag{2.17}$$

The next order of the mean-field approximation is given by the quadratic fluctuations of the density and the current. The Hamiltonian (2.2) must be expanded quadratically and normal-ordered in the thermodynamic limit $N \to \infty$. The latter limit involves some subtle boundary effects which actually determine the strength of the effective *local* interaction of fluctuations. Actually, a more precise expression of (2.16) for large, but finite, N can be obtained from (2.4,2.14) [17],

$$\langle\Omega|\rho(x)|\Omega\rangle \simeq \rho_0 \; \Theta\left(\frac{N}{\rho_0} - \pi|x|^2\right) \ , \qquad \langle\Omega|J_i(x)|\Omega\rangle = -\frac{1}{2m}\, \epsilon_{ij}\; \partial_j\langle\Omega|\rho(x)|\Omega\rangle \ . \tag{2.18}$$

Namely, the density of a filled, finite, Landau level has the shape of a droplet, with a chiral edge current, due to eq. (2.15). The contribution of this edge current to the ground-state value of the Hamiltonian in the form (2.9) is non-vanishing for $N \to \infty$, and correctly gives the ground-state energy $E_0^{(0)}$ (2.17). This boundary effect can be removed by rewriting the Hamiltonian. Using an algebraic identity of the Bogomol'nyi type [4],

$$i\epsilon^{ij}(D_i\Psi)^\dagger D_j\Psi + m\epsilon^{ij}\partial_i J^j + B\rho \equiv 0 \ , \tag{2.19}$$

the Hamiltonian (2.2) can be rewritten

$$\mathcal{H} = \int d^2x \; \frac{1}{2m}\left[(D_i\Psi)^\dagger(D_i\Psi) + i\alpha \; \epsilon_{ij}(D_i\Psi)^\dagger D_j\Psi - \frac{\alpha}{\kappa}\rho^2\right] \tag{2.20}$$

for any value of α. For $\alpha = 1$, the derivative terms in (2.20) vanish on the ground state, due to eq. (2.15); thus, there are no boundary effects. The ground-state energy (2.17) is given by the local term ρ^2 only. The new expression (2.20) of the Hamiltonian can be easily normal ordered in the thermodynamic limit as follows:

$$:\mathcal{H}: \; \equiv \mathcal{H} - \langle\Omega|\mathcal{H}^{(0)}|\Omega\rangle = \mathcal{H} - \frac{\pi}{nm}\int d^2x\rho_0^2 \ . \tag{2.21}$$

Therefore, the Hamiltonian (2.20) with $\alpha = 1$ is adopted for studying the quadratic fluctuations. Note that the normal-ordering procedure has produced an effective local interaction, which is the mean field approximation of the long-range "statistical repulsion" of anyons. This statistical repulsion generates a positive energy density $E_0 > 0$ as in the case of free fermions.

Finally, we note that an attractive local interaction $(-g\rho^2/2)$ can also be included in the Hamiltonian for the anyon fluid (2.2), (2.20) [4][18]. The previous analysis can be extended for generic values of $g < 1/(m|\kappa|)$. At the "self-dual" point $g = 1/(m|\kappa|)$, the local attraction exactly balances the statistical repulsion, and there is phase transition for the anyon fluid: non-trivial classical solutions with $E_0 = 0$ were found in ref.[4] and conformal invariance was shown to hold to three-loop order in ref.[18]. The nature of the other phase $g > 1/(m|\kappa|)$ is not presently understood.

2.3 Quadratic fluctuations

Following the approach of ref.[9], we study the quadratic fluctuations using the variables (ρ, J_i), satisfying the algebra (2.8). Actually, we are only interested in representing this algebra to leading order in the fluctuations, as well as expanding the Hamiltonian (2.20) to quadratic order. To this effect, we introduce a small parameter ϵ which keeps track of the size of fluctuations,

$$\rho(x) = \rho_0 + \epsilon\hat{\rho}(x) + O(\epsilon^2) \ , \qquad J(x) = \epsilon\hat{J}(x) + O(\epsilon^2) \ . \tag{2.22}$$

By inserting this expansion in the second of the current commutators (2.8), we obtain

$$\epsilon^2 \left[\hat{\rho}(x), \hat{J}(y)\right] = \hbar\frac{\rho_0}{im}\left(\frac{\partial}{\partial x^i}\delta(x-y) + O(\epsilon)\right) \ . \tag{2.23}$$

This shows that ϵ^2 is of order $O(\hbar)$, so that we can neglect the fluctuations in the r.h.s. of the commutators. The third commutator in (2.8) can be similarly estimated:

$$\epsilon^2 \left[\hat{J}_i(x), \hat{J}_j(y)\right] = O(\epsilon^3) \sim 0 \ . \tag{2.24}$$

In general, the use of this ϵ-expansion yields consistent results for multiple commutators of the approximate algebra.

This approximate algebra, given by (2.23), (2.24) and $[\rho(x), \rho(y)] = 0$, can be represented by a bosonic canonical field ϕ, satisfying $\left[\phi(x), \phi^\dagger(y)\right] = \delta(x-y)$, as follows,

$$\hat{\rho}(x) = \sqrt{\rho_0}(\phi + \phi^\dagger) \ , \qquad \hat{J}_i(x) = \frac{\sqrt{\rho_0}}{2im}\partial_i(\phi - \phi^\dagger) \ . \tag{2.25}$$

(Note that the zero mode of ϕ is absent due to the condition $\int d^2x \ \rho(x) = N$.) The Hamiltonian (2.20), with $\alpha = 1$, can be written in terms of currents, similarly to (2.9), and then expanded to quadratic order in the fluctuations. The result is [9]

$$\mathcal{H}^{(2)} = \int d^2x : \frac{1}{2m}\left[\frac{1}{4\rho_0}(\hat{K}_1 + i\hat{K}_2)^\dagger(\hat{K}_1 + i\hat{K}_2) + \frac{2\pi}{n}\hat{\rho}^2(x)\right] :, \tag{2.26}$$

where $K_i \equiv \partial_i\rho + 2imJ_i$. By inserting the Fourier modes

$$\phi(x) = \int \frac{d^2p}{2\pi}e^{ip\cdot x} \ a_p, \qquad \left[a_p, a_q^\dagger\right] = \delta^2(p-q) \tag{2.27}$$

one obtains,

$$\begin{aligned} \mathcal{H}^{(2)} &= \int d^2p : \left[A_p a_p^\dagger a_p + B(a_p a_{-p} + a_p^\dagger a_{-p}^\dagger)\right] : , \\ A_p &= \frac{p^2}{2m} + \frac{2\pi\rho_0}{nm}, \qquad B = \frac{\pi\rho_0}{nm} \end{aligned} \tag{2.28}$$

The terms coming from the repulsive interaction can be normal-ordered by the Bogoliubov transformation

$$\begin{aligned} a_p &= \cosh\chi\ \alpha_p + \sinh\chi\ \alpha^\dagger_{-p}\ , \quad \chi = \chi(p)\ , \\ a^\dagger_{-p} &= \sinh\chi\ \alpha_p + \cosh\chi\ \alpha^\dagger_{-p}\ . \end{aligned} \tag{2.29}$$

This is an $SO(1,1)$ rotation in the $(a_p, a^\dagger_{-p})$ space which preserves the commutation relations. The result is

$$\mathcal{H}^{(2)} = \Delta E_0^{(2)} + \int d^2p\ E_p\ \alpha^\dagger_p \alpha_p\ , \tag{2.30}$$

with

$$E_p = \frac{|p|}{m}\sqrt{\frac{2\pi\rho_0}{n} + \frac{p^2}{4}} \ \stackrel{p\to 0}{\longrightarrow}\ v_s\ |p|\ , \tag{2.31}$$

and

$$\Delta E_0^{(2)} = -\frac{A}{(2\pi)^2}\int d^2p\ \frac{p^2}{4m}\left(1+\frac{\eta}{p^2} - \sqrt{1+\frac{2\eta}{p^2}}\right)\ , \qquad \eta = \frac{4\pi\rho_0}{n} \tag{2.32}$$

The quadratic fluctuations show a massless longitudinal excitation with sound velocity [9],

$$v_s = \frac{1}{m}\sqrt{\frac{2\pi\rho_0}{n}}\ , \qquad \left(boson - based\ anyons\ \ \kappa = -\frac{n}{2\pi}\ ,\ \frac{\theta}{\pi} = \frac{1}{n}\right) \tag{2.33}$$

The use of the Bogoliubov transformation [9] makes manifest the striking similarity between the anyon fluid and the usual superfluid [11]. The latter is the canonical example for spontaneous breaking of a global symmetry , the U(1) symmetry for particle number conservation. Actually, the same spontaneous symmetry breaking occurs in the anyon fluid, because the Bogoliubov rotated ground state, satisfying $\alpha_p|\tilde{\Omega}\rangle = 0$, does not have a well defined particle number. The broken symmetry is the global $U(1)$ subgroup of the gauge group[†]. Therefore, the anyon fluid gives an interesting new realization of the Goldstone mechanism in non-relativistic field theory. Note that the "microscopic" mechanism leading to $\langle\rho\rangle = \rho_0$ is different from the Bose-Einstein condensation, and that there is no Higgs phenomenon associated to the Chern-Simons field. We shall discuss these differences in section five, together with the results of the non-Abelian case. The anyon fluid becomes a superconductor [6] when is coupled to an external electro-magnetic field, because the massless mode gives mass to the photon by the usual Higgs mechanism.

2.4 Fermion-based anyons

It is interesting to extend these results to fermion-based anyons. Suppose now that the matter field Ψ satisfies canonical anti-commutation relations. The mean field is again self-consistent (eq. (2.11)) for uniform fillings of the Landau levels, because each filled level

[†]Besides, it is the unique part of the gauge symmetry which can break [19].

contributes a constant value to the density, away from the boundary [20]. Due to Fermi statistics, we can put two spin-1/2 fermions per Landau orbital, at most, and uniformly fill the lowest $n/2$ Landau levels, where $n = 2p + \sigma$, $\sigma = 0, 1$; if n is odd ($\sigma = 1$), we fill the top level with one electron per orbital. The resulting ground-state density is again given by (2.16), and the allowed values of the Chern-Simons coupling constant are $\kappa = -n/2\pi$, which correspond now to the fractional statistics $\theta/\pi = 1 - 1/n$. The ground-state energy, obtained by (2.13), is:

$$
\begin{aligned}
E_0^{(0)} &= \frac{B_0 A}{2\pi} \sum_{k=0}^{p-1} (2\epsilon_k + \sigma\epsilon_p) = \begin{cases} \frac{A\rho_0^2\pi}{2m} , & n\ even, \\ \frac{A\rho_0^2\pi}{2m}\left(1 + \frac{1}{n^2}\right) , & n\ odd, \end{cases} \\
& \left(fermion - based\ anyons, \quad \kappa = -\frac{n}{2\pi} \right) . \qquad (2.34)
\end{aligned}
$$

This ground-state energy oscillates between even and odd values of n and correctly reproduces the energy of the filled Fermi sea for $\kappa \to \infty$.

The Hamiltonian must be normal-ordered differently from (2.21), because the ground-state energy is higher for fermion-based anyons than for boson-based ones. Again, we can dispose of the boundary terms in the ground-state expectation value of $\mathcal{H}$ by choosing the parameter α in eq. (2.20) which gives vanishing derivative terms. This is found to be $\alpha = n/2 \ + (\sigma/2n)$ by using some equations similar to (2.15). As a consequence, fermion-based anyons have an effective local repulsion different from the boson-based ones. The discussion of quadratic fluctuations is the same as in the previous bosonic case, because the current algebra is independent of the statistics. We obtain the Hamiltonian (2.28) with

$$
A_p^F = \frac{p^2}{2m} + \frac{\pi\rho_0}{m}\left(1 + \frac{\sigma}{n^2}\right) , \qquad B^F = \frac{\pi\rho_0}{2m}\left(1 + \frac{\sigma}{n^2}\right) , \qquad \sigma = n\ mod\ 2 , \qquad (2.35)
$$

leading to a massless mode with sound velocity,

$$
v_s^F = \begin{cases} \frac{1}{m}\sqrt{\rho_0\pi} , & n\ even , \\ \frac{1}{m}\sqrt{\rho_0\pi\left(1 + \frac{1}{n^2}\right)} , & n\ odd , \end{cases} \left(fermion - based\ anyons\ , \ \kappa = -\frac{n}{2\pi}\ , \ \frac{\theta}{\pi} = 1 - \frac{1}{n} \right) . \qquad (2.36)
$$

This value is different (always lower) than the result of ref.[6] for fermion-based anyons, because these authors considered spinless fermions. In the free fermion limit, $v_s \to v_F/\sqrt{2}$, where v_F is the velocity of particle-hole excitations at the Fermi surface: thus, the mean field approximation picks up one particular value of the continuum of massless particle-hole excitations with velocities $0 < v_s < v_F$. Note also that this approach gives the same result for boson-based and fermion-based anyons at the common midpoint of semions, with statistics $\theta/\pi = 1/2$, for both $E_0^{(0)}$ and v_s (eqs. (2.17,2.33) and (2.34,2.36), respectively). This shows that the current-algebra approach can describe both types of anyon constructions, with greater accuracy in the regions $\theta/\pi \simeq 0$ and $\theta/\pi \simeq 1$, respectively, corresponding to $1/\kappa = 2\pi/n \to 0$ in both cases [9].

2.5 Ground-state energy

The best approximation for the ground-state energy $E_0^{(0)} + \Delta E_0^{(2)}$ is obtained by combining the boson-based expression for $0 \le \theta/\pi \le 1/2$ and the fermion-based one for $1/2 \le \theta/\pi \le 1$. As discussed in ref.[9], the second-order contribution $\Delta E_0^{(2)}$ (2.32) is ultraviolet divergent and must be regularized by allowing a *finite size* to anyons, $a \equiv 1/\Lambda$, where Λ is the momentum cut-off. Anyons are collective excitations which naturally have a finite size; however, this length does not appear in the effective Chern-Simons Lagrangian and must be supplemented otherwise. Possibly, it could be self-consistently determined in the exact solution of this theory, which is, however, not known at present. Within the mean-field approximation, anyons have the size given by the magnetic length $O\left(1/\sqrt{B_0}\right)$, which is the minimal localization of particles in the Landau levels. Therefore, we have,

$$a = \frac{1}{\Lambda} = \frac{\delta}{\sqrt{B_0}} = \delta\,\sqrt{\frac{n}{2\pi\rho_0}}\ , \qquad \delta = O(1)\ , \tag{2.37}$$

where δ is a proportionality constant. This defines the cut-off Λ for both the boson-based ($\theta/\pi = 1/n_B$) and fermion-based ($\theta/\pi = 1 - 1/n_F$) anyons,

$$\Lambda_B^2 = \frac{2\pi\rho_0}{\delta_B^2 n_B}\ , \qquad \Lambda_F^2 = \frac{2\pi\rho_0}{\delta_F^2 n_F}\ . \tag{2.38}$$

The boson-based and fermion-based ground-state energies match at the semion point $\theta/\pi = 1/2$ for the natural choice $\delta_F = \delta_B = \delta$; the parameter $\delta = O(1)$ is left free.

We can integrate $\Delta E_0^{(2)}$ (2.32) with the respective cut-offs, use $\eta_B = 4\pi\rho_0/n_B$ (respectively, $\eta_F = 2\pi\rho_0\,(1+\sigma/n_F^2)$), and obtain,

$$\begin{aligned} E_0^{(2)} &= E_0^{(0)} + \Delta E_0^{(2)} \\ &= \begin{cases} \frac{\pi\rho_0^2 A}{m}\left[\frac{\theta}{\pi} - \left(\frac{\theta}{\pi}\right)^2 F\left(\left(2\delta^2\right)^{-1}\right)\right]\ , & 0 \le \frac{\theta}{\pi} = \frac{1}{n_B} \le \frac{1}{2}\ , \\ \frac{\pi\rho_0^2 A}{2m}\left[1 + \frac{\sigma}{n_F^2} - \frac{1}{2}\left(1+\frac{\sigma}{n_F^2}\right)^2 F\left(\left(n_F\delta^2\left(1+\frac{\sigma}{n_F^2}\right)\right)^{-1}\right)\right]\ , & \frac{1}{2} \le \frac{\theta}{\pi} = 1 - \frac{1}{n_F} \le 1\ , \end{cases} \end{aligned} \tag{2.39}$$

where $\sigma = n_F \ mod\ 2$, and

$$2\ F(y) = y^2 + 2y - (y+1)\sqrt{y^2+2y} + \log\left|y + 1 + \sqrt{y^2+2y}\right|\ . \tag{2.40}$$

It is interesting to discuss the qualitative behavior of the ground-state energy as a function of θ/π. The quadratic correction is negative definite, as it should, and vanishes at the end points $\theta/\pi = 0, 1$, where the leading expressions $E_0^{(0)}$ already gives the exact result. Near free bosons, $\theta/\pi \sim 0$, the ground-state energy is quadratic in θ/π: this is a natural second-order result for bosons interacting with strength $O(\theta/\pi)$. Near the fermionic end, $\theta/\pi = 1 - 1/n_F \to 1$, the oscillations $O(1/n_F^2)$ between even and odd n_F values become subleading, and the ground-state energy approaches linearly the Fermi energy from below, $E_0^{(2)} \simeq E_0^F (1 - |1-\theta/\pi|/2\delta^2)$. The shape of the ground-state energy as a function of statistics has been much investigated in the quantum mechanics of a finite number N of anyons [21]. A direct comparison of these results with (2.39) is, however, difficult, because the quantum-mechanical excited states form a continuum in the large N limit. We thus find that the field theoretic approach gives the best result for this quantity.

3 The $U(2)$ non-Abelian anyon fluid

We now consider non-relativistic matter carrying an *isospin* $1/2$ representation of index $r = 1, 2$, which interacts with an $U(1) \times SU(2)$ Chern-Simons gauge field $\mathcal{A}_\mu = (A_\mu, A_\mu^a)$, $a = 1, 2, 3$, with couplings $\kappa_{U(1)} \equiv \kappa$ and $\kappa_{SU(2)} \equiv \tilde{\kappa}$. The Lagrangian reads

$$\mathcal{L} = i\Psi^\dagger(D_0\Psi) - \frac{1}{2}(D_i\Psi)^\dagger(D_i\Psi) + \frac{\tilde{\kappa}}{2}\epsilon^{\alpha\beta\gamma}\left(A_\alpha^a\partial_\beta A_\gamma^a - \frac{1}{3}\epsilon_{abc}A_\alpha^a A_\beta^b A_\gamma^c\right) + \frac{\kappa}{2}\epsilon^{\alpha\beta\gamma}A_\alpha\partial_\beta A_\gamma \ , \quad (3.1)$$

where the summation over the isospin index is implicit and the covariant derivative is

$$D_\mu = \partial_\mu + i\mathcal{A}_\mu = (\partial_\mu + iA_\mu)\mathbb{I} + iA_\mu^a\frac{\sigma^a}{2} \ , \quad (3.2)$$

and σ^a are the Pauli matrices. By proceeding as in the Abelian case, we find the Hamiltonian

$$\mathcal{H} = \int d^2x \ \frac{1}{2m} \ (D_i\Psi)^\dagger(D_i\Psi) \ . \quad (3.3)$$

The equations of motion for the Abelian part of the gauge field are again given by (2.3) and those of the $SU(2)$ part are,

$$\begin{aligned} -F_{12}^a \ &\equiv \ B^a = \epsilon_{ij}\left(\partial_i A_j^a + \frac{1}{2}\epsilon^{abc}A_i^b A_j^c\right) = -\frac{1}{\tilde{\kappa}} \ \rho^a \qquad (Gauss \ \ law), \\ F_{0i}^a \ &= \ \partial_0 A_i^a + \partial_i A_0^a - \epsilon_{abc}A_0^b A_i^c = -\frac{1}{\tilde{\kappa}}\epsilon_{ij}J_j^a \ , \end{aligned} \quad (3.4)$$

where the $SU(2)$ isospin density and current are given by,

$$\begin{aligned} \rho^a \ &= \ \Psi^\dagger \ \frac{\sigma^a}{2} \ \Psi \ , \\ J_i^a \ &= \ \frac{1}{2im}\left(\Psi^\dagger\frac{\sigma^a}{2}D_i\Psi - (D_i\Psi)^\dagger\frac{\sigma^a}{2}\Psi\right) \ . \end{aligned} \quad (3.5)$$

The covariant conservation law is

$$(\mathcal{D}_\mu J^\mu)^a \equiv \partial_\mu J^{\mu,a} - \epsilon^{abc}A_\mu^b J^{\mu,c} = 0 \ . \quad (3.6)$$

3.1 U(2) mean field approximation

Let us look for a self-consistent approximation of the ground state which displays uniform densities of both matter and isospin,

$$\begin{aligned} \langle\rho\rangle = \rho_0, \quad & \langle J_i\rangle = 0 \\ \langle\rho^a\rangle = \rho^a_0, \quad & \langle J^a_i\rangle = 0 \end{aligned} \tag{3.7}$$

Given that local gauge invariance cannot break spontaneously [19], a nonvanishing mean isospin density ρ^a_0 is not rigorously true in the exact theory. However, it is possible in the mean-field approximation, where local gauge invariance is reduced to the global one. Therefore, we shall argue that the mean field theory correctly describes the breaking of the $U(2)$ *global* gauge symmetry down to a $U(1)$ subgroup. Correspondingly, the isospin quantum number will no longer be conserved.

We can rotate the isospin axes so that the mean isospin density become $\langle\rho^a\rangle = \delta^a_3\ \tilde{\rho}_0$. Constant densities imply constant magnetic and iso-magnetic fields, eq. (3.4) and

$$\langle\Omega|B^a|\Omega\rangle \equiv \delta^a_3\ \widetilde{B}_0\ = -\frac{1}{\tilde{\kappa}}\ \delta^a_3\ \tilde{\rho}_0\ , \tag{3.8}$$

respectively, and the corresponding $A^{(0)}_\mu$ fields. The zeroth-order mean-field Hamiltonian thus reads

$$\mathcal{H} \to \mathcal{H}^{(0)} = \int d^2x\ \frac{1}{2m}\ (D^{(0)}_i\Psi)^\dagger(D^{(0)}_i\Psi)\ , \tag{3.9}$$

where

$$D^{(0)}_i = \partial_i - iA^{(0)}_i\ , \quad A^{(0)}_i = \epsilon_{ij}\frac{x^j}{2}\left(\frac{\rho_0}{\kappa}1\!\!I + \frac{\tilde{\rho}_0}{2\tilde{\kappa}}\ \sigma^3\right)\ . \tag{3.10}$$

This Hamiltonian describes two copies of Landau levels, one for each isospin orientation, with different values of the magnetic field $B_+ = (B_0 + \ \widetilde{B}_0/2)$ and $B_- = (B_0 - \ \widetilde{B}_0/2)$, respectively. The number of Landau orbitals in an area A is then $N^\pm_L = |B_\pm|A/2\pi$. Following the same steps of the Abelian case, we test the consistency of the mean field hypothesis by constructing uniform ground states for these Landau level problems. For boson-based anyons, we fill uniformly the pair of first Landau levels with n_+ isospin-up and n_- isospin-down particles per orbital, respectively. The equations relating densities and magnetic fields are

$$\begin{aligned} -\kappa B_0 = \rho_0 \quad &= \quad n_+\frac{N^+_L}{A} + n_-\frac{N^-_L}{A} = n_+\frac{|B_0 + \frac{\widetilde{B}_0}{2}|}{2\pi} + n_-\frac{|B_0 - \frac{\widetilde{B}_0}{2}|}{2\pi} \\ -\tilde{\kappa}\widetilde{B}_0 = \tilde{\rho}_0 \quad &= \quad \frac{1}{2}\left(n_+\frac{|B_0 + \frac{\widetilde{B}_0}{2}|}{2\pi} - n_-\frac{|B_0 - \frac{\widetilde{B}_0}{2}|}{2\pi}\right)\ . \end{aligned} \tag{3.11}$$

These equations relate the unknown quantities $(n_+, n_-, \tilde{\rho}_0)$ to the external parameters $(\rho_0, \kappa, \tilde{\kappa})$. In the Abelian case, the uniform filling was only possible for certain values of (ρ_0, κ). Here, instead, there is a one-parameter freedom, which we fix by minimizing the ground-state energy‡. This can be expressed in terms of the known data as follows:

$$\begin{aligned} \frac{E_0^{(0)}}{A} &= \frac{(B_0 + \frac{\tilde{B}_0}{2})^2}{4\pi m} n_+ + \frac{(B_0 - \frac{\tilde{B}_0}{2})^2}{4\pi m} n_- \\ &= \frac{1}{2m}\left[\left|\frac{\rho_0}{\kappa} + \frac{\tilde{\rho}_0}{2\tilde{\kappa}}\right|\left(\frac{\rho_0}{2} + \tilde{\rho}_0\right) + \left|\frac{\rho_0}{\kappa} - \frac{\tilde{\rho}_0}{2\tilde{\kappa}}\right|\left(\frac{\rho_0}{2} - \tilde{\rho}_0\right)\right] \ . \end{aligned} \tag{3.12}$$

We study this expression as a function of the unknown $\tilde{\rho}_0$, with range $|\tilde{\rho}_0| \leq \rho_0/2$, and locate its minima. Choosing for convenience $\kappa < 0$ and varying $\tilde{\kappa}$, we find three phases for the non-Abelian theory:

i) $\tilde{\kappa} < 0$. The minimum of energy is found for $\tilde{\rho}_0 = 0$;

ii) $\tilde{\kappa} > 0$ and $1/|\tilde{\kappa}| > 4/|\kappa|$. The minimum is found for $\tilde{\rho}_0 = \pm\rho_0\ 2\tilde{\kappa}/\kappa$;

iii) $\tilde{\kappa} > 0$ and $1/|\tilde{\kappa}| < 4/|\kappa|$. The minimum is found for $\tilde{\rho}_0 = \pm\rho_0/2$, with energy

$$E_0^{(0)}\Big|_{\tilde{\rho}_0 = \pm\rho_0/2} = \frac{\rho_0^2 A}{2m}\left(\frac{1}{|\kappa|} - \frac{1}{4|\tilde{\kappa}|}\right) \ . \tag{3.13}$$

We see that the zero-th order mean field approximation manufactures a classical potential with non-trivial minima, as in the standard cases of spontaneously broken symmetry (see fig. 1).

The general feature of the non-Abelian problem is the presence of anyonic particles with both isospin charges and opposite contributions to the average iso-magnetic field $\tilde{B}_0$. If the two couplings have the same sign, the minimal energy ground-state configuration has equal populations of isospin up and down particles $(n_+ = n_-)$. This is the phase (i). In the phase (ii), the difference of populations depends on the ratio $\tilde{\kappa}/\kappa$. In these two cases, the non-Abelian mean field approximation is not self-consistent: in the first one, $\tilde{\rho}_0 = 0$ does not reproduce (3.7); in the second case, the non-vanishing value of $\tilde{\rho}_0$ is achieved in the singular limit $n_+ \to \infty$ and vanishing magnetic field in the corresponding Landau level $(B + B_3) \to 0$. This case is not further analyzed here. These two phases can be continuously connected to the theory with $SU(2)$ gauge interaction only, by letting $1/|k| \ll 1/|\tilde{\kappa}|$. In particular, we find that the non-Abelian mean-field approximation (3.7) is not consistent for Chern-Simons theories with symmetry $SU(2)$ only, or for other semi-simple Lie algebras. This $SU(2)$ invariant anyon fluid ground state is actually P and T invariant, due to the vanishing of the iso-magnetic field (although the fluctuations are not invariant). Similar models with a pair of oppositely charged anyons have been introduced [12], for explaining

‡Note that gauge invariance requires $4\pi\tilde{\kappa}$ to be an integer [4].

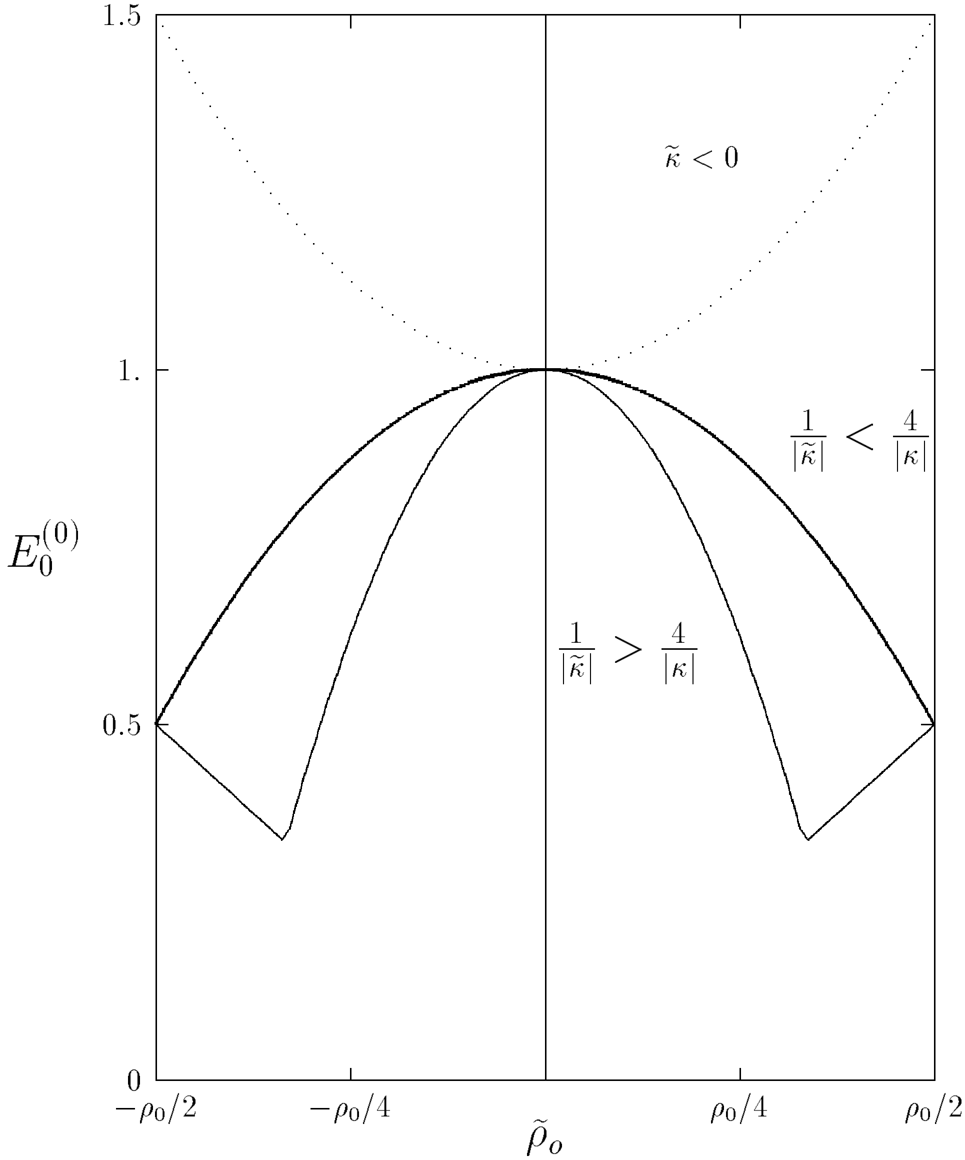

Figure 1: Mean field ground-state energy (3.12) of the non-Abelian anyon fluid, plotted as a function of $\widetilde{\rho}_0$, for one value of $\widetilde{\kappa}$ in each phase.

the lack of P and T violations in the high-temperature superconductivity. These models cannot be analyzed within this mean field approximation.

Here we shall discuss the phase (iii), where the $U(1)$ interaction is dominating the $SU(2)$ one. Actually, this phase is continuously connected with the previous Abelian model by letting $1/|\tilde{\kappa}| \ll 1/|\kappa|$. In this phase, the lowest Landau level with field $|B_0 - \tilde{B}_0/2|$ is empty, $n_- = 0$, and has higher energy than the other Landau level with field $|B_0 + \tilde{B}_0/2|$, which is populated. A gap makes stable this mean-field ground state against (n_+, n_-) fluctuations. Note that the non-Abelian interaction lowers the ground-state energy of the pure Abelian theory (2.17), due to the cancellation mechanism discussed above.

3.2 U(2) current algebra

The analysis of low-energy, quadratic fluctuations around the non-Abelian mean-field ground state is similar to the Abelian case in section (2.3): we must derive the non-Abelian current algebra and rewrite the Hamiltonian (3.3) in terms of currents. The non-Abelian Chern-Simon field can be solved completely in terms of the matter density at equal time by using the radial gauge [15],

$$x^i \mathcal{A}_i(x) = 0 \ . \tag{3.14}$$

These gauge conditions eliminate the commutator of non-Abelian gauge fields appearing in the Gauss law (3.4), which can be solved as in the Abelian case (2.6),

$$\begin{aligned} A_0^a &= -\frac{1}{\tilde{\kappa}}\,\frac{1}{x\cdot\partial}\ \epsilon_{ij} x^i J_j^a \ , \\ A_i^a &= \frac{1}{\tilde{\kappa}}\,\frac{1}{1+x\cdot\partial}\ \epsilon_{ij} x^j \rho^a \ , \end{aligned} \tag{3.15}$$

where the operators $(x\cdot\partial)^{-1}$ will be better defined afterwards. Actually, in (2+1)-dimensions, the solution of the Gauss law is possible in any axial gauge $n^\mu A_\mu = 0$, at the expenses of breaking either rotation or translation invariance. In our problem, it is preferable to maintain explicit rotation invariance, because the Chern-Simons interaction is chiral.

The non-Abelian current algebra can be obtained again by quantizing the bosonic matter field $\Psi_i(x)$ only. The commutation relations between the gauge-invariant currents (ρ, J_i) are still independent of the Chern-Simons coupling and are given by the eqs. (2.8). The commutation relations of ρ^a are ,

$$\begin{aligned} [\hat{\rho}^a(x), \hat{\rho}(y)] &= 0 \ , \\ \left[\hat{\rho}^a(x), \hat{J}_i(y)\right] &= \frac{1}{im}\,\frac{\partial}{\partial x^i}\left(\delta(x-y)\rho^a(x)\right) + \frac{1}{im\tilde{\kappa}}\ \epsilon_{ij}\ y^j \epsilon_{abc}\rho^b(y)\ \left(\frac{1}{2+x\cdot\partial}\rho^c(x)\right) \end{aligned}$$

$$-\frac{1}{im\tilde{\kappa}}\epsilon_{ij}\ x^j\epsilon_{abc}\rho^b(x)\delta(x-y)\ \left(\frac{1}{2+y\cdot\partial}\rho^c(y)\right)\ ,$$

$$\left[\hat{\rho}^a(x),\hat{\rho}^b(y)\right] \ = \ i\ \epsilon_{abc}\ \rho^c(x)\ \delta(x-y)\ . \tag{3.16}$$

We use some algebraic identities relating the matter and isospin (1/2) currents, which follows from the completeness of the basis of (2×2) isospin matrices. These are obtained by "gauging" the identities found in ref.[10]:

$$4\rho^a\rho^a=\rho^2\ ,\qquad \rho J_i^a=\rho^a J_i-\frac{1}{m}\epsilon_{abc}\rho^b(\mathcal{D}_i\rho)^c \tag{3.17}$$

Therefore, we can consider the currents (ρ^a, J_i) as independent variables and (ρ, J_i^a) as dependent ones. In particular, we do not need the explicit form of the commutators involving J_i^a.

Next, we study the normal ordering of the Hamiltonian (3.3). The analysis is similar to the Abelian case (eqs. (2.20)-(2.21)) because the non-Abelian ground state has the same filling of the lowest Landau level. Therefore, we must add to (3.3) a term proportional to the $U(2)$ Bogomol'nyi identity, analogous to (2.19), with coefficient $\alpha=1$. We then find:

$$\mathcal{H}=\int\ d^2x\ \frac{1}{2m}\ \left[(D_i\Psi)^\dagger(D_i\Psi)+i\epsilon_{ij}(D_i\Psi)^\dagger D_j\Psi-\frac{1}{\kappa}\rho^2-\frac{1}{\tilde{\kappa}}(\rho^a)^2\right]\ . \tag{3.18}$$

This can be written in terms of the currents (ρ^a, J_i), by using the identities (3.17) and another one for $(D_i\Psi)^\dagger(D_j\Psi)$ [10]:

$$\mathcal{H}=\int d^2x\ \frac{1}{2m}\ \left[\frac{1}{\rho}\left((\mathcal{D}_i\rho)^a(\mathcal{D}_i\rho)^a+m^2J_iJ_i\right)\ -\ \epsilon^{ij}\frac{1}{\rho}\left(m(\partial_i\rho)J_j+2\ \epsilon^{abc}\ \frac{\rho^a}{\rho}\ (\mathcal{D}_i\rho)^b(\mathcal{D}_i\rho)^c\right)\right.$$
$$\left.-\ \frac{1}{\kappa}\rho^2-\frac{1}{\tilde{\kappa}}(\rho^a)^2\right]. \tag{3.19}$$

3.3 Quadratic fluctuations

We expand the current algebra (3.16) and the Hamiltonian (3.19) to leading order by plugging in the expansions

$$\rho^a\simeq\frac{\rho_0}{2}\delta_3^a+\hat{\rho}^a(x)\ ,\qquad J(x)\simeq\hat{J}(x)\ , \tag{3.20}$$

$$A_i^a\simeq\frac{1}{\tilde{\kappa}}\ \epsilon_{ij}x^j\left(\delta_3^a\ \frac{\rho_0}{4}+\frac{1}{2+x\cdot\partial}\ \hat{\rho}^a\right)\ , \tag{3.21}$$

where the operators with hat are much smaller that their mean field values, (see eqs. (2.22)-(2.24)). The expansion of the Hamiltonian involves the covariant derivative of the isospin density ,

$$(\mathcal{D}_i\hat{\rho})^a\simeq\partial_i\hat{\rho}^a+\frac{\rho_0}{2\tilde{\kappa}}\ \epsilon_{ij}x^j\ \epsilon^{ab3}\Lambda\ \hat{\rho}^b\ ,\qquad \Lambda=\frac{1}{2+x\cdot\partial}-\frac{1}{2}\ . \tag{3.22}$$

The quadratic Hamiltonian splits into two terms (we omit the hats from now on),

$$\mathcal{H}^{(2)} = \mathcal{H}^{(2)}_A \left[\rho^3, J_i\right] \ + \ \mathcal{H}^{(2)}_{NA} \left[\rho^1, \rho^2\right] . \tag{3.23}$$

The first term is

$$\mathcal{H}^{(2)}_A = \int \ d^2x \ \frac{1}{2m\rho_0} \left[\left(\partial_i \rho^3\right)^2 + m^2 (J_i)^2 - 2m\epsilon_{ij}\partial_i\rho^3 J_j - \rho_0 \left(\frac{4}{\kappa} + \frac{1}{\widetilde{\kappa}}\right) \left(\rho^3\right)^2 \right] , \tag{3.24}$$

and is similar to the Abelian Hamiltonian (2.26). It describes local density fluctuations of the isospin-up particles in the lowest Landau level, without isospin flips. The second term in the Hamiltonian is

$$\begin{aligned}\mathcal{H}^{(2)}_{NA} = \int d^2x \ \frac{1}{2m\rho_0} \Big[\left(\partial_i\rho^\alpha\right)^2 \ &- \ \rho_0 \left(\frac{4}{\kappa} + \frac{1}{2\widetilde{\kappa}}\right) \left(\rho^\alpha\right)^2 - \frac{\rho_0}{\widetilde{\kappa}} \left(\epsilon_{ij} x^i \partial_j \rho^\alpha\right) \epsilon_{\alpha\beta} \ \Lambda \ \rho^\beta \\ &+ \ \left(\frac{\rho_0}{2\widetilde{\kappa}}\right)^2 x^2 (\Lambda\rho^\alpha)(\Lambda\rho^\alpha) \Big] ,\end{aligned} \tag{3.25}$$

where the indices $\alpha, \beta = 1, 2$ are a subset of the adjoint isospin indices. This term describes the non-Abelian dynamics of isospin-flip excitations.

The approximated $U(2)$ current algebra (3.16) of the independent variables (ρ^a, J_i) also decouples into two orthogonal subalgebras of (ρ^3, J_i) and (ρ^1, ρ^2), respectively:

$$\begin{aligned}\left[\rho^3(x), \rho^3(y)\right] \ &= \ \left[J_i(x), J_j(y)\right] = 0 \ , \\ \left[\rho^3(x), J_i(y)\right] \ &= \ \frac{\rho_0}{2im} \ \partial_i \delta(x-y) \ ,\end{aligned} \tag{3.26}$$

and

$$\begin{aligned}\left[\rho^\alpha(x), \rho^\alpha(y)\right] \ &= \ 0 \qquad \alpha = 1, 2 \ , \\ \left[\rho^1(x), \rho^2(y)\right] \ &= \ i\frac{\rho_0}{2}\delta(x-y) \ .\end{aligned} \tag{3.27}$$

Therefore, the two terms of the quadratic Hamiltonian (3.23) are decoupled at the quantum level. The (ρ^3, J_i) subalgebra is isomorphic to the $U(1)$ current algebra (2.23,2.24), thus the analysis of section 2.3 can be completely repeated for the Hamiltonian (3.24). After the Bogoliubov transformation, one obtains a massless mode with sound velocity depending on both Chern-Simons coupling constants,

$$v_s = \frac{1}{m}\sqrt{\rho_0\left(\frac{1}{|\kappa|} - \frac{1}{4\widetilde{\kappa}}\right)} \ , \qquad \left(0 < \frac{1}{\widetilde{\kappa}} < \frac{4}{|\kappa|} \ , \quad \kappa < 0\right) \ . \tag{3.28}$$

Therefore, the density fluctuations which do not change isospin behaves as the Abelian ones, with sound velocity related to the ground-state energy (3.13). Their physical interpretation will be discussed in section five.

In the next section, we shall find the spectrum of isospin-flip fluctuations. The (ρ^1, ρ^2) approximate current algebra can be represented by a non-relativistic bosonic field χ as follows,

$$\rho^1 = \frac{\sqrt{\rho_0}}{2}\left(\chi + \chi^\dagger\right) , \qquad \rho^2 = \frac{\sqrt{\rho_0}}{2} i\left(\chi^\dagger - \chi\right) \ , \qquad \left[\chi(x), \chi^\dagger(y)\right] = \delta(x-y) \ . \tag{3.29}$$

Their Hamiltonian can be expressed in terms of the integro-differential operator Δ acting on χ,

$$\begin{aligned} \mathcal{H}^{(2)}_{NA} &= \int d^2x\ \chi^\dagger\ \frac{1}{2m}\left[-\partial^2 + c + \frac{\rho_0}{2\widetilde{\kappa}}(J\Lambda + \Lambda^\dagger J) + \left(\frac{\rho_0}{2\widetilde{\kappa}}\right)^2 \Lambda^\dagger x^2 \Lambda\right] \chi \\ &= \int d^2x\ \chi^\dagger\ \Delta\ \chi \end{aligned} \tag{3.30}$$

where $J = -i\epsilon_{ij}x_j\partial_i$ is the angular momentum and $c = -\rho_0\left(4/\kappa + 1/2\widetilde{\kappa}\right) > 0$, is a positive constant. Note that this Hamiltonian is normal-ordered, thus there is no need of the Bogoliubov transformation in this case; the spectrum of these excitations is given by the eigenvalues of the Hermitean operator Δ.

4 The spectrum of non-Abelian fluctuations

4.1 General properties and gauge invariance

The radial gauge condition (3.14) breaks translation invariance, because it selects a preferred origin of the coordinates of the plane. This also happens in the Landau levels, due to the choice of the background potential $A_i = \frac{B}{2}\epsilon_{ij}x^j$. Clearly, both systems have translation invariance, which can be achieved by combining translations with compensating gauge transformations, the so-called *magnetic* translations. Non-observable non-gauge invariant quantities, like the eigen-functions, transform covariantly under magnetic translations, while physical quantities, like the energy levels, are invariant.

The magnetic translation operators for the non-Abelian Hamiltonian (3.30) are obtained by exploiting its close analogy with the Landau level Hamiltonian [16]. We introduce the eigen-functions $\psi_{E,\ell}$ for one-particle states with energy E and angular momentum ℓ,

$$\chi(x) = \sum_{E,\ell} \psi_{E,\ell}(x)\ c_{E,\ell}\ , \tag{4.1}$$

and we find that they satisfy the Schrödinger-like equation,

$$\begin{aligned} \Delta\ \psi_{E,l} &= E\psi_{E,l} = \left(-\frac{1}{2m}\sum_{i=1}^{2} D_i^\dagger D_i + \frac{c}{2}\right)\psi_{E,l} \\ D_i &= \partial_i - i\hat{A}_i = \partial_i - i\epsilon_{ij}x^j\frac{\rho_0}{2\widetilde{\kappa}}\left(\frac{1}{2 + x\cdot\partial} - \frac{1}{2}\right)\ , \end{aligned} \tag{4.2}$$

which is similar to the Landau-level one, were it not for the additional non-local operators in the covariant derivatives. In the Landau level problem, the gauge transformation which restores the gauge condition after translation is ($B=2$),

$$x_i^{'}=x_i+b_i,\qquad A_i^{'}-A_i=\partial_i\theta,\quad \theta=\epsilon_{ij}x^ib^j,\quad A_i^{'}=\epsilon_{ij}(x^j+b^j) \tag{4.3}$$

The eigen-functions are translated and rotated

$$\begin{aligned}\psi_L^{'}(x^{'}) &= e^{-i\theta}\psi_L(x+b)\simeq(1+b^i\pi_i^L+O(b^2))\psi_L(x)\\ \pi_i^L &= \frac{\partial}{\partial x_i}+i\epsilon_{ij}x^j\end{aligned} \tag{4.4}$$

while the Hamiltonian $H^L=-(\partial/\partial x^i-iA_i)^2/2m$ is covariant. This implies

$$[H^L,\pi_i^L]=0\ . \tag{4.5}$$

In the non-Abelian problem (4.2), the compensating gauge transformation from $x^iA_i^a=0$ to $(x^i+b^i)A_i^{\prime a}=0$, with A_i^a given by (3.21), is similarly found to be

$$\begin{aligned}A_i^{\prime a} &= A_i^a+\partial_i\theta^a-\epsilon_{abc}\theta^bA_i^c\\ \theta^a &= \frac{1}{\widetilde{\kappa}}\epsilon_{ij}x^ib^j\left(\delta_3^a\frac{\rho_0}{4}+\frac{1}{(1+x\cdot\partial)(2+x\cdot\partial)}\rho^a\right)\end{aligned} \tag{4.6}$$

to leading order in the fluctuating densities ρ^a. Their approximate gauge transformations are found to be

$$\begin{aligned}\rho^{\prime 3} &= \rho^3\\ \chi^{'} &= e^{-i\hat{\theta}}\chi\simeq\left[1-i\epsilon_{ij}x^ib^j\frac{\rho_0}{2\widetilde{\kappa}}\left(\frac{1}{2}-\frac{1}{(1+x\cdot\partial)(2+x\cdot\partial)}\right)\right]\chi\ ,\end{aligned} \tag{4.7}$$

thus χ acquires an operator-valued phase. One can show that the derivatives D_i in (4.2) are "covariant" in the operator sense: $D_i^{'}e^{-i\hat{\theta}}=e^{-i\theta}D_i$, where θ is given by (4.3). Thus, in analogy with (4.4,4.5), the operator Δ commutes with the following magnetic translation operator,

$$\pi_i=\partial_i+i\epsilon_{ij}x^j\ \frac{\rho_0}{2\widetilde{\kappa}}\left(\frac{1}{2}-\frac{1}{(1+x\cdot\partial)(2+x\cdot\partial)}\right)\ ,\qquad [\Delta,\pi_i]=0\ . \tag{4.8}$$

It is convenient to introduce the holomorphic components $\pi,\pi^\dagger$,

$$\pi=\frac{1}{2}(\pi_1-i\pi_2)=\frac{\partial}{\partial z}+\frac{\overline{z}}{2}-z\frac{1}{(1+x\cdot\partial)(2+x\cdot\partial)} \tag{4.9}$$

(hereafter, the length scale given by the non-Abelian mean field $\rho_0/2\widetilde{\kappa}$ is put equal to 2 and the mass $m=1$). The operators π and $\pi^\dagger$ satisfy the same algebra as their simpler

Landau-level analogues, because the non-local interaction mediated by the Chern-Simons field is translation invariant,

$$[\pi, \pi^\dagger] = 1, \qquad [J, \pi] = -\pi, \tag{4.10}$$

These relations imply that the spectrum of Δ is infinitely degenerate in angular momentum, because $\pi\psi_{E,\ell} \propto \psi_{E,\ell-1}$ or $\pi\psi_{E,\ell} = 0$. Another analogous identity is

$$2\pi^\dagger\pi = \left(\Delta - \frac{c}{2} - 1\right) + 2J \tag{4.11}$$

which shows that π is invertible apart from a special line in the (E, J) plane of the spectrum, which we shall discuss later on. Away from this line, $\pi^\dagger$ can be used to generate the eigenfunctions of arbitrary positive ℓ starting from any given value, say from $\ell = 0$.

As in the Landau-level problem, the operator Δ can be put into a manifest positive-definite form, by introducing the holomorphic components of the covariant derivatives D_i (4.2), as follows,

$$\begin{aligned} \Delta &= 2a^\dagger a + \frac{c}{2} - 1 \ , \\ a &= \frac{\partial}{\partial \bar{z}} + z\left(\frac{1}{2} - \frac{1}{2 + x\cdot\partial}\right), \quad a^\dagger = -\frac{\partial}{\partial z} + \left(\frac{1}{2} + \frac{1}{x\cdot\partial}\right)\bar{z} \end{aligned} \tag{4.12}$$

Note, however, that the eigenvalue problem is much harder, because the operators $a, a^\dagger$ do not satisfy the simple harmonic oscillator algebra. The form (4.12) for Δ allows to put a positive lower bound on the energy,

$$E \geq \frac{c}{2} - \frac{\rho_0}{4\widetilde{\kappa}} \equiv \rho_0\left(\frac{2}{|\kappa|} - \frac{1}{2\widetilde{\kappa}}\right) > 0 \tag{4.13}$$

which implies that the spectrum for the non-Abelian fluctuations is positive definite and has a mass gap. In the following discussion, we shall show that this bound is saturated and explain its physical origin.

4.2 Physical interpretation of the spectrum

In the previous section, we have shown that:

i) the operator Δ looks like a non-local deformation of the Landau level problem of electrons in the average effective magnetic field $B_{eff} = \rho_0/2\widetilde{\kappa}$;

ii) the spectrum has a gap $M = \rho_0(2/|\kappa| - 1/2\widetilde{\kappa})$, proportional to both couplings.

Let us try to explain these results in simple terms before entering in the more technical analysis of the eigenvalue problem.

The zero-th order mean field approximation has provided us with the physical picture of two effective Landau level structures for up and down isospin particles, with effective magnetic fields $\rho_0(1/|\kappa| - 1/4\tilde{\kappa})$ and $\rho_0(1/|\kappa| + 1/4\tilde{\kappa})$, respectively. The ground state, corresponding to the homogeneous filling of the lowest isospin-up level, has the energy $\mathcal{E}_0 = \rho_0(1/|\kappa| - 1/4\tilde{\kappa})/2$ per particle (3.13), which is given by the expectation value of the two-body repulsive term,

$$\int d^2x \, \frac{1}{2} \left(B\rho + B^a \rho^a \right) = \mathcal{E}_0 \rho_0 A + 4\mathcal{E}_0 \int d^2x \, \left(\hat{\rho}^a \right)^2 \ , \tag{4.14}$$

in the Hamiltonian (3.19). This repulsive interaction affects both the density fluctuations which are isospin diagonal ($\rho^3 \propto a^\dagger_\uparrow a_\uparrow$) and isospin rotating ($\rho^1 + i\rho^2 \propto \psi \propto a^\dagger_\downarrow a_\uparrow$). The former are Bogoliubov transformed, and therefore are gapless, with sound velocity proportional to $\sqrt{\mathcal{E}_0}$. The latter excitations are not transformed and then acquire the gap $M = 4\mathcal{E}_0$ from (4.14).

In more physical terms, the isospin diagonal phonons are local fluctuations in the filling of the lowest Landau level, which do not feel a net magnetic field. On the other hand, the isospin rotating fluctuations are made of individual isospin flips, which move one electron from the filled up-level to an empty down-level. This can be thought of as leading to two effects: the *hole* in the filled up-level propagates as a phonon in a magnetic field, the *magneto-phonon*, which is gapful [22]. The jump of the electron to any empty down-level gives a discrete spectrum with steps proportional to the non-Abelian magnetic field $\rho_0/2\tilde{\kappa}$. Therefore, we expect that the operator Δ has a discrete, Landau-like spectrum above the magneto-phonon gap M.

4.3 Eigenvectors and eigenvalues

After the separation of variables,

$$\psi_{E,\ell}(r,\theta) = e^{i\ell\theta} \psi_{E,\ell}(r) \tag{4.15}$$

the eigenvalue problem (4.2) can be rewritten, for the radial part,

$$\begin{aligned} 2(E-\Delta)\psi_{E,\ell}(r) \;\; = \;\; & \left[\frac{1}{r^2}((r\partial_r)^2 - \ell^2) + 2\ell + 2\lambda - r^2 + \right. \\ & \left. + \, 4\frac{1}{r\partial_r}\left(r^2 + \ell\right)\frac{1}{2 + r\partial_r} \right] \psi_{E,l}(r) \ , \end{aligned} \tag{4.16}$$

where we parametrized $E = \lambda + c/2 = \lambda + 1 + M$. The operator in the first line of this equation is the Landau Hamiltonian, which would yield the spectrum $\lambda = 2k+1, k \geq 0$; the second line is the additional non-local term. It is convenient to transform (4.16) into a fourth-order differential equation for the reduced wave-function $\psi = (2 + r\partial_r)\varphi$. By multiplying also on the left by $(r^3\partial_r)$, we obtain

$$\begin{aligned} &\left[(r\partial_r)^4 + (r\partial_r)(-r^4 + 2(\lambda+\ell)r^2 - \ell^2 - 4)(r\partial_r) + 4\ell(\ell + r^2)\right]\varphi = 0 \ , \\ &\psi = (2 + r\partial_r)\varphi \ . \end{aligned} \tag{4.17}$$

Although the operator $(2 + r\partial_r)$ has a non-trivial kernel, we shall find that it is invertible in the subspace of physically (normalizable) wave-functions, for which the two eigenvalue problems (4.16) and (4.17) are actually equivalent.

The analysis of the characteristic equations for the solutions of (4.17) around $r = 0$ and $r = \infty$, leads to the following asymptotic behaviors,

$$\begin{array}{cccc|cccc} & r \to 0 & & & & r \to \infty & & \\ \hline \varphi & \simeq r^{|\ell|} \ , & \psi & \simeq r^{|\ell|} & \varphi & \simeq e^{-r^2/2} \ , & \psi & \simeq e^{-r^2/2} \\ & \simeq r^{-|\ell|} \ , & & \simeq r^{-|\ell|} & & \simeq e^{r^2/2} \ , & & \simeq e^{r^2/2} \\ & \simeq r^{2} \ , & & \simeq r^{2} & & \simeq r^{0} \ , & & \simeq r^{0} \\ & \simeq r^{-2} \ , & & \simeq r^{0} & & \simeq r^{-4} \ , & & \simeq r^{-4} \end{array} \tag{4.18}$$

The first two behaviors for both $r \to 0$ and $r \to \infty$ in this table are found in the Landau problem, while the last two ones are new. Note that these four behaviors can also be obtained from the integro-differential form (4.16), by introducing two constants for the homogeneous solutions of the integral operators.

Since the $r \to \infty$ asymptotics of free waves are not found in (4.18), we conclude that the physical solutions are square-integrable and that the spectrum is discrete. The integrable behaviors are $\psi \simeq (r^{|\ell|}, r^0, r^2)$ for $r \to 0$ and $\psi \simeq (e^{-r^2/2}, r^{-4})$ for $r \to \infty$.

Next, we analyze the action of the magnetic translation operator π (4.9). The multiple action of π generates all the integral operators

$$\frac{1}{m + r\partial_r}\ \psi(r) = \frac{1}{r^m}\ \frac{1}{r\partial_r}\ r^m\ \psi(r), \qquad m \in Z \ , \tag{4.19}$$

whose action should be well-defined and unique. We first need the precise definition of these integral operators [15]:

$$F = \frac{1}{r\partial_r} f(r) = \int_a^1 \frac{d\lambda}{\lambda}\ f(\lambda r) \ . \tag{4.20}$$

In this equation, the constant a parametrizes the homogeneous solution of $r\partial_r F = F$, which corresponds to the residual gauge freedom within the radial gauge (3.14); we take a global complete gauge fixing by setting $a = \infty$ in (4.20), where ψ and φ vanish. This choice leads to a consistent solution of the eigenvalue problem[§]. This gauge choice enforces the physical condition that matter fluctuations vanishing at infinity should not produce a gauge field at infinity. In this gauge, the integral operators (4.19) are well defined on wave-functions with asymptotics $\psi \simeq e^{-r^2/2}$ $(r \to \infty)$, but can be singular on $\psi \simeq r^{-4}$; therefore, we neglect the latter type of solutions. We have shown that gauge invariance (the action of π) imposes further conditions on the physical solutions, which also ensure the invertibility of the relation between ϕ and ψ. The explicit action of the integral operators (4.20) on the basis of polynomials times the exponential asymptotics is,

$$\frac{1}{r\partial_r}\ r^\beta e^{-r^2/2} \equiv \int_\infty^1 \frac{d\lambda}{\lambda}\ (\lambda r)^\beta\ e^{-\lambda^2 r^2/2} = -\frac{e^{-r^2/2}}{2}\ r^\beta\ \Psi\left(1, 1+\frac{\beta}{2}; \frac{r^2}{2}\right)\ , \tag{4.21}$$

where Ψ is the confluent hypergeometric function vanishing at $r = \infty$ [24]. This is also the incomplete gamma function $\Gamma(\beta/2, r^2/2)$, which is polynomial for $\beta = 2, 4, 6, ...$, and has logarithmic terms for $\beta = 0, -2, -4, ...$ Its expansion for $r \to 0$ is,

$$\frac{1}{r\partial_r}\ r^\beta e^{-r^2/2} \xrightarrow{r\to 0} \begin{cases} e^{-r^2/2}\ \frac{r^\beta}{\beta}\left(1+\frac{r^2}{2+\beta}+O(r^4)\right) - \Gamma(\frac{\beta}{2})2^{\beta/2-1}\ , & \beta \neq 0, -2, -4, ..., \\ \frac{1}{2}\ \log r^2 + O(1)\ , & \beta = 0, -2, -4, ... \end{cases} \tag{4.22}$$

In conclusion, we accept integrable eigen-functions with asymptotics $\psi \simeq e^{-r^2/2}$, for $r \to \infty$, and $\psi \simeq (r^0, r^2, r^{|\ell|})$ for $r \to 0$. The counting of free parameters is as follows: there are four physical independent solutions in total (three at $r \simeq 0$ and one at $r \simeq \infty$), plus the energy, minus the four matching conditions $(d/dr)^n \psi(r_0)$, $n = 0, 1, 2, 3$, at $0 < r_0 < \infty$, and the normalization condition. This counting is consistent with a unique physical solution and a discrete spectrum. Actually, we shall find that the system of conditions is over-determined (-1 free parameters), because the logarithmic solution corresponding to the degenerate exponents $\alpha = 0, 2$, at $r = 0$, will not be present.

For $\ell = 0$, the subset of physical solutions of the differential equation (4.17) are also solutions of the following second order equation,

$$\left[(r\partial_r)^2 - r^4 + 2\lambda r^2 - 4\right]\phi = 0\ , \qquad \psi = (2 + r\partial_r)\,\varphi = \frac{2 + r\partial_r}{r\partial_r}\phi\ , \tag{4.23}$$

for the function ϕ. Indeed, the asymptotic behaviors of this equation, $\phi \simeq r^{\pm 2}$ $(r \to 0)$ and $\phi \simeq e^{\pm r^2/2}$ $(r \to \infty)$ correspond to $\psi \simeq r^0, r^2$, and $\psi \simeq e^{\pm r^2/2}$, respectively. The general solutions of (4.23) are readily found in terms of confluent hypergeometric functions Ψ and Φ: the unique regular solution at infinity is given by,

$$\phi = r^2\ e^{-r^2/2} \begin{cases} \Phi(1-k, 3; r^2), & k = 1, 2, 3, ..., \\ \Psi(1-k, 3; r^2), & k \neq 1, 2, 3, ..., \end{cases} \tag{4.24}$$

[§]Here, we do not find any analogous of the obstruction to fixing completely the axial gauge described in ref. [23].

as a function of the energy parameter $E = 2k + 2 + M$. The Φ solutions corresponds to the discrete spectrum

$$E = \frac{\rho_0}{2\tilde{\kappa}m}(k+1) + M \ , \qquad M = \frac{\rho_0}{m}\left(\frac{2}{|\kappa|} - \frac{1}{2\tilde{\kappa}}\right) \ , \qquad k = 1, 2, 3, ..., \ . \tag{4.25}$$

(Physical units were restored in (4.25)). Their wave-functions are polynomial, because $\Phi(1-k, 3; r^2)$ truncates at the k-th term and behaves as $\Phi \simeq r^2$ $(r \to 0)$. The corresponding ψ functions are also polynomial, as shown by using (4.22). Therefore, these solutions are acceptable.

The second type of solutions exists for the complementary continuous range of energy, and behaves as $\Phi \simeq r^{-2}$ for $r \to 0$. As a consequence, the corresponding ψ functions are logarithmic for $r \to 0$,

$$\begin{aligned} \psi(r) &= \frac{2 + r\partial_r}{r\partial_r}\frac{1}{\Gamma(1-k)}\left(\frac{1}{r^2} + k + 1 + O(r^2)\right)e^{-r^2/2} \\ &\stackrel{r\to 0}{\longrightarrow} -\frac{1}{2\Gamma(1-k)} \log r^2 \ , \qquad k \neq 1, 2, 3, ... \end{aligned} \tag{4.26}$$

Although the logarithmic behavior is square-integrable, it produces a $\delta(r)$ term in the r.h.s. of the eigenvalue equation (4.16), due to $(\partial_i^2 + ...)\psi \propto \delta(x)$, which cannot be accepted. Another reason for rejecting these solutions is that they are mapped by $\pi^\dagger$ and π into non-integrable solutions $\psi \simeq r^{-|\ell|}$ $(r \to 0)$ with $\ell = \pm 1$, as shown in the appendix.

In conclusion, the spectrum for $\ell = 0$ is discrete and given by (4.25). Let us add some remarks:

i) The number of free parameters for the reduced second-order problem is equal to zero, because there are two physical independent solutions, plus the energy, minus two matching conditions and the normalization. This is correct for a discrete spectrum and show no sign of the $\ell \neq 0$ over-determination mentioned before.

ii) Eigenfunctions of (4.17) with asymptotic $\psi \simeq r^{-4}$ $(r \to \infty)$ correspond for $\ell = 0$ to the solutions of the inhomogeneous equation $[(r\partial_r)^2 - r^4 + 2\lambda r^2 - 4]\phi = 1$. In the appendix, we show that they behave as $\psi = O(\log r^2)(r \to 0)$ and should be rejected by the same arguments given for eq. (4.26). This is another reason for discarding this type of solutions, independent of (4.19).

iii) The physical solutions (4.25) are of the form $\psi = (2 + r\partial_r)\varphi$, with φ regular for $r = 0$, thus they have vanishing zero mode

$$\int d^2x \ \psi\Big|_{\ell=0} = 2\pi \int_0^\infty dr \partial_r (r^2 \varphi) = 0 \tag{4.27}$$

as required for density fluctuations around the mean field.

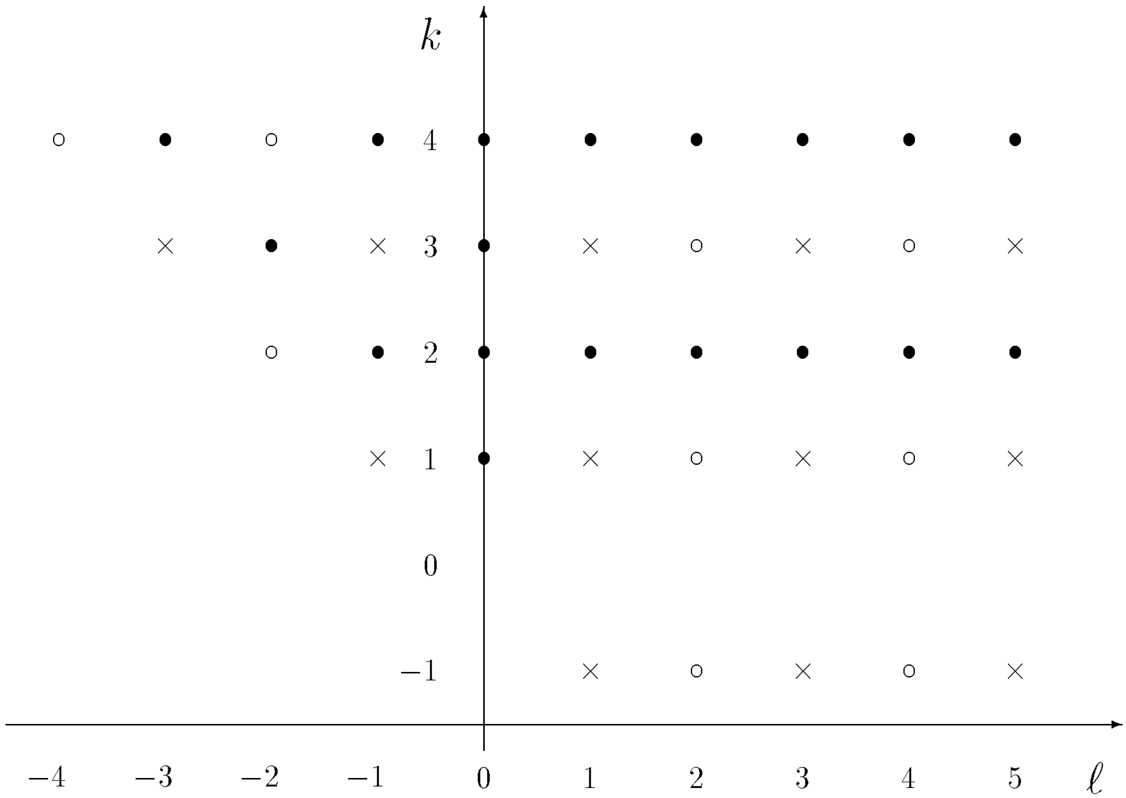

Figure 2: Synopsis of the physical eigenstates in the angular momentum (ℓ) versus energy (k) plane (The meaning of symbols ($\bullet$), ($\circ$) and ($\times$) is given in the text).

The $\ell = 0$ spectrum (4.25) extends for $\ell \neq 0$ into Landau -like levels, whose eigenfunctions can be obtained in principle by applying the π and $\pi^\dagger$ magnetic translation operators (4.9,4.10) to the $\ell = 0$ functions. For $\ell \neq 0$, we do not have a general explicit method of solution and we made a case by case analysis. The properties of the $\ell \neq 0$ solutions are summarized in fig. 2 and will be briefly discussed hereafter, leaving the details to the appendix. The allowed values of ℓ are bounded from below by the line $k = -\ell$, because for these values π is not invertible, by eq. (4.11), and annihilates the physical wave functions.

Let us first note that the condition (4.27) is trivially satisfied by $\ell \neq 0$ eigen-functions, due to their angular dependence, while it eliminates the zero mode for $\ell = 0$. This implies that another line of eigenstates should exist for $\ell > 0$, which is not connected to the previous

$\ell = 0$ states. This is possible for the energy $E = M, (k = -1)$, because π is not invertible at $\ell = 1$, $\pi\psi_{-1,1} = 0$, where the line stops. Moreover, the bound (4.13) forbids states of lower energy. Indeed, such eigenstates are found by explicit analysis of the differential equation (4.17).

The general properties of $(\ell \neq 0)$ solutions are the following (see fig. 2). There are polynomial solutions, which were found by solving the four-term recursion relation with Mathematica [25]. These are of two types:

i) for $k \geq 2$ even, and $\ell \geq 0$,

$$\psi = e^{i\ell\theta} r^{|\ell|} \left(a_0 + a_1 r^2 + \ldots + a_k r^{2k} \right) e^{-r^2/2} , \qquad (\ \textit{points}\ (\bullet)), \tag{4.28}$$

ii) for $k \geq -1$ and $\ell \geq -k$, ℓ even,

$$\psi = e^{i\ell\theta} \left(b_0 + b_1 r^2 + \ldots + b_{k+\ell/2} r^{2k+\ell} \right) e^{-r^2/2} , \qquad (\ \textit{points}\ (\circ)), \tag{4.29}$$

Moreover, for odd ℓ, there are non-polynomial solutions, represented by crosses $(\times)$ in fig. 2, which display a double power expansion:

$$\psi = e^{i\ell\theta} \ \left\{ r^{|\ell|} \left(a_0 + a_1 r^2 + \cdots \right) + b_0 + b_1 r^2 + \cdots \right\} e^{-r^2/2} , \tag{4.30}$$

which is similar to the confluent hypergeometric function Ψ. These solutions can only be obtained by applying π or $\pi^\dagger$ to a neighbor polynomial solution. In the appendix, we report a table of the polynomial eigen-functions (4.28),(4.29) for the first few values of k and ℓ, and give examples of the action of π. Note that no normalizable solutions are found with energy $k = 0$; thus, there is the double of the Landau-level gap between the lowest available level $(k = -1)$ and all the higher ones $(k = 1, 2, \ldots)$. We do not have a physical interpretation of this result.

In conclusion, the complete energy spectrum is discrete and given by eq. (4.25) for $k = -1, 1, 2, 3, \ldots$, with each level infinitely degenerate in angular momentum $(\ell \geq -k)$.

Let us discuss more precisely the action of the magnetic translation operators π and $\pi^\dagger$ and show that it closes on these solutions, thus ensuring their translation (and gauge) invariance. Equations (4.10,4.11) imply $\pi^\dagger\pi\psi_{k,-k} = 0$ and $\pi\pi^\dagger\psi_{k,-k-1} = 0$: by explicit calculation (see the appendix), we actually find $\pi\psi_{k,-k} = 0$, leading to the following pattern,

$$\begin{array}{ccccccc} \cdots \longrightarrow & \psi_{k,-k-1} & \xrightarrow{\pi^\dagger} & \psi_{k,-k} \longrightarrow & \psi_{k,-k+1} \longrightarrow \cdots \\ & 0 & \xleftarrow{\pi} & \psi_{k,-k} \longleftarrow & \psi_{k,-k+1} \longleftarrow \cdots \end{array} \tag{4.31}$$

Therefore, the representation of the $(\pi, \pi^\dagger)$ gauge algebra is not *fully decomposable* into normalizable $(\ell \geq -k)$ and non-normalizable $(\ell < -k)$ states: the former are mapped by

π and $\pi^\dagger$ into themselves, while the latter are also mapped by $\pi^\dagger$ into normalizable ones. Nevertheless, the projection of the non-normalizable states to zero is consistent[¶]. Actually, the same non-decomposable representation occurs in the elementary Landau levels, because the action of the operators π^L and $(\pi^L)^\dagger$ on the Landau wave-functions is the same as (4.31). Therefore, non-decomposability seems to be a rather general property of gauge invariance in the Hamiltonian formalism.

We now discuss the completeness of the basis of eigen-functions we have found. For $\ell = 0$, the eigen-functions (4.24) are linear combinations of the polynomials $\{r^{2n}e^{-r^2/2}, n > 0\}$, which form a complete basis for square integrable functions of (r^2) with vanishing zero mode. For $\ell \neq 0$, the space of solutions cannot be easily defined in mathematical terms, due to the involved $r = 0$ boundary conditions, which can be either $\psi \simeq e^{i\ell\theta} r^0$ or $\psi \simeq e^{i\ell\theta} r^{|\ell|}$. Therefore, the issue of completeness cannot be easily addressed for the $\ell \neq 0$ subspaces, which are, nevertheless, isomorphic to the $\ell = 0$ complete basis by the π action. Note that the singular behavior $\psi \simeq e^{i\ell\theta} r^0$ at $r = 0$, is fully acceptable, because it cancels in gauge invariant quantities like $\rho^a\rho^a \propto \psi^\dagger\psi$.

5 Spontaneous symmetry breaking and the excitations of the anyon fluid

We have been describing the Abelian and non-Abelian $U(2)$ anyon fluids, which are non-relativistic gauge theories of the Chern-Simons type, and we have shown the spontaneous breaking of the corresponding global symmetries U(1) and $U(2) \to U(1)$, respectively. It is interesting to discuss the analogies and differences with the four-dimensional Yang-Mills theory of the Standard Model of electroweak interactions, and identify the excitations of the anyon fluid with the non-relativistic analogues of Goldstone and Higgs particles, if possible. There are two basic differences:

i) the non-relativistic matter fields have half of the degrees of freedom of their relativistic counterparts, because the latter describe both particles and antiparticles;

ii) the Chern-Simons gauge field has no propagating physical degrees of freedom and thus cannot lead to the Higgs phenomenon.

Let us first recall the superfluid, which is the canonical example of spontaneous symmetry

[¶]Note that our analysis does not exclude the existence of other solutions for some $\ell \neq 0$ isolated values. However, these would not be acceptable because they would not close under the action of π.

breaking in non-relativistic field theory:

$$H = \int d^2x \left(\frac{1}{2m} |\partial_i \Psi|^2 + \frac{g}{2} |\Psi|^4 \right) \ , \qquad g > 0 \ . \tag{5.1}$$

Due to Bose condensation, the field acquires the ground-state expectation value $\langle \Psi \rangle = \sqrt{\rho_0}$, which breaks the $U(1)$ global symmetry of particle number conservation. Small excitations around the mean field are diagonalized by the Bogoliubov transformation. This leads to a massless excitations with sound velocity $v_s \propto \sqrt{g\rho_0}$, controlled by the repulsive interaction; moreover, the ground state does not have a definite particle number, due to the Bogoliubov rotation.

The usual description [11] in terms of radial ($\hat{\rho}$) and phase (θ) components of the field, $\Psi = \sqrt{\rho_0 + \hat{\rho}}\ e^{i\theta}$, can be easily compared to the current algebra description (2.8). The current is represented as,

$$J_i = \frac{1}{m}\ Im\ \left(\psi^\dagger \partial_i \psi \right) = \frac{\rho_0 + \hat{\rho}}{m}\ \partial_i \theta \ , \tag{5.2}$$

and the approximate commutation relations (2.23), (2.24) become,

$$[J_i, \hat{\rho}] \simeq \frac{i\rho_0}{m}\ \partial_i \delta(x-y) \ \longrightarrow [\theta, \hat{\rho}] = i\delta(x-y) \ . \tag{5.3}$$

Therefore, the would-be relativistic Higgs ($\hat{\rho}$) and Goldstone (θ) fields are conjugate variables in the non-relativistic theory, the superfluid massless mode is a Goldstone particle and there is no non-relativistic analogue of the Higgs particle.

The Abelian anyon fluid is very similar to the superfluid. The "microscopic" mechanism leading to $\langle \rho \rangle = \rho_0$ is not the Bose condensation - there is no macroscopic occupation of a single energy level, rather a macroscopic number of particles at the same energy. Nevertheless, there is spontaneous breaking of the global $U(1)$ symmetry, because the Bogoliubov transformed ground state has no definite particle number (see sect 2.3).

One can find a closer relation with the usual superfluid by formally integrating out the Chern-Simons field. The resulting self-interacting matter theory can possibly reduce to (5.1) for small fluctuations around the saddle-point approximation. In this sense, we can consider the anyon superfluid as a non-relativistic example of *dynamical* symmetry breaking [13]. However, this is a peculiar example, where the self-interaction should have special normal-ordering effects, as in sec. 2.3, 2.4, which determine different effective $|\Psi|^4$ interactions for boson-based ($g \propto 1/\kappa$) and fermion-based ($g \propto 1$) anyons.

The current algebra approach is general enough to handle this non-standard mechanism of spontaneous symmetry breaking. By expanding again Ψ into (ρ, θ) components, one obtains the current $J_i \simeq \rho_0 \left(\partial_i\theta - A_i\right)/m$; this is still purely longitudinal, because the Chern-Simons gauge field can be (locally) reduced to a pure gauge, $A_i = \partial_i\alpha$,

$$J_i \simeq \frac{\rho_0}{m}\ \partial_i(\theta-\alpha)\ . \tag{5.4}$$

Therefore, the current algebra (2.23), (2.24) is the same as in the superfluid (5.3) and the anyon massless mode is a Goldstone particle. Note that the current, i.e. $(A_i - \partial_i\theta)$, is the fundamental quantity of the Landau-Ginzburg theory of superconductivity [26], but it has there a completely different meaning, because the gauge field has transverse degrees of freedom.

The $U(2)$ non-Abelian anyon fluid can be described in similar terms. The mean field value $\langle\rho^a\rangle = \delta_3^a\ \rho_0/2$ breaks the $SU(2)$ global symmetry to the $U(1)$ subgroup generated by $(1-\sigma_3)/2$. Although Ψ_r has vanishing ground-state value, it is still convenient to introduce the parametrization of the Standard Model [13],

$$\Psi_r = \exp\left(i\frac{\theta^a\sigma_a}{2}\right)\begin{pmatrix}\sqrt{\rho_0+\hat\rho}\\ 0\end{pmatrix}\ , \tag{5.5}$$

where θ^a are the three would-be relativistic Goldstone particles and $\hat\rho$ the would-be Higgs one. The non-relativistic field Ψ_r describes only two of these degrees of freedom, while the other two are conjugate momenta. The approximate current algebra (3.26,(3.27) can be rewritten as follows:

$$\begin{aligned} &J_i \simeq \frac{\rho_0}{m}\ \partial_i\theta^3\ , \quad \rho^3 \simeq \rho_0 + \hat\rho \quad \longrightarrow \quad \left[\theta^3(x), \hat\rho(y)\right] \simeq i\delta(x-y)\ , \\ &\rho^1 \simeq \rho_0\ \theta^2 \quad\ , \quad \rho^2 \simeq -\rho_0\theta^1 \quad \longrightarrow \quad \left[\theta^1(x), \theta^2(y)\right] \simeq \frac{1}{\rho_0}\delta(x-y)\ . \end{aligned} \tag{5.6}$$

The density $\hat\rho$ and the isospin-diagonal phase θ^3 represent the Goldstone particle as in the Abelian case; this excitation does not have a well-defined isospin number. The other pair of would-be Goldstones are conjugate variables; they do not undergo the Bogoliubov transformation, because the isospin-down number is conserved by the remaining $U(1)$ symmetry. The effective repulsive interaction $(4/|\kappa| - 1/\tilde\kappa)\,(\rho^a)^2$ induced by the Chern-Simons field gives a mass gap to this excitation.

6 Concluding remarks

In this paper, we analyzed the $U(2)$ Chern-Simons theory coupled to non-relativistic matter with isospin 1/2. We applied the mean field approximation developed in the refs. [6],[9], and uncover a phase of the theory where the global $U(2)$ symmetry breaks spontaneously to the

$U(1)$ one. Besides one Goldstone excitation already present in the Abelian model, we found a massive excitation with non-trivial non-Abelian dynamics. Therefore, the phenomenon of superfluidity and superconductivity, originally discovered by Laughlin [6], extends smoothly into this phase of the non-Abelian anyon fluid.

This theory can also be consider as a toy model of the Standard Model of the electroweak interactions: in section five, we clarified the analogies and differences between the Chern-Simons theory and the four-dimensional Yang-Mills theory. Although there are many simplifying features, the non-Abelian anyon fluid is an instructive example where a non-Abelian gauge theory can be explicitly solved in the low-energy limit.

Another interesting aspect of this theory is its close relation with gravity in (2+1) dimension. Actually, the Chern-Simons action for the non-Abelian group $ISO(2,1)$ (respectively $SO(4)$) can be rewritten as the Einstein-Hilbert action (respectively with cosmological constant) [27]. A suitable coupling of matter fields to gravity might lead to a theory similar to the non-Abelian anyon fluid: the mean-field approximation might describe a phase of semi-classical cosmology, where the metric is "induced" by the (dynamical) symmetry breaking [28].

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A Details of the eigenvalue problem

Examples of $\ell \neq 0$ eigen-functions

The polynomial solutions (4.28), denoted by the points (•) in fig. 2, are of the form,

$$\psi = e^{il\theta} \left(2 + r\partial_r\right) \varphi \ , \quad \varphi(r) = r^\ell \ \left(a_0 + a_1 r^2 + \cdots + a_{k-1}\ r^{2k-2}\right)\ e^{-r^2/2} \ , \qquad \text{(A.1)}$$

and exist for $k \geq 2$, even, and $\ell \geq 0$. The simplest ones are listed hereafter ($\ell_n \equiv (\ell + n)$):

k	$a_0(\ell)$	a_1	a_2	a_3	a_4	a_5
2	$-\ell_1$	1				
4	$-\ell_4\ell_3\ell_1$	$\ell_3(3\ell+8)$	$-3\ell_3$	1		
6	$-\ell_6\ell_5\ell_4\ell_3\ell_1$	$\ell_6\ell_5\ell_3(5\ell+12)$	$-2\ell_5\ell_3(5\ell+26)$	$2\ell_5(5\ell+22)$	$-5\ell_5$	1

(A.2)

The other type of polynomial solutions, denoted by ($\circ$) in fig. 2, are of the form,

$$\psi = e^{il\theta}\ (2 + r\partial_r)\ \varphi\ , \quad \varphi(r) = \left(\frac{b_{-1}}{r^2} + b_0 + b_1 r^2 + \cdots + b_{k-1+\ell/2}\ r^{2k+\ell-2} \right)\ e^{-r^2/2}\ , \qquad (A.3)$$

and exist for $k \geq -1$ and $\ell \geq -k$, ℓ even. The simplest ones are listed hereafter:

k	ℓ	b_{-1}	b_0	b_1	b_2
-1	2	1			
-1	4	4	1		
-1	6	24	8	1	
1	2	-2	0	1	
1	4	-16	-4	0	1

(A.4)

Note that for some even, negative, values of ℓ, these two types of solutions actually merge.

Examples of the action of π and $\pi^\dagger$

Let us derive some of the non-polynomial solutions, denoted by ($\times$) in fig. 2, by applying π to a polynomial solution with neighbor value of ℓ. Consider for example the lower energy level $k = -1$, and find $\psi_{-1,1}$:

$$\begin{aligned} \psi_{-1,1} \propto \pi\psi_{-1,2} &= \left(\partial_z + \frac{\bar{z}}{2} - \bar{z}\frac{1}{(1 + r\partial_r)(2 + r\partial_r)} \right)\ (2 + r\partial_r)\ \frac{e^{i2\theta}}{r^2}\ e^{-r^2/2} \\ &= -e^{-r^2/2}\ \partial_z e^{i2\theta}\ -\ e^{i\theta}\ \frac{1}{r\partial_r}\ \frac{1}{r}\ e^{-r^2/2} \\ &= e^{i\theta}\frac{1}{r\partial_r}\ r\ e^{-r^2/2} = -\ e^{i\theta}\ \frac{r}{2}\ \Psi\left(1, \frac{3}{2}, \frac{r^2}{2}\right)\ . \end{aligned} \qquad (A.5)$$

In this derivation, we used some formal properties of $(r\partial_r)^{-1}$ which follows by integration by parts of (4.20), and, at the very end, its explicit form (4.21). The further application of π yields:

$$\pi\psi_{-1,1} = \left(\partial_z + \frac{\bar{z}}{2} - \bar{z}\frac{1}{(1 + r\partial_r)(2 + r\partial_r)} \right) \frac{z}{r}\ \frac{1}{r\partial_r}\ r\ e^{-r^2/2} =\ 0\ . \qquad (A.6)$$

Actually, this vanishing result can be found more easily by collecting the common denominator $1/(1 + r\partial_r)$. Equation (A.6) verifies the closure of the gauge algebra on the physical solutions with energy $k = -1$, as indicated in the diagram (4.31). Another non-trivial action in this diagram is given by $\pi^\dagger$ applied to the unphysical logarithmic solution $\psi_{-1,0}$ found in (4.24), which has the explicit form,

$$\psi^{(log)}_{-1,0} = \frac{2 + r\partial_r}{r\partial_r}\ \frac{1}{r^2}\ e^{-r^2/2} = -\frac{1}{r\partial_r}\ e^{-r^2/2}\ . \qquad (A.7)$$

The action of $\pi^\dagger$ is computed as follows,

$$\pi^\dagger \psi^{(log)}_{-1,0} = \frac{z}{2}\left(\frac{1}{2+r\partial_r} + \frac{1}{r\partial_r} - \frac{2}{(1+r\partial_r)(2+r\partial_r)}\right) e^{-r^2/2} = \psi_{-1,1} \ . \qquad (A.8)$$

One can similarly compute that the ($\ell = 0$) logarithmic solutions Ψ in (4.24), for any value of $k \neq 1, 2, 3 \ldots$, are mapped by π and $\pi^\dagger$ into $\ell = \pm 1$ solutions with non-integrable behavior $\psi \simeq r^{-|\ell|}$; actually, it is sufficient to use the ($r \to 0$) expansion of these eigenfunctions [24] which is reported in (4.26).

Solutions with r^{-4} asymptotics at ($r \to \infty$)

The study of the asymptotic behaviors shows that these solutions are also solutions of the inhomogeneous reduced problem (4.23), $[(r\partial_r)^2 - r^4 + 2\lambda r^2 - 4]\phi^{(-4)} = 1$. It is sufficient to solve it in the case $k \neq 1, 2, 3 \ldots$. The inhomogeneous solutions can be obtained from the Green function,

$$G(r,\rho) = \frac{u_1(r)u_2(\rho)\Theta(\rho - r) + u_1(\rho)u_2(r)\Theta(r-\rho)}{a_2(\rho)W(\rho)} \ . \qquad (A.9)$$

In this equation, u_1 and u_2 are the two independent, homogeneous solutions vanishing at ($r \to 0$) and ($r \to \infty$), which are given by the Φ and Ψ confluent Hypergeometric functions in (4.24), respectively. Moreover, a_2 is the coefficient of second-order term in the differential equation and W is the wronskian, $a_2(\rho)W(\rho) \propto \rho$. The resulting expression for the Green function integrated against the source can be expanded for asymptotic values of r. For ($r \to \infty$), one recover the r^{-4} behavior by the cancellation of the positive and negative exponentials of Φ and Ψ. For ($r \to 0$), one finds the behavior,

$$\phi^{(-4)}_{k,0} \propto \frac{1}{\Gamma(1-k)}\left[1 + O\left(r^2\right) + O\left(r^2 \log r^2\right)\right] e^{-r^2/2} \ , \qquad (A.10)$$

which leads to a logarithmic behavior for $\psi^{(-4)}_{k,0}$ and a non-normalizable one for $\psi^{(-4)}_{k,1} = \pi^\dagger \psi^{(-4)}_{k,0}$, by the same mechanism discussed above.

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