

LOOP VARIABLES FOR A CLASS OF CONICAL SPACETIMES*

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Abstract

Loop variables are used to describe the presence of topological defects in spacetime. In particular we study the dependence of the holonomy transformation on angular momentum and torsion for a multi-chiral cone. We also compute the holonomies for multiple moving crossed cosmic strings and two plane topological defects-crossed by a cosmic string.

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1 INTRODUCTION

In the loop space formalism for gauge theories [1] the fields depend on paths rather than on spacetime points, and a gauge field is described by associating with each path in spacetime an element of the corresponding gauge groups. The fundamental quantity that arises from this path-dependent approach, the non-integrable phase factor [2] (or loop variable) represents the electromagnetic field or a general gauge field more adequately than the field strength or the integral of the vector potential [2]. In the electromagnetic case, for example, as observed by Wu and Yang [2], in a situation where global aspects are taken into consideration the field strength underdescribes the theory and the integral of the vector potential for every loop overdescribes it. The exact description is given by the factor $\exp\left(\frac{ie}{\hbar c} \oint_c A_\mu dx^\mu\right)$.

The extension of the loop formalism to the theory of gravity was first considered by Mandelstam [3] who established several equations involving the loop variables, and also by Voronov and Makeenko [4]. Recently, Bollini et al. [5] computed the loop variables for the gravitational field corresponding to the Kerr metric.

The loop variables in the theory of gravity are matrices representing parallel transport along contours in a spacetime with a given affine connection. They are connected with the holonomy transformations which contain important topological information. These mathematical objects contain information, for example, about how vectors change when parallel transported around a closed curve. They also can be thought of as measuring the failure of a single coordinate patch to extend the way around a closed curve.

Suppose that we have a vector v^α at a point P of a closed curve C in spacetime. Then, one can produce a vector \bar{v}^α at P which, in general, will be different from v^α , by parallel transporting v^α around C . In this case, we associate with the point P and the curve C a linear map U_β^α such that for any vector v^α at P , the vector \bar{v}^α at P results from parallel transporting v^α around C and is given by $\bar{v}^\alpha = U_\beta^\alpha v^\beta$. The linear map U_β^α is called holonomy

transformation associated with the point P and the curve C . If we choose a tetrad frame and a parameter $\lambda \in [0, 1]$ for the curve C such that $C(0) = C(1) = P$, then in parallel transporting a vector v^α from $C(\lambda)$ to $C(\lambda + d\lambda)$, the vector components change by $\delta v^\alpha = M_\beta^\alpha[x(\lambda)]v^\beta$, where M_β^α is a linear map which depends on the tetrad, the affine connection of the spacetime and the value of λ . Then, it follows that the holonomy transformation U_β^α is given by the ordered matrix product of the N linear maps

$$U_\beta^\alpha = \lim_{N \rightarrow \infty} \prod_{i=1}^N \left\{ \delta_{\alpha\beta} + \frac{1}{N} M_\beta^\alpha[x(\lambda)]_{\lambda=i/N} \right\} \quad (1.1)$$

One often writes the expression in Eq.(1.1) as

$$U(C) = P \exp \left(\int_C M \right) \quad (1.2)$$

where P means ordered product along a curve C . Equation (1.2) should be understood as an abbreviation of the right hand side of Eq.(1.1). Note that if M_β^α is independent of λ , then it follows from Eq.(1.1) that U_β^α is given by $U_\beta^\alpha = (\exp M)_\beta^\alpha$.

In this paper we shall use the notation

$$U_{BA}(C) = P \exp \left(\int_A^B \Gamma_\mu(x(\lambda)) \frac{dx^\mu}{d\lambda} d\lambda \right) \quad (1.3)$$

where Γ_μ is the tetradic connection and A, B are the initial and final points, respectively, of the path. Then, associated with every path C from a point A to point B , we have a loop variable given by Eq.(1.3) which is a function of the path C as a geometrical object.

The purpose of this paper is to examine the possibility of the use of loop variables to characterize the presence of topological defects of spacetime. In particular, we study a topological defect corresponding to the multiple parallel chiral strings in the context of Einstein [6] and Einstein-Cartan theories [7], the multiple moving crossed comic strings [8] in order to test if the strings are really moving and crossing. Finally, we consider the topological defects corresponding to two domain walls crossed by a cosmic string [9] and two plane topological defects crossed by a cosmic string [9] also.

2 LOOP VARIABLES IN THE SPACE-TIME OF MULTIPLE MOVING CROSSED COSMIC STRINGS.

Recently, Letelier and Gal'tsov [8] found an exact solution of the Einstein equations describing an arbitrary number of non-parallel straight infinitely long cosmic strings moving with different constant velocities. The metric corresponding to this configuration of strings is given by

$$ds^2 = -e^{-4V}(dx + F_1 dt + G_1 dz)^2 - e^{-4V}(dy + F_2 dt + G_2 dz) - dz^2 + dt^2 \quad (2.1)$$

where $V = 2\sum_{i=1}^N \mu_i \ln r_i$, with $r_i = |\zeta - \alpha_i|$ and $\zeta = x + iy$, $\alpha_j = v_{xj}t + m_{xj}z + x_{0j} + i(v_{yj}t + m_{yj}z + y_{0j})$. The functions $F_1(G_1)$ and $F_2(G_2)$ are the real and imaginary parts of two analytic functions on the variable ζ also this functions depend on t and z through the combinations α_i . The explicit form of this functions will not play a major role in our analysis, they can be found in [8].

Our interest in compute the holonomies for this spacetime is to test if really the strings are non-parallel and move in different directions. To do this let us introduce a set of four vectors $e_{(a)}^\mu$ ($a = 1, 2, 3, 4$ is a tetradic index) which are orthonormal at each point with respect to the metric with Minkowski signature, that is, $g_{\mu\nu}e_{(a)}^\mu e_{(b)}^\nu = \eta_{ab} = \text{diag}(-1, -1, -1, +1)$. We assume that the $e_{(a)}^\mu$'s are matrix invertible, that is, that there exists an inverse frame $e_\mu^{(a)}$ given by $e_\mu^{(a)} e_{(a)}^\nu = \delta_\mu^\nu$ and $e_\mu^{(a)} e_{(b)}^\mu = \delta_b^a$.

Now define the 1-forms θ^a as

$$\begin{aligned} \theta^1 &= e^{-2V}(dx + F_1 dt + G_1 dz) \\ \theta^2 &= e^{-2V}(dy + F_2 dt + G_2 dz) \\ \theta^3 &= dz \\ \theta^4 &= dt. \end{aligned} \quad (2.2)$$

Then, in the coordinate system ($x^1 = x, x^2 = y, x^3 = z, x^4 = t$), the tetrad

frame defined by $\theta^a = e^{(a)}_\mu dx^\mu$ is given by

$$\begin{aligned}
e_1^{(1)} &= e^{-2V}, & e_3^{(1)} &= e^{-2V} G_1, & e_4^{(1)} &= e^{-2V} F_1 \\
e_2^{(2)} &= e^{-2V}, & e_3^{(2)} &= e^{-2V} G_2, & e_4^{(2)} &= e^{-2V} F_2 \\
e_3^{(3)} &= 1, & e_4^{(4)} &= 1.
\end{aligned} \tag{2.3}$$

Using Cartan's structure equations $d\theta^a + \omega^a_b \wedge \theta^b = 0$, for arbitrary functions V, F_1, G_1, F_2 and G_2 we get the following expressions for the tetradic connection

$$\begin{aligned}
\Gamma^1_{\mu 3} dx^\mu &= -e^{-2V} \chi_1 dx + \frac{1}{2} e^{-2V} \eta_1 dy - \frac{1}{2} e^{-2V} (\chi_2 - F_2 \eta_1 + 2F_1 \chi_1) dt + \\
&\quad \frac{1}{2} e^{-2V} (G_2 \eta_1 - 2G_1 \chi_1) dz = -\Gamma^3_{\mu 1} dx^\mu \\
\Gamma^2_{\mu 1} dx^\mu &= -2 \frac{\partial V}{\partial y} dx - 2 \frac{\partial V}{\partial x} dy + \frac{1}{2} \left(\xi_1 + 4F_2 \frac{\partial V}{\partial x} - 4F_1 \frac{\partial V}{\partial y} \right) dt + \\
&\quad \left(2G_2 \frac{\partial V}{\partial x} - 2G_1 \frac{\partial V}{\partial y} + \xi_2 \right) dz = -\Gamma^1_{\mu 2} dx^\mu \\
\Gamma^2_{\mu 3} dx^\mu &= \frac{1}{2} e^{-2V} (F_1 \eta_1 + 2F_2 \eta_1) dt + \frac{1}{2} e^{-2V} \eta_1 dx + e^{-2V} (\chi_3 dy + \\
&\quad \frac{1}{2} e^{-2V} (G_1 \eta_1 + 2G_2 \chi_3) dz = -\Gamma^3_{\mu 2} dx^\mu \\
\Gamma^3_{\mu 4} dx^\mu &= \frac{1}{2} e^{-4V} \eta_2 (dy + F_2 dt + G_2 dz) + \frac{1}{2} e^{-4V} \chi_2 (dx + F_1 dt + G_1 dz) = \Gamma^4_{\mu 3} dx^\mu \\
\Gamma^4_{\mu 1} dx^\mu &= e^{-2V} \chi_4 dx + \frac{1}{2} e^{-2V} \eta_3 dy + \frac{1}{2} e^{-2V} (F_2 \eta_3 + 2F_1 \chi_4) dt + \\
&\quad \frac{1}{2} e^{-2V} (G_2 \eta_3 + \chi_2 + 2G_2 \chi_4) dz = \Gamma^1_{\mu 4} dx^\mu \\
\Gamma^4_{\mu 2} dx^\mu &= \frac{1}{2} e^{-2V} (2F_2 \eta_4 + F_1 \eta_3) dt + \frac{1}{2} e^{-2V} \eta_3 dx + e^{-2V} \eta_4 dy + \\
&\quad \frac{1}{2} e^{-2V} (2G_2 \eta_4 + \eta_2 + G_1 \eta_3) dz = \Gamma^2_{\mu 4} dx^\mu,
\end{aligned} \tag{2.4}$$

where

$$\eta_1 = \frac{\partial G_1}{\partial y} + \frac{\partial G_2}{\partial x}$$

$$\begin{aligned}
\eta_2 &= -\frac{\partial G_2}{\partial t} + \frac{\partial G_2}{\partial x} F_1 + \frac{\partial G_2}{\partial y} F_2 - \frac{\partial F_2}{\partial x} G_1 - \frac{\partial F_2}{\partial y} G_2 + \frac{\partial F_2}{\partial z} \\
\eta_3 &= \frac{\partial F_1}{\partial y} + \frac{\partial F_2}{\partial x} \\
\eta_4 &= 2\frac{\partial V}{\partial t} - 2\frac{\partial V}{\partial x} F_1 - 2\frac{\partial V}{\partial y} F_2 + \frac{\partial F_2}{\partial y} \\
\chi_1 &= 2\frac{\partial V}{\partial x} G_1 + 2\frac{\partial V}{\partial y} G_2 - 2\frac{\partial V}{\partial z} - \frac{\partial G_1}{\partial x} \\
\chi_2 &= -\frac{\partial G_1}{\partial t} + \frac{\partial G_1}{\partial x} F_1 + \frac{\partial G_1}{\partial y} F_2 - \frac{\partial F_1}{\partial x} G_1 - \frac{\partial F_1}{\partial y} G_2 + \frac{\partial F_1}{\partial z} \\
\chi_3 &= 2\frac{\partial V}{\partial x} G_1 + 2\frac{\partial V}{\partial y} G_2 - 2\frac{\partial V}{\partial z} - \frac{\partial G_2}{\partial y} \\
\chi_4 &= 2\frac{\partial V}{\partial t} - 2\frac{\partial V}{\partial x} F_1 - 2\frac{\partial V}{\partial y} F_2 + \frac{\partial F_1}{\partial x} \\
\xi_1 &= \frac{\partial F_1}{\partial y} + \frac{\partial F_2}{\partial x} \\
\xi_2 &= \frac{\partial G_1}{\partial y} - \frac{\partial F_2}{\partial x}.
\end{aligned} \tag{2.5}$$

The property of analyticity of the function $F(x + iy) = F_1 + iF_2$ and similarly of $G = G_1 + iG_2$ tells us that

$$\eta_1 = \eta_3 = 0, \quad \chi_1 = \chi_3. \tag{2.6}$$

We shall not use explicitly this conditions in order to get general expressions for future reference.

Using the tetradic connections given by Eq.(2.4) we can compute the loop variables. In our case we are interested in computing the loop variables for segments in the t and z directions in order to detect if the strings are moving with respect to each other and if they are parallel or not. For a translation in time $\Gamma_\mu dx^\mu = \Gamma_t dt$ with Γ_t being

$$\Gamma_t = \begin{pmatrix} 0 & B & C & A \\ B & 0 & -F & -D \\ C & F & 0 & E \\ A & D & -E & 0 \end{pmatrix}$$

$$= -iFJ_{12} - iDJ_{13} - iEJ_{23} - iBJ_{41} - iCJ_{42} - iAJ_{43}, \quad (2.7)$$

where the boost parameters A, B, C are given by

$$\begin{aligned} A &= \frac{1}{2}e^{-4V}(\eta_2F_2 + \chi_2F_1) \\ B &= \frac{1}{2}e^{-2V}(\eta_3F_2 + 2\chi_4F_1) \\ C &= \frac{1}{2}e^{-2V}(2\eta_4F_2 + \eta_3F_1) \end{aligned} \quad (2.8)$$

and the rotation parameters D, E, F are

$$\begin{aligned} D &= -\frac{1}{2}e^{-2V}(\chi_2 - \eta_1F_2 + 2F_1\chi_1) \\ E &= \frac{1}{2}e^{-2V}(\eta_1F_1 + 2\chi_3F_2) \\ F &= -\frac{1}{2}\left(\xi_1 + 4F_2\frac{\partial V}{\partial x} - 4F_1\frac{\partial V}{\partial y}\right). \end{aligned} \quad (2.9)$$

Then, for a segment in the time direction, the loop variable is a combination of boosts in all directions and rotations. Note that the conditions (2.6) for the generic case do not eliminate any of functions A, \dots, F .

Now, let us consider a segment in the z -direction. In this case we have $\Gamma_{\mu 3}dx^\mu = \Gamma_z dz$, where

$$\begin{aligned} \Gamma_z &= \begin{pmatrix} 0 & B' & C' & A' \\ B' & 0 & -F' & -D' \\ C' & F' & 0 & E' \\ A' & D' & -E' & 0 \end{pmatrix} \\ &= -iF'J_{12} - iD'J_{13} - iE'J_{23} - iB'J_{41} - iC'J_{42} - iA'J_{43} \end{aligned} \quad (2.10)$$

where

$$\begin{aligned} A' &= \frac{1}{2}e^{-4V}(\eta_2G_2 + \chi_2G_1), \quad B' = \frac{1}{2}e^{-2V}(\eta_3G_2 + \chi_2 + 2\chi_4G_1) \\ C' &= \frac{1}{2}e^{-2V}(2\chi_4G_2 + \eta_2 + \eta_3G_1), \quad D' = \frac{1}{2}e^{-2V}(\eta_1G_2 - 2\chi_1G_1) \\ E' &= \frac{1}{2}e^{-2V}(\eta_1G_1 + 2\chi_3G_2), \quad F' = -2\left(G_2\frac{\partial V}{\partial x} - G_1\frac{\partial V}{\partial y}\right). \end{aligned} \quad (2.11)$$

As in the previous case, the loop variable along the z direction is a combination of boosts and rotations which depends on the z coordinate indicating that the cosmic strings crosses at some points, and then they are not parallel. Similar result concerning the segment in t -direction indicates that the strings are moving with respect to each other.

3 LOOP VARIABLES IN A MULTIPLE CHIRAL CONICAL SPACETIME.

In a recent paper Gal'tsov and Letelier [6] showed that the chiral conical spacetime arises naturally from the spinning particle solution of (2+1)-dimensional gravity by an appropriate boost. This chiral conical spacetime provides the gravitational counterpart for the infinitely thin straight chiral strings in the same way that an ordinary conical spacetime is associated with the usual string [10] The metric associated to the chiral conical spacetime [6] (spinning string with cosmic dislocation) is given by

$$ds^2 = -\bar{r}^{8\mu}(dr^2 + r^2d\varphi^2) - (dz + 4J^z d\varphi)^2 + (dt + 4J^t d\varphi)^2, \quad (3.1)$$

where J^t represents the string angular momentum, $2J^z/\pi$ is the analogous of the Burgers-vector of dislocation and μ is the linear mass density of the string. The angle φ takes the values $0 \leq \varphi \leq 2\pi$, and the other variables: $-\infty < t < \infty, 0 < r < \infty$, and $-\infty < z < \infty$. If we consider a cartesian system of coordinates $x = r \cos \varphi, y = r \sin \varphi$, we can write Eq.(3.1) as

$$ds^2 = -e^{-4V}(dx^2 + dy^2) - (dz + 4J^z \frac{xdy - ydx}{r^2})^2 + (dt - 4J^t \frac{xdy - ydx}{r^2})^2 \quad (3.2)$$

with $V = 2\mu \ln r$.

The generalization of the chiral cone to a multiple chiral cone can [6] be obtained by introducing the parameters $\mu_i, J_i^t, J_i^z, i = 1, 2, \dots, N$, defining each chiral string located at the points $\vec{r} = \vec{r}_i$ of the plane $z = 0$. The

resulting metric has the form of Eq.(3.2) with the following interchanges

$$\begin{aligned}
J^t \frac{xdy - ydx}{r^2} &\rightarrow \sum_{i=1}^N J_i^t \frac{(x - x_i)dy - (y - y_i)dx}{|\vec{r} - \vec{r}_i|^2} \\
J^z \frac{xdy - ydx}{r^2} &\rightarrow \sum_{i=1}^N J_i^z \frac{(x - x_i)dy - (y - y_i)dx}{|\vec{r} - \vec{r}_i|^2} \\
V = 2\mu \ln r &\rightarrow V = \sum_{i=1}^N \mu_i \ln[r^2 - 2rr_i \cos(\varphi - \varphi_i) + r_i^2] \quad (3.3)
\end{aligned}$$

As a consequence of Eq.(3.3), the spacetime generated by N multiple chiral cosmic string can be written as [6]

$$\begin{aligned}
ds^2 = -e^{-4V}(dx^2 + dy^2) - [dz - \sum_{i=1}^N B_i(W_i^1 dy - W_i^2 dx)]^2 + \\
[dt - \sum_{i=1}^N A_i(W_i^1 dy - W_i^2 dx)]^2, \quad (3.4)
\end{aligned}$$

where

$$\begin{aligned}
A_i = 4J_i^t, \quad B_i = 4J_i^z \\
W_i^1 = \frac{x - x_i}{|\vec{r} - \vec{r}_i|^2}, \quad W_i^2 = \frac{y - y_i}{|\vec{r} - \vec{r}_i|^2}. \quad (3.5)
\end{aligned}$$

In the previous case, static one, the holonomy transformation was calculated directly from the metric. For the present case, stationary one, it is possible to do the same, but with a slightly redefinition of the loop variables. This can be done because this solution of the Einstein equation can be patched together from flat coordinates patches but connected by some additional matching condition in order to take into account the helical structure and the shift in the z -direction. First of all let us recover a previous result concerning the holonomy in the static case [11], specifically in the space of a multiple cosmic string [12]. In this case one calculates the holonomy transformation corresponding to circles in the xy -plane directly from the metric. Then, when we parallel transport a vector around multiple cosmic strings

at rest at $\vec{r} = \vec{r}_i$ along a circle, this vector acquires a phase given by [11] $U(C) = \exp[-8\pi i(\sum_{j=1}^N \mu_j)J_{12}]$, where J_{12} is the generator of rotations in the xy -plane, around the z -axis. Therefore, when we go around the multiple cosmic string from the point (\vec{x}, t) to (\vec{x}', t') , the column vectors (\vec{x}, t) and (\vec{x}', t') are related by

$$\begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix} = \begin{pmatrix} \cos(8\pi\tilde{\mu}) & \sin(8\pi\tilde{\mu}) & 0 & 0 \\ -\sin(8\pi\tilde{\mu}) & \cos(8\pi\tilde{\mu}) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}, \quad (3.6)$$

where we have set

$$\tilde{\mu} = \sum_{j=1}^N \mu_j. \quad (3.7)$$

Since the space-time outside the multiple cosmic string is locally flat, we can describe the analytic solution purely in terms of spacetime patches with Minkowski metric, but connected by some matching conditions which are given by Eq.(3.6), that relates points (\vec{x}, t) and (\vec{x}', t') along the edges.

As in the multiple cosmic string case, the space-time of the multiple chiral cosmic string is locally flat, and consequently we can describe it in terms of space-time patches with Minkowski metric, but connected by some conditions which are the same as in the static multiple string case, except those concerning the t and z coordinates. These conditions are expressed by relating points (\vec{x}, t) and (\vec{x}', t') as follows

$$\begin{aligned} x' &= \cos(8\pi\tilde{\mu})x + \sin(8\pi\tilde{\mu})y \\ y' &= -\sin(8\pi\tilde{\mu})x + \cos(8\pi\tilde{\mu})y \\ z' &= z + 8\pi\left(\frac{r}{|\vec{r} - \vec{r}_i|^2}\right)^2 J_i^z \\ t' &= t + 8\pi\left(\frac{r}{|\vec{r} - \vec{r}_i|^2}\right)^2 J_i^t, \end{aligned} \quad (3.8)$$

where we are considering as paths circles in the xy -plane.

The transformations given by Eq.(3.8) can cast in the form of a homogeneous matrix multiplication as follows: let M_A^B be a five dimensional matrix, with A and B running from 1 to 5. We take M_ν^μ equal to the rotation

matrix given by [11], $U(C) = \exp(-8\pi i \tilde{\mu} J_{12})$, $M_5^A = 8\pi (\frac{r}{|\vec{r}-\vec{r}_i|^2})^2 J_i^t$ and $M_5^z = 8\pi (\frac{r}{|\vec{r}-\vec{r}_i|^2})^2 J_i^z$, so that

$$\begin{aligned} \begin{pmatrix} x' \\ y' \\ z' \\ t' \\ 1 \end{pmatrix} &= \begin{pmatrix} \cos(8\pi \tilde{\mu}) & \sin(8\pi \tilde{\mu}) & 0 & 0 & 0 \\ -\sin(8\pi \tilde{\mu}) & \cos(8\pi \tilde{\mu}) & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 8\pi (\frac{r}{|\vec{r}-\vec{r}_i|^2})^2 J_i^z \\ 0 & 0 & 0 & 1 & 8\pi (\frac{r}{|\vec{r}-\vec{r}_i|^2})^2 J_i^t \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \\ 1 \end{pmatrix} \\ &= \exp[-8\pi (\frac{r}{|\vec{r}-\vec{r}_k|^2})^2 J_k^z M_3] \exp[-8\pi (\frac{r}{|\vec{r}-\vec{r}_k|^2})^2 J_k^t M_4] \times \\ &\quad \exp(-8\pi \tilde{\mu} J_{12}) \begin{pmatrix} x \\ y \\ z \\ t \\ 1 \end{pmatrix}, \end{aligned} \quad (3.9)$$

where M_3 and M_4 are the following matrices

$$M_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad M_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (3.10)$$

Equation (3.9) is the exact expression for the holonomy for circles in the space-time. By defining $y^A = (y^\mu, 1)$ we can cast the conditions (3.8) as $y'^A = M_B^A y^B$ which tells us that the points (\vec{x}, t) and (\vec{x}', t') along the edges are related by the phase given by Eq. (3.9) that depends on the parameters μ_j, J_j^t and J_j^z .

The existence of locally flat coordinates in this space-time permits us to consider Eq. (3.9) as a ‘‘parallel transport’’ matrix. Then we can say that when we carry a vector along a circle in this space-time it acquires a phase that depends on μ_j, J_j^t and J_j^z which prevents it to be equal to the unit matrix. This effect is exclusively due to the non trivial topology of the space-time under consideration. This is a gravitational analogue [11, 13] of the Aharanov-Bohm effect [14], but in this case, purely at the classical level.

We can also compute the holonomy transformations for circles in the multiple chiral conical space-time in the context of the Einstein-Cartan theory. In this case the connection 1-forms appropriately chosen give us [7]

$$\Gamma_{\mu 2}^1 dx^\mu = 2(\partial_x V dy - \partial_y V dx) = -\Gamma_{\mu 1}^2 dx^\mu \quad (3.11)$$

which can be written in cylindrical coordinates ($x^1 = r, x^2 = \varphi, x^3 = z, x^4 = t$) as

$$\Gamma_{\mu 2}^1 dx^\mu = -\frac{2}{r} \frac{\partial V}{\partial \varphi} dr - \left(1 - 2r \frac{\partial V}{\partial r}\right) d\varphi = -\Gamma_{\mu 1}^2 dx^\mu. \quad (3.12)$$

Now, consider the same previous circle, at constant time. In this case $U_{2\pi,0}(C)$ is given by

$$U_{2\pi,0}(C) = \exp\left(\int_0^{2\pi} \Gamma_\varphi d\varphi\right) = \exp\left[-8\pi i \left(\sum_{j=1}^N \mu_j\right) J_{12}\right] \quad (3.13)$$

where

$$\Gamma_\varphi = i \left[1 - 4 \sum_{j=1}^N \mu_j \frac{R(R - r_j \cos(\varphi - \varphi_j))}{(R^2 - 2Rr_j \cos(\varphi - \varphi_j) + r_j^2)}\right] J_{12}, \quad (3.14)$$

R being the radius of the circle. Into Eq. (3.12) we have dropped out the factor $\exp(-2\pi i J_{12})$ which is equal to the 4×4 identity matrix. Note that in this case the holonomy transformation has a simple expression. It does not carry information concerning angular momentum and torsion and coincides with a previous result [11] concerning the multiple cosmic string solution [12]. Then, the concept of holonomy can be used to detect different connections that come out from Einstein and Einstein-Cartan theories.

4 LOOP VARIABLES IN OTHER CONICAL SPACE-TIME.

The purpose of this section is to complete a previous work [15] in which we examine the possibility of the use of loop variables to characterize the presence of topological defects of spacetime. We considered a single domain

wall crossed by multiple cosmic strings. Other simple examples of conical spacetimes include two domain walls crossed by a cosmic string of linear mass density μ and two planes topological defects plus a cosmic string. A variety of conical space-time of different topologies can be found in [9].

Now let us consider two domain walls parallel to the xy -plane that intersect the z -axis at $\pm h$ and crossed by a single cosmic string. The metric corresponding to the spacetime generated by this configuration is

$$ds^2 = e^{-4\pi\sigma|h^2-z^2|} - e^{4\pi\sigma t} \rho^{-4\mu} (d\rho^2 + \rho^2 d\varphi^2) - 4z^2 dz^2 + dt^2, \quad (4.1)$$

where σ is the matter density of the wall and μ is the linear mass density of the cosmic string.

Proceeding in the same way of the previous cases let us define the appropriate 1-forms θ^a that give the usual flat spacetime limit $\sigma = 0, \mu = 0$ as

$$\begin{aligned} \theta^1 &= e^{-2\pi\sigma|h^2-z^2|+2\pi\sigma t} (\rho^{-2\mu} \cos \varphi d\rho - \rho^{-2\mu+1} \sin \varphi d\varphi) \\ \theta^2 &= e^{-2\pi\sigma|h^2-z^2|+2\pi\sigma t} (\rho^{-2\mu} \sin \varphi d\rho + \rho^{-2\mu+1} \cos \varphi d\varphi) \\ \theta^3 &= 2ze^{-2\pi\sigma|h^2-z^2|} dz \\ \theta^4 &= e^{-2\pi\sigma|h^2-z^2|} dt \end{aligned} \quad (4.2)$$

In a coordinate system ($x^1 = \rho, x^2 = \varphi, x^3 = z, x^4 = t$), the tetrad vectors are given by

$$\begin{aligned} e_1^{(1)} &= \rho^{-2\mu} e^{-2\pi\sigma|h^2-z^2|+2\pi\sigma t} \cos \varphi, & e_2^{(1)} &= -\rho^{-2\mu+1} e^{-2\pi\sigma|h^2-z^2|+2\pi\sigma t} \sin \varphi \\ e_1^{(2)} &= \rho^{-2\mu} e^{-2\pi\sigma|h^2-z^2|+2\pi\sigma t} \sin \varphi, & e_2^{(2)} &= \rho^{-2\mu+1} e^{-2\pi\sigma|h^2-z^2|+2\pi\sigma t} \cos \varphi \\ e_3^{(3)} &= 2ze^{-2\pi\sigma|h^2-z^2|}, & e_4^{(4)} &= e^{-2\pi\sigma|h^2-z^2|}. \end{aligned} \quad (4.3)$$

Using these results we can compute the tetradic connections which are

$$\begin{aligned} \Gamma_{\mu 2}^1 dx^\mu &= (1 - 4\mu) d\varphi = -\Gamma_{\mu 1}^2 dx^\mu \\ \Gamma_{\mu 3}^1 dx^\mu &= 2\pi\sigma \frac{|h^2 - z^2|}{h^2 - z^2} e^{2\pi\sigma t} \rho^{-4\mu} d\varphi = -\Gamma_{\mu 1}^3 dx^\mu \end{aligned}$$

$$\begin{aligned}
\Gamma_{\mu 3}^2 dx^\mu &= 2\pi\sigma \frac{|h^2 - z^2|}{h^2 - z^2} e^{2\pi\sigma t} \rho^{-4\mu+1} d\varphi = -\Gamma_{\mu 2}^3 dx^\mu \\
\Gamma_{\mu 1}^4 dx^\mu &= 2\pi\sigma e^{2\pi\sigma t} \rho^{-4\mu} d\varphi = \Gamma_{\mu 4}^1 dx^\mu \\
\Gamma_{\mu 2}^4 dx^\mu &= 2\pi\sigma e^{2\pi\sigma t} \rho^{-4\mu+1} d\varphi = \Gamma_{\mu 4}^2 dx^\mu \\
\Gamma_{\mu 3}^4 dx^\mu &= 2\pi\sigma \frac{|h^2 - z^2|}{h^2 - z^2} dt = \Gamma_{\mu 4}^3 dx^\mu.
\end{aligned} \tag{4.4}$$

Let us consider the path C as a circle centered on the z axis with radius R lying on a plane parallel to the xy plane at a fixed time. In this case we have that $\Gamma_\mu dx^\mu = \Gamma_\varphi d\varphi$ where

$$\Gamma_\mu = i(1 - 4\mu)J_{12} - 2\pi i\sigma e^{2\pi\sigma t} \rho^{-4\mu+1} \left(J_{24} + \frac{|h^2 - z^2|}{h^2 - z^2} J_{23} \right). \tag{4.5}$$

From Eq.(4.5) we get

$$U_{2\pi,0}(C) = \exp \left[-8\pi i\mu J_{12} - 4\pi^2 i\sigma e^{2\pi\sigma t} \rho^{-4\mu+1} \left(J_{24} + \frac{|h^2 - z^2|}{h^2 - z^2} J_{23} \right) \right]. \tag{4.6}$$

Equation (4.6) is the exact expression for the holonomy transformation for a circle with center at the cosmic string and that is parallel to the domain walls.

The holonomy transformation associated to the circle C that correspond to the domain walls only ($\mu = 0$) and to the cosmic string only ($\sigma = 0$) are given, respectively, by

$$U_{2\pi,0}(C) = \exp \left[-4\pi^2 i\sigma e^{2\pi\sigma t} \rho \left(J_{24} - \frac{|h^2 - z^2|}{h^2 - z^2} J_{23} \right) \right] \tag{4.7}$$

and

$$U_{2\pi,0}(C) = \exp(-8\pi i\mu J_{12}). \tag{4.8}$$

From these results we see that the holonomy transformations detect the topological defects in all cases. In particular, for the domain walls plus a cosmic string, the value of $U_{2\pi,0}(C)$ depends on the radius of the circle. Note that $U_{2\pi,0}(C)$ distinguishes.

The regions $z < -h$, $-h < z < h$ and $z > h$ in the cases of two domain walls crossed by a cosmic string and two domain walls only.

In the case of two planes topological defects crossed by a cosmic string with equation of state $p = -\gamma\sigma$ ($\gamma < 1$), the metric so given by

$$ds^2 = -\rho^{-8\mu} e^H (d\rho^2 + \rho^2 dy^2) + e^F (-4z^2 dz^2 + dt^2), \quad (4.9)$$

where

$$\begin{aligned} e^H &= (t - |h^2 - z^2|)^{\frac{1}{2(1-\gamma)}} \\ e^F &= \frac{1}{8\pi\sigma(1-\gamma)} \frac{1}{t - |h^2 - z^2|} \left[\frac{t - |h^2 - z^2|}{t + |h^2 - z^2|} \right]^{\frac{1}{4(\gamma-1)}}. \end{aligned} \quad (4.10)$$

The holonomy transformation for the same circle C described above is given by

$$U_{2\pi,0}(C) = \exp \left[-8\pi i \mu J_{12} - \frac{\pi i e^{(H-F)/2}}{4(1-\gamma)(t - |h^2 - z^2|)} \rho^{-4\mu+1} \left(J_{24} - \frac{|h^2 - z^2|}{h^2 - z^2} J_{23} \right) \right]. \quad (4.11)$$

To get Eq(4.11), we have used the following relations between the tetradic connections

$$\begin{aligned} \Gamma_{\mu 2}^1 dx^\mu &= (1 - 4\mu) dy = -\Gamma_{\mu 1}^2 dx^\mu \\ \Gamma_{\mu 3}^1 dx^\mu &= \frac{e^{-F/2}}{2z} \frac{\partial}{\partial z} (e^{H/2}) \rho^{-4\mu} d\rho = -\Gamma_{\mu 1}^3 dx^\mu \\ \Gamma_{\mu 3}^2 dx^\mu &= \frac{e^{-F/2}}{2z} \frac{\partial}{\partial z} (e^{H/2}) \rho^{-4\mu+1} dy = -\Gamma_{\mu 2}^3 dx^\mu \\ \Gamma_{\mu 1}^4 dx^\mu &= e^{-F/2} \frac{\partial}{\partial t} (e^{H/2}) \rho^{-4\mu} d\rho = \Gamma_{\mu 4}^1 dx^\mu \\ \Gamma_{\mu 2}^4 dx^\mu &= e^{-F/2} \frac{\partial}{\partial t} (e^{H/2}) \rho^{-4\mu+1} dy = \Gamma_{\mu 4}^2 dx^\mu \\ \Gamma_{\mu 3}^4 dx^\mu &= \frac{e^{-F/2}}{2z} \frac{\partial}{\partial z} (e^{F/2}) dt + 2ze^{-F/2} \frac{\partial}{\partial t} (e^{F/2}) dz = \Gamma_{\mu 4}^3 dx^\mu. \end{aligned} \quad (4.12)$$

As in the previous case corresponding to two domain walls plus a string, the holonomy transformation detects the topological defects and distinguishes the different regions $z < -h$, $-h < z < h$ and $z > h$.

5 CONCLUDING REMARKS

We have shown by explicit computation from the metric corresponding to a multiple parallel chiral cosmic strings that the loop variables are combinations of rotations around the three axis and boosts with appropriate parameters that depend on the characteristics of each chiral string defined by μ_i, J_i^t and J_i^z . The holonomy transformation, in this spacetime, assumes a simple form in the context of the Einstein-Cartan theory which recover an expression for the case of multiple cosmic strings. The loop variables associated with the multiple moving crossed cosmic strings are also combinations of rotations and boosts in the directions of the three spatial axis, and permit us to conclude that these are not parallel and are moving with respect to each other.

In the of two domain walls and two planes topological defects plus cosmic string, the holonomy transformation distinguishes the presence of strings and membranes and depends on whether the loop encircles the strings and in which side of the planes topological defects are located.

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