A metric of Yukawa potential as an exact solution to the field equations of general relativity

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Abstract

It is shown that, by defining a suitable energy momentum tensor, the field equations of general relativity admit a line element of Yukawa potential as an exact solution. It is also shown that matter that produces strong force may be negative, in which case there would be no Schwarzschild-like singularity.

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1. Approximate solution

It has been shown that the line element of Yukawa potential of the form

$$e^{-\nu} = 1 - \alpha\frac{e^{-\beta r}}{r} \tag{1}$$

approximately satisfies, at short range $\beta r \ll 1$, the vacuum field equations of general relativity, $T_{\mu\nu} = 0$,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa T_{\mu\nu}, \tag{2}$$

assuming a centrally symmetric spacetime metric

$$ds^2 = e^{\mu}dt^2 - e^{\nu}dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2). \tag{3}$$

In general, the quantity $R = 1/\beta$ specifies a range so that the line element of Yukawa potential can be approximated as a solution to the field equations of general relativity for the region $r \ll R$. It is seen that for the maximal possible range of $R \to \infty$, the metric of Yukawa form reduces to the familiar Schwarzschild metric, which is used to describe the gravitational field. In the case of short range of nuclear physics, the quantity R can be assigned a value in terms of the fundamental constants $\hbar$ and c, and the rest mass of Yukawa force carrier. The result has shown that within the short range of strong force, the field equations of general relativity admit a line element that takes the form of Yukawa potential for strong interaction. This leads to the conclusion that if there is no other form of matter, besides the mass and the charge, that characterises strong interaction, then it would be possible to consider strong interaction also a manifestation of general relativity at short range.

Assume the motion of a particle in a strong force field is governed by the geodesic equations

$$\frac{d^2x^{\mu}}{ds^2} + \Gamma^{\mu}_{\nu\sigma}\frac{dx^{\nu}}{ds}\frac{dx^{\sigma}}{ds} = 0, \tag{4}$$

then with the metric of Yukawa potential of the form

$$ds^2 = \left(1 - \alpha\frac{e^{-\beta r}}{r}\right)c^2dt^2 - \left(1 - \alpha\frac{e^{-\beta r}}{r}\right)^{-1}dr^2 - r^2\left(d\theta^2 + sin^2\theta d\phi^2\right), \tag{5}$$

the equations for the geodesics can be written explicitly as [1, 2]

$$\left(1 - \alpha\frac{e^{-\beta r}}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\theta}{d\tau}\right)^2 + r^2 sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2 - c^2\left(1 - \frac{\alpha e^{-\beta r}}{r}\right)\left(\frac{dt}{d\tau}\right)^2 = -c^2, \tag{6}$$

$$\frac{d}{d\tau}\left(r^2\frac{d\theta}{d\tau}\right) - r^2\sin\theta\cos\theta\left(\frac{d\phi}{d\tau}\right)^2 = 0, \tag{7}$$

$$\frac{d}{d\tau}\left(r^2\sin^2\theta\frac{d\phi}{d\tau}\right) = 0, \tag{8}$$

$$\frac{d}{d\tau}\left[\left(1 - \alpha\frac{e^{-\beta r}}{r}\right)\frac{dt}{d\tau}\right] = 0. \tag{9}$$

By choosing spherical polar coordinates and considering the motion in the plane $\theta = \pi/2$, the equations (8) and (9) are reduced to

$$\frac{d\phi}{d\tau} = \frac{l}{r^2}, \qquad \frac{dt}{d\tau} = \frac{kr}{r - \alpha e^{-\beta r}}, \tag{10}$$

where l and k are constants of integration. With these relations, the equation for the orbit can be obtained from the equation (6) as

$$\left(\frac{l}{r^2}\frac{dr}{d\phi}\right)^2 + \frac{l^2}{r^2} = c^2(k^2 - 1) + \frac{\alpha c^2}{r}e^{-\beta r} + \frac{\alpha l^2}{r^3}e^{-\beta r}. \tag{11}$$

In the case when the condition $\beta r \ll 1$ is satisfied, with the approximation $e^{-\beta r} = 1 - \beta r$, the equation for the orbit becomes

$$\left(\frac{l}{r^2}\frac{dr}{d\phi}\right)^2 + \frac{l^2(1+\alpha\beta)}{r^2} = c^2(k^2 - (1+\alpha\beta)) + \frac{\alpha c^2}{r} + \frac{\alpha l^2}{r^3}. \tag{12}$$

By letting $u = 1/r$ and differentiating the resulting equation with respect to the variable ϕ, it is found

$$\frac{d^2u}{d\phi^2} + (1+\alpha\beta)u = \frac{\alpha c^2}{2l^2} + \frac{3\alpha l^2}{2}u^2. \tag{13}$$

Hence, the dynamics of a particle under the influence of strong force of Yukawa potential is similar to that of a particle in the Schwarzschild gravitational field.

2. Exact solution

Unlike the assumed massless force carriers of the gravitational field, hence the vacuum solution to the field equations of general relativity could be justified, the force carriers of the strong field are assumed to be massive. Therefore, any solutions to the field equations of general relativity that may be used to describe the strong field should be a nonvacuum solution. This means that in order to find a more appropriate solution to describe strong force, a strong energy momemtum tensor must be specified. In the following it will be discussed a particular form of strong energy momentum tensor so that the field equations of general relativity will admit the line element of Yukawa potential as an exact solution.

Consider a strong energy momentum tensor of the form

$$T_\mu^\nu = \begin{pmatrix} -\frac{\alpha\beta}{\kappa}\frac{e^{-\beta r}}{r^2} & 0 & 0 & 0 \\ 0 & -\frac{\alpha\beta}{\kappa}\frac{e^{-\beta r}}{r^2} & 0 & 0 \\ 0 & 0 & \frac{\alpha\beta^2}{2\kappa}\frac{e^{-\beta r}}{r} & 0 \\ 0 & 0 & 0 & \frac{\alpha\beta^2}{2\kappa}\frac{e^{-\beta r}}{r} \end{pmatrix} \tag{14}$$

With this energy momentum tensor, the field equations of general relativity reduce to the system of equations [3]

$$e^{-\nu}\left(\frac{d\nu}{dr}-\frac{1}{r}\right)+\frac{1}{r} = -\alpha\beta\frac{e^{-\beta r}}{r}, \tag{15}$$

$$-e^{-\nu}\left(\frac{d\mu}{dr}+\frac{1}{r}\right)+\frac{1}{r} = -\alpha\beta\frac{e^{-\beta r}}{r}, \tag{16}$$

$$\frac{d\nu}{dt} = 0, \tag{17}$$

$$-e^{-\nu}\left(2\frac{d^2\mu}{dr^2}+\left(\frac{d\mu}{dr}\right)^2+\frac{2}{r}\left(\frac{d\mu}{dr}-\frac{d\nu}{dr}\right)-\frac{d\mu}{dr}\frac{d\nu}{dr}\right)+e^{-\mu}\left(2\frac{d^2\nu}{dt^2}+\left(\frac{d\nu}{dt}\right)^2-\frac{d\nu}{dt}\frac{d\mu}{dt}\right)=\alpha\beta^2\frac{e^{-\beta r}}{r}. \tag{18}$$

These equations are not independent since it can be verified that the last equation follows from the first three equations. Furthermore, the first two equations give $d\nu/dr + d\mu/dr = 0$ that also leads to $\nu + \mu = 0$. This system of equations when integrated gives the metric of Yukawa potential.

The form of the energy momentum tensor considered above is mathematically acceptable since it can be verified to satisfy the conservation laws

$$T^{\nu}_{\mu;\nu} = \frac{1}{\sqrt{-g}}\frac{\partial T^{\nu}_{\mu}\sqrt{-g}}{\partial x^{\nu}} - \frac{1}{2}\frac{g_{\lambda\sigma}}{\partial x^{\mu}}T^{\lambda\sigma} = 0, \tag{19}$$

using the metric $g_{\mu\nu}$ of Yukawa form.

However, there emerges an important feature that relates to the nature of the quantity α in the line element and that of the energy momentum tensor. That is, since the quantities κ and β are positive, the energy component T^0_0 and the quantity α always have opposite signs. Therefore, since g^{00} is positive, if the energy component T_{00} is considered to be positive, then the energy component $T^0_0 = g^{00}T_{00}$ must be positive, and, in this case, the quantity α must be negative and there would be no Schwarzschild-like singularity.

References

[1] Bergmann, P. G. *Introduction to the Theory of Relativity* (Dover, 1976).

[2] Lawden, D. F. *An Introduction to Tensor Calculus, Relativity and Cosmology* (John Wiley & Sons, 1982).

[3] Landau, L. D. & Lifshitz, E. M. *The Classical Theory of Fields* (Pergamon Press, 1975).