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Involutions on the Algebra of Physical Observables from Reality Conditions

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Abstract

Some aspects of the algebraic quantization programme proposed by Ashtekar are revisited in this article. It is proved that, for systems with first-class constraints, the involution introduced on the algebra of quantum operators via reality conditions can never be projected unambiguously to the algebra of physical observables, ie, of quantum observables modulo constraints. It is nevertheless shown that, under sufficiently general assumptions, one can still induce an involution on the algebra of physical observables from reality conditions, though the involution obtained depends on the choice of particular representatives for the equivalence classes of quantum observables and this implies an additional ambiguity in the quantization procedure suggested by Ashtekar.

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I. Introduction

Recently, Ashtekar *et al*[1–3] have ellaborated a programme for the non-perturbative quantization of dynamical systems with first-class constraints. This programme is specially designed to deal with the problem of quantizing general relativity, and has already been carried out successfully in a number of lower dimensional gravitational models, including minisuperspaces,[4,5] midisuperspaces[6] and 2+1 gravity.[1,3,7] The programme proposed by Ashtekar is an extension, based on the algebraic approach to quantum mechanics,[8] of Dirac's canonical quantization method.[9] One of the main novelties with respect to Dirac's procedure is the introduction of a prescription to find the inner product in the space of quantum states. This allows one to adhere to the standard probabilistic interpretation of quantum mechanics when the quantization can be achieved.

Ashtekar's programme consists of a series of steps that, after completion, should provide us with a consistent quantum theory. It can be applied, in principle, to any classical system whose phase space Γ is a real symplectic manifold.[1]

One must first choose a subspace S of the vector space of smooth complex functions on Γ. This subspace must contain the unit function and be closed both under complex conjugation and Poisson brackets.[2] In addition, S has to be complete, in the sense that any sufficiently regular complex function on phase space should be expressable as a sum of products of elements in S (or as a limit of this type of sums).[2]

Each element X in S is to be regarded as an elementary classical variable which is unambiguously associated with an abstract operator $\hat{X}$. One then constructs the free associative algebra generated by these elementary quantum operators. On this algebra, one imposes the commutation relations that follow from the classical Poisson brackets, namely, if $X, Y \in S$, one must demand that $[\hat{X}, \hat{Y}] = i\hbar\widehat{\{X, Y\}}$

(at least up to terms proportional to $\hbar^2$). If there exist algebraic relations between the elements in S (eg, when the dimension of S is greater than that of Γ), such relations have also to be imposed on the corresponding quantum operators, with a suitable choice of factor ordering, if needed.[2] The algebra of operators obtained in this way will be called $\mathcal{A}$.

At this point one should promote the complex conjugation relations in S to an involution on $\mathcal{A}$. We recall that an involution $\star$ on the algebra $\mathcal{A}$ is a map $\star : \mathcal{A} \rightarrow \mathcal{A}$ that satisfies

$$(\hat{X}^{\star})^{\star} = \hat{X} \ , \tag{1.1}$$

$$(\hat{X} + \lambda \hat{Y})^{\star} = \hat{X}^{\star} + \overline{\lambda}\, \hat{Y}^{\star} \ , \qquad (\hat{X}\hat{Y})^{\star} = \hat{Y}^{\star}\hat{X}^{\star} \ , \tag{1.2}$$

for all $\hat{X}, \hat{Y} \in \mathcal{A}$ and complex numbers λ. Here, $\overline{\lambda}$ is the complex conjugate to λ. To introduce the desired involution on $\mathcal{A}$, one can proceed in the following manner. For every $X, Y \in S$ such that Y is the complex conjugate to X, define $\hat{X}^{\star} = \hat{Y}$, and use properties (1.2) to extend this definition to all the operators in $\mathcal{A}$. It is not difficult to check then that one gets an involution on $\mathcal{A}$ provided that the $\star$-operation is compatible with the structure of this algebra. This amounts to require that the commutation and algebraic relations between elementary operators which have been imposed on $\mathcal{A}$ are stable under the $\star$-operation, in the sense that their $\star$-conjugates do not supply any new relation which is not implied by the original ones. We will assume hereafter that this is in fact the case, and denote the resulting $\star$-algebra by $\mathcal{A}^{(\star)}$. The $\star$-relations in $\mathcal{A}^{(\star)}$ are usually called reality conditions,[1] for they capture the complex conjugation relations between elementary classical variables.

The next step in the quantization consists in finding a faithful representation for the abstract algebra $\mathcal{A}$ by linear operators acting on a complex vector space V. If the classical system possesses first-class constraints $\{C_i\}$, these constraints must now be explicitly represented by operators $\{\hat{C}_i\}$. In general, a choice of factor ordering, and of regularization in infinite dimensional systems,[2,3] are needed at this point in order to get a consistent algebra of quantum constraints,[9] that is, to guarantee that

$$[\hat{C}_i, \hat{C}_j] = \hat{f}_{ij}{}^k \hat{C}_k \ , \tag{1.3}$$

where $\hat{f}_{ij}{}^k \in \mathcal{A}$ and we use the convention that pairs of contracted indices are summed over.

The kernel $V_p \subset V$ of the constraints $\{\hat{C}_i\}$ supplies the vector subspace of quantum states. One must then determine the subalgebra $\mathcal{A}_p \subset \mathcal{A}$ of operators which leave V_p invariant. These operators commute weakly with the quantum constraints,

$$\hat{A} \in \mathcal{A}_p \quad \Longleftrightarrow \quad [\hat{A}, \hat{C}_i] = \hat{h}_i{}^j \hat{C}_j \qquad (\hat{h}_i{}^j \in \mathcal{A}) \ . \tag{1.4}$$

Let us define now

$$\mathcal{I}_C \equiv \{\hat{X}^i \hat{C}_i ; \ \hat{X}^i \in \mathcal{A}\} \ . \tag{1.5}$$

Using Eqs. (1.3,4) one can show that $\mathcal{I}_C \subset \mathcal{A}_p$ and that, $\forall \hat{I} \in \mathcal{I}_C$ and $\forall \hat{A} \in \mathcal{A}_p$, both $\hat{A}\hat{I}$ and $\hat{I}\hat{A}$ belong to $\mathcal{I}_C$, so that $\mathcal{I}_C$ is an ideal of $\mathcal{A}_p$. On the other hand, if $\hat{A} \in \mathcal{A}_p$, all the operators of the form $\hat{B} = \hat{A} + \hat{I}$, with $\hat{I} \in \mathcal{I}_C$, have exactly the same action on quantum states, for V_p is anihilated by the quantum constraints. In order to obtain the algebra $\mathcal{A}'_p$ of operators with a well-defined action on V_p, one should therefore take the quotient of $\mathcal{A}_p$ by the ideal $\mathcal{I}_C$:

$$\mathcal{A}'_p \equiv \mathcal{A}_p / \mathcal{I}_C \ . \tag{1.6}$$

The operators in $\mathcal{A}'_p$ are the quantum physical observables of the system.[2]

The quantization programme presented so far leaves a certain freedom in the following steps: a) the selection of the subspace S of elementary classical variables, b) the construction of the linear representation for the algebra $\mathcal{A}$ of quantum operators, and c) the choice of factor ordering in the quantum constraints $\{\hat{C}_i\}$. The final result of the quantization process will depend on these inputs.[2] In particular, Ashtekar and Tate[2] assumed at this stage that, with a judicious choice of such inputs (and at least for a large variety of physical systems), the involution defined on $\mathcal{A}^{(\star)}$ would unambiguously induce an involution on $\mathcal{A}'_p$. Actually, the $\star$-relations will project unambiguously to the algebra of physical observables only if two conditions are fulfilled. On the one hand, $\mathcal{A}_p \subset \mathcal{A}$ must be invariant under the $\star$-operation: $\forall \hat{A} \in \mathcal{A}_p$, $\hat{A}^\star \in \mathcal{A}_p$. On the other hand, it is necessary that $\mathcal{I}_C \subset \mathcal{A}_p$ be a $\star$-ideal of $\mathcal{A}_p$: $\forall \hat{I} \in \mathcal{I}_C$, $\hat{I}^\star \in \mathcal{I}_C$. When this is the case, the $\star$-operation provides a uniquely defined map between equivalence classes in $\mathcal{A}'_p$ which satisfies the properties (1.1,2) of an involution. Such an involution will be denoted again by $\star$, and the resulting $\star$-algebra of physical observables by ${\mathcal{A}'_p}^{(\star)}$.

The idea suggested by Ashtekar[1−3] is to employ the involution on ${\mathcal{A}'_p}^{(\star)}$ to select the inner product $<,>$ on V_p and, therefore, the Hilbert space $\mathcal{H}$ of physical states (normalizable quantum states). More specifically, he proposed to determine the inner product on V_p by demanding that the $\star$-relations between physical observables are realized as adjoint relations on the Hilbert space $\mathcal{H}$, ie,

$$< \Psi, \hat{A}'\Phi > = < \hat{B}'\Psi, \Phi > \qquad \forall\, \Phi, \Psi \in \mathcal{H}, \quad \forall \hat{A}', \hat{B}' = (\hat{A}')^\star \in {\mathcal{A}'_p}^{(\star)}. \tag{1.7}$$

Rendall showed[10] that this condition is such a severe restriction on the inner product that, if an admissible inner product exists, it is unique (up to a positive global factor) under very general assumptions.

This completes the quantization programme put forward by Ashtekar. If this programme can be carried out for a given classical system, one would arrive at a mathematically consistent quantum theory in which real physical observables would be represented by self-adjoint operators acting on a Hilbert space of physical states.

The purpose of this work is to demonstrate however that one of the steps of the above quantization method can never be achieved. We will prove in Section II that the $\star$-relations in $\mathcal{A}^{(\star)}$ can never be projected unambiguously to the algebra of physical observables. This problem can be nonetheless overcome by slightly modifying

Ashtekar's programme, as we will show in Section III. The price to be paid is to allow a new freedom in the quantization process. A particular procedure to introduce an involution on $\mathcal{A}'_p$ from reality conditions should then be adopted. The subtleties that arise in defining such an involution are illustrated in Section IV by considering some simple physical systems. We finally discuss the physical implications of our results and conclude in Section V.

II. Ambiguities in the Reality Conditions on Physical Observables

We want to prove that reality conditions (ie, the $\star$-relations between quantum operators) never project unambiguously to the algebra of physical observables when there exist first-class constraints on the system. We will assume that the faithful, linear representation constructed for the algebra $\mathcal{A}$ of quantum operators is irreducible. Otherwise, one should decompose it in irreducible components, and apply the proof to follow to each component separately.

We have seen that, in order to obtain a uniquely defined involution on physical observables from reality conditions, it is necessary that both $\mathcal{A}_p$ and $\mathcal{I}_C$ be invariant under the $\star$-operation. In particular, we should have

$$\forall \hat{I} \in \mathcal{I}_C, \quad \hat{I}^{\circ} \equiv \hat{I}^{\star} \in \mathcal{I}_C \ . \tag{2.1}$$

Taking $\hat{I}$ equal to each of the quantum constraints and recalling definiton (1.5), we hence get

$$\hat{C}_i^{\star} = \hat{Y}_i^{\ j} \hat{C}_j \ , \tag{2.2}$$

for some $\hat{Y}_i^{\ j} \in \mathcal{A}$. Select now one of the quantum constraints, eg, $\hat{C}_1$, and consider all the operators of the form $\hat{I}_1 = \hat{Z}\hat{C}_1 \in \mathcal{I}_C$, with $\hat{Z} \in \mathcal{A}$. Employing again condition (2.1), and using Eq. (2.2), we obtain

$$(\hat{I}_1)^{\star} = \hat{C}_1^{\star} \hat{Z}^{\star} = \hat{Y}_1^{\ j} \hat{C}_j \hat{Z}^{\star} \equiv \hat{I}_1^{\circ} = \hat{X}_1^{\ k} \hat{C}_k \ , \tag{2.3}$$

where we have expressed $\hat{I}_1^\circ \in \mathcal{I}_C$ as a combination of quantum constraints.

On the other hand, the image $\hat{Z}^\star$ of all the operators $\hat{Z} \in \mathcal{A}$ is again the whole algebra $\mathcal{A}$, because the $\star$-operation is an involution. Relation (2.3) therefore implies that, $\forall \hat{Z} \in \mathcal{A}$, there exist $\hat{X}_1^{\ k} \in \mathcal{A}$ such that

$$\hat{Y}_1^{\ j} \hat{C}_j \hat{Z} = \hat{X}_1^{\ k} \hat{C}_k \ . \tag{2.4}$$

This identity between operators must hold on any element of V, the vector space on which $\mathcal{A}$ has been represented. Choosing then $\Phi \in V_p \subset V$ with Φ different from zero, it follows from Eq. (2.4) that, $\forall \hat{Z} \in \mathcal{A}$,

$$\hat{Y}_1^{\ j} \hat{C}_j (\hat{Z}\Phi) = \hat{X}_1^{\ k} \hat{C}_k \Phi = 0 \ , \tag{2.5}$$

for the physical state Φ is anihilated by all quantum constraints. Besides, since the representation constructed is irreducible and $\Phi \neq 0$, the range of $\hat{Z}\Phi$ $(\forall \hat{Z} \in \mathcal{A})$ must be the whole vector space V. So, the above equation states that V is the kernel of the operator $\hat{Y}_1^{\ j} \hat{C}_j$. Being the representation for $\mathcal{A}$ faithful, we then must have

$$\hat{Y}_1^{\ j} \hat{C}_j = \hat{0} \ . \tag{2.6}$$

But this is clearly inconsistent with the fact that the $\star$-operation is an involution, because, using Eqs. (2.2) and (2.6), we get that $\hat{C}_1 = (\hat{C}_1^\star)^\star = \hat{0}$. In this way, we conclude that, when there exist first-class constraints, $\mathcal{I}_C$ is never a $\star$-ideal of $\mathcal{A}_p$ and, therefore, reality conditions do not project unambiguously to the algebra of physical observables.

Thus, the $\star$-operation never provides a uniquely defined map between equivalence classes in $\mathcal{A}'_p$. Moreover, even though one could find a representative $\hat{A}$ for a given physical observable $\hat{A}'$ such that $\hat{A}^\star \in \mathcal{A}_p$, it is not yet true that the $\star$-conjugates of all the operators in the equivalence class $\hat{A}'$ (ie, the operators $\hat{A} + \hat{I}$, with $\hat{I} \in \mathcal{I}_C$) belong at least to the algebra $\mathcal{A}_p$.

For the sake of an example, let us consider a classical system whose phase space admits a set of global coordinates of the form $s \equiv \{t, H, x, p\}$, with $t, H, x, p \in \mathbb{R}$, and H and p the momenta canonically conjugate to t and x, respectively. Suppose, in addition, that there exists only one first-class constraint on the system, given by $H = 0$. This extremely simple example describes, for instance, a Kantowski-Sachs model with positive cosmological constant.[5]

As elementary classical variables, we can choose the complex vector space spanned by s and the unity. The $\star$-operation on the corresponding algebra $\mathcal{A}$ of quantum operators is defined by

$$\hat{t}^{\star} = \hat{t}\ , \qquad \hat{H}^{\star} = \hat{H}\ , \tag{2.7}$$

$$\hat{x}^{\star} = \hat{x}\ , \qquad \hat{p}^{\star} = \hat{p}\ , \qquad \hat{1}^{\star} = \hat{1}\ , \tag{2.8}$$

and the properties (1.2) of an involution. The only quantum constraint is $\hat{H} = 0$. On the other hand, it is not difficult to prove that the equivalence classes in $\mathcal{A}'_p$ of the operators $\hat{1}$, $\hat{x}$ and $\hat{p}$ form a complete set of physical observables. Using Eq. (2.8), it then follows that each equivalence class of observables possesses at least a representative whose $\star$-conjugate belongs to the algebra $\mathcal{A}_p$. However, the $\star$-image of different representatives do not coincide in general (not even modulo the constraint $\hat{H} = 0$). Let us take, for instance, the operators $\hat{x}$, $\hat{x} + \hat{t}\hat{H}$ and $\hat{x} + (\hat{t})^2\hat{H}$, all of them in the same equivalence class of physical observables. From Eqs. (2.7,8) and the commutator $[\hat{t}, \hat{H}] = i\hbar\hat{1}$, we get

$$\hat{x}^{\star} = \hat{x}, \quad \left(\hat{x} + \hat{t}\hat{H}\right)^{\star} = \left(\hat{x} + \hat{t}\hat{H}\right) - i\hbar\hat{1}, \quad \left(\hat{x} + (\hat{t})^2\hat{H}\right)^{\star} = \left(\hat{x} + (\hat{t})^2\hat{H}\right) - 2i\hbar\hat{t}. \tag{2.9}$$

Hence, the $\star$-conjugate to $\hat{x}$ and to $\hat{x} + \hat{t}\hat{H}$ belong to different classes of observables, whereas the $\star$-conjugate to $\hat{x} + (\hat{t})^2\hat{H}$ is not even in $\mathcal{A}_p$.

III. Involutions on Physical Observables

We have seen that the $\star$-relations in $\mathcal{A}^{(\star)}$ do not project unambiguously to $\mathcal{A}'_p$, because the $\star$-operation never maps all the representatives of a class of physical observables into another equivalence class. In order to define the $\star$-conjugate to a physical observable, one is therefore forced to choose first a particular representative for it. We now want to discuss under which circumstances it is possible to introduce an involution on $\mathcal{A}'_p$ by this procedure, namely, by selecting a particular representative for each equivalence class in $\mathcal{A}'_p$.

To construct an involution $\star$ on $\mathcal{A}'_p$, it actually suffices to define the $\star$-operation on an (over-)complete set of physical observables, and demand that this operation verifies conditions (1.2). Suppose then that $\{\hat{U}'_a\}$ is a complete set in $\mathcal{A}'_p$, that is, that $\mathcal{A}'_p$ can be obtained from the free associative algebra $\mathcal{B}'$ generated by $\{\hat{U}'_a\}$ by imposing the commutation relations between the observables $\hat{U}'_a$, as well as any algebraic relation that could exist between them. Assume also that one can find representatives $\{\hat{U}_a\}$ of the observables $\{\hat{U}'_a\}$ such that their $\star$-conjugates $\{\hat{U}^\star_a\}$ belong to $\mathcal{A}_p$. One might then hope that the $\star$-operation on $\mathcal{A}'_p$ could be defined by

$$(\hat{U}'_a)^\star = (\hat{U}^\star_a)' \ , \tag{3.1}$$

where $(\hat{U}^\star_a)'$ denotes the equivalence class of $\hat{U}^\star_a$. However, we will prove that the assumptions introduced above do not guarantee that Eq. (3.1) leads to a well-defined involution on the algebra of physical observables.

The proof makes use of the fact that, being $\{\hat{U}'_a\}$ complete in $\mathcal{A}'_p$, any operator in the algebra $\mathcal{A}_p$ should be expressable, modulo an element in the ideal $\mathcal{I}_C$ (1.5), as (possibly a limit of) a sum of products of the representatives $\{\hat{U}_a\}$. In particular, since every $\hat{U}^\star_a \in \mathcal{A}_p$, one gets

$$\hat{U}^\star_a = \sum_n \lambda_a^{b_1 \ldots b_n} \hat{U}_{b_1} \ldots \hat{U}_{b_n} + \hat{X}_a^{\ i} \hat{C}_i \ , \tag{3.2}$$

with $\hat{X}_a^{\,i} \in \mathcal{A}$ and the $\lambda_a^{b_1...b_n}$'s some complex numbers. Hence, from Eq. (3.1),

$$(\hat{U}'_a)^\star = \sum_n \lambda_a^{b_1...b_n} \hat{U}'_{b_1}...\hat{U}'_{b_n}. \tag{3.3}$$

This $\star$-operation will be an involution on $\mathcal{A}'_p$ only if $((\hat{U}'_a)^\star)^\star = \hat{U}'_a$ for all $\hat{U}'_a$. This, together with Eqs. (1.2), (3.1) and (3.3), implies

$$\hat{U}'_a = \sum_n \overline{\lambda}_a^{b_1...b_n} (\hat{U}^\star_{b_n})'...(\hat{U}^\star_{b_1})'. \tag{3.4}$$

On the other hand, we have from Eq. (3.2)

$$\hat{U}_a = \sum_n \overline{\lambda}_a^{b_1...b_n} \hat{U}^\star_{b_n}...\hat{U}^\star_{b_1} + \hat{C}^\star_i \hat{X}_a^{\,i\,\star}, \tag{3.5}$$

since the $\star$-operation is an involution on $\mathcal{A}^{(\star)}$. Consistency of Eq. (3.4) with (3.5) requires then

$$\hat{C}^\star_i \hat{X}_a^{\,i\,\star} = \hat{Y}_a^{\,i} \hat{C}_i, \tag{3.6}$$

for some operators $\hat{Y}_a^{\,i} \in \mathcal{A}$. This condition will not be satisfied by generic operators $\hat{X}_a^{\,i}\hat{C}_i \in \mathcal{I}_C$, because the ideal $\mathcal{I}_C$ is not invariant under the $\star$-operation when there exist first-class constraints on the system. Therefore, the $\star$-relations (3.3) will not supply in general an involution on $\mathcal{A}'_p$. To obtain that involution, it is necessary that both conditions (3.2) and (3.6) are satisfied by the representatives of our complete set of physical observables.

We will study now the case in which these requirements hold for our particular choice of representatives. Our previous discussion shows that the $\star$-operation defined by Eqs. (3.3) and (1.2) is then an involution on $\mathcal{B}'$, the free associative algebra generated by $\{\hat{U}'_a\}$. Recalling that the algebra $\mathcal{A}'_p$ of physical observables can be obtained from $\mathcal{B}'$ by imposing on its generators the commutation relations and any existing algebraic relations, we conclude that the $\star$-operation introduced on $\mathcal{B}'$ straightforwardly supplies an involution on $\mathcal{A}'_p$ provided that such an operation is compatible with the relations imposed on the generators $\{\hat{U}'_a\}$. In other words, the $\star$-conjugate to those relations should not lead to any new restriction on $\mathcal{B}'$. When this requisite is fulfilled, one gets an involution on $\mathcal{A}'_p$ which captures the reality conditions on quantum operators.

Notice that the involution at which one arrives depends, nevertheless, on two choices: the complete set of physical observables and the representatives for them. In general, distinct choices may lead to different involutions on the algebra of physical observables. We will comment on this point further in Section V.

A situation which is often encountered in physical applications[4,5] is that one can find a complete set in $\mathcal{A}'_p$ admitting representatives $\{\hat{U}_a\}$ such that the complex vector space spanned by them is closed under reality conditions, ie,

$$\hat{U}^{\star}_a = \lambda_a^{\ b} \hat{U}_b \ . \tag{3.7}$$

In this case, assumption (3.2) holds with $\hat{X}_a^{\ i} \hat{C}_i = \hat{0}$, so that Eq. (3.6) is trivially satisfied. It is then at least possible to obtain an involution on the free algebra $\mathcal{B}'$ by replacing the operators $\hat{U}_a$ in Eq. (3.7) with their corresponding equivalence classes of physical observables.

IV. Examples

Let us illustrate our discussion by dealing with some examples. Consider, for instance, the physical system that was analysed at the end of Section II. A complete set of physical observables for this system is $\mathcal{O}' \equiv \{\hat{1}', \hat{x}', \hat{p}'\}$, where $\hat{1}'$, $\hat{x}'$ and $\hat{p}'$ are the equivalence classes of the operators $\hat{1}$, $\hat{x}$ and $\hat{p}$, respectively. We can select these operators as the representatives of $\mathcal{O}'$. The associated reality conditions, which are given by Eq. (2.8), have the form (3.7). So, hypotheses (3.2) and (3.6) apply. We can therefore try to induce an involution on $\mathcal{A}'_p$ by the procedure explained in Section III. Since there exist no algebraic relations in $\mathcal{O}'$, the only consistency requirement that must be satisfied in order to get the desired involution is that reality conditions (2.8) are compatible with the commutators of the physical observables in $\mathcal{O}'$. There is just one commutator different from zero: $[\hat{x}',\hat{p}'] = i\hbar\hat{1}'$. On the other hand, we obtain from Eqs. (2.8) and (3.3)

$$(\hat{x}')^{\star} = \hat{x}' \;, \tag{4.1}$$

$$(\hat{p}')^{\star} = \hat{p}' \;, \qquad (\hat{1}')^{\star} = \hat{1}' \;. \tag{4.2}$$

Taking then the $\star$-conjugate to $[\hat{x}',\hat{p}']$, we get

$$([\hat{x}',\hat{p}'])^{\star} = [(\hat{p}')^{\star},(\hat{x}')^{\star}] = [\hat{p}',\hat{x}'] = -i\hbar\hat{1}' \;, \tag{4.3}$$

which is precisely $(i\hbar\hat{1}')^{\star}$. All other commutators between $(\hat{1}')^{\star}$, $(\hat{x}')^{\star}$ and $(\hat{p}')^{\star}$ vanish identically. Hence, the $\star$-operation constructed is compatible with the structure of $\mathcal{A}'_p$, and provides an involution on this algebra.

Let us consider now other choices of representatives of $\mathcal{O}'$. Adopt, eg, the choice $\{\hat{1},\hat{x}+\hat{t}(\hat{H})^2,\hat{p}\}$. It follows from Eqs. (2.7,8) that

$$\left(\hat{x}+\hat{t}(\hat{H})^2\right)^{\star} = \left(\hat{x}+\hat{t}(\hat{H})^2\right) - 2i\hbar\hat{H}, \qquad \hat{p}^{\star} = \hat{p}, \qquad \hat{1}^{\star} = \hat{1}. \tag{4.4}$$

These reality conditions are of the type (3.2), with $\hat{X}_a^{\,i}\hat{C}_i = -2i\hbar\hat{H}$ for $\hat{U}_a = \hat{x}+\hat{t}(\hat{H})^2$, vanishing otherwise. In particular, assumption (3.6) is verified. Therefore, one can introduce a $\star$-operation on $\mathcal{A}'_p$ by applying Eq. (3.3) to the present case. In this way, one recovers the $\star$-relations (4.1,2), and thus the same involution on the algebra of physical observables that was obtained above.

Choose now the operators $\hat{1}$, $\hat{x}+\hat{t}\hat{H}$ and $\hat{p}$ as representatives of $\mathcal{O}'$. The reality conditions are then given by

$$(\hat{x}+\hat{t}\hat{H})^{\star} = (\hat{x}+\hat{t}\hat{H}) - i\hbar\hat{1}, \qquad \hat{p}^{\star} = \hat{p}, \qquad \hat{1}^{\star} = \hat{1}. \tag{4.5}$$

These reality conditions are of the form (3.7), and induce on $\mathcal{A}'_p$ the $\star$-operation defined through Eq. (4.2) and

$$(\hat{x}')^{\star} = (\hat{x}') - i\hbar\hat{1}' \;. \tag{4.6}$$

Since Eqs. (4.2) and (4.6) imply again relation (4.3), and $(\hat{1}')^{\star}$ commutes with $(\hat{x}')^{\star}$ and $(\hat{p}')^{\star}$, the introduced $\star$-operation is compatible with the commutators of the physical observables, and is therefore an involution on $\mathcal{A}'_p$. However, this involution differs from that obtained in Eqs. (4.1,2). This proves that the involution induced on $\mathcal{A}'_p$ from reality conditions depends on the particular selection of representatives made for the complete set of physical observables under consideration.

Suppose that we can represent the $\star$-relations on $\mathcal{A}'_p$ as adjoint relations on a Hilbert space of physical states, as suggested by Ashtekar. From the involution provided by Eqs. (4.1,2), we would then arrive at a quantum theory in which the observable $\hat{x}'$ would be self-adjoint. On the other hand, the involution defined through Eqs. (4.2) and (4.6) would lead to a quantum theory in which $\hat{x}'$ would not be represented by a self-adjoint operator, so that it should not correspond to a real physical observable of the system. This ambiguity in the quantization can be nonetheless removed by insisting, for instance, on that the real classical variable x should be represented by the quantum observable $\hat{x}'$. One would thus expect that the spectrum of $\hat{x}'$ should be real to guarantee that this observable has always real expectation values. Hence, $\hat{x}'$ should be self-adjoint. By itself, this condition supports the use of involution (4.1,2) in the quantization, and elliminates other possible $\star$-relations on $\mathcal{A}'_p$, like, eg, relation (4.6).

To close this section, we will present an example in which the involution induced on $\mathcal{B}'$ via reality conditions is not compatible with the structure of the algebra of physical observables. Let us consider a physical system with a first-class constraint of the form $H = 0$, where $H \in I\!\!R$ is the momentum canonically conjugate to a certain variable $t \in I\!\!R$. We will assume that the reduced phase space of the system is the cotangent bundle over the unit circle S^1. As elementary variables, we can choose the complex vector space spanned by $\{1, t, H, c_\theta \equiv \cos\theta, s_\theta \equiv \sin\theta, p_\theta\}$. Here, $\theta \in S^1$, and $p_\theta \in I\!\!R$ is the momentum conjugate to θ. The reality conditions on the corresponding algebra $\mathcal{A}^{(\star)}$ of quantum operators are given by Eq. (2.7) and

$$\hat{c}_\theta^\star = \hat{c}_\theta, \qquad \hat{s}_\theta^\star = \hat{s}_\theta, \qquad \hat{p}_\theta^\star = \hat{p}_\theta, \qquad \hat{1}^\star = \hat{1}\ . \tag{4.7}$$

Besides, since $\cos^2\theta + \sin^2\theta = 1$, we will impose the algebraic relation

$$(\hat{c}_\theta)^2 + (\hat{s}_\theta)^2 = \hat{1}\ . \tag{4.8}$$

A complete set of physical observables is $\mathcal{O}' \equiv \{\hat{1}', \hat{c}'_\theta, \hat{s}'_\theta, \hat{p}'_\theta\}$, the prime denoting equivalence classes in $\mathcal{A}'_p$. The only non-vanishing commutators in $\mathcal{O}'$ are

$$[\hat{c}'_\theta, \hat{p}'_\theta] = -i\hbar\hat{s}'_\theta, \qquad [\hat{s}'_\theta, \hat{p}'_\theta] = i\hbar\hat{c}'_\theta. \tag{4.9}$$

In addition, relation (4.8) implies that the physical observables in $\mathcal{O}'$ must satisfy

$$(\hat{c}'_\theta)^2 + (\hat{s}'_\theta)^2 = \hat{1}'\ . \tag{4.10}$$

If one chooses $\hat{1}$, $\hat{c}_\theta$, $\hat{s}_\theta$ and $\hat{p}_\theta$ as the representatives of $\mathcal{O}'$, the procedure explained in Section III allows one to obtain an $\star$-operation on $\mathcal{B}'$ (the free associative algebra generated by $\mathcal{O}'$) which is compatible with the commutators (4.9) and the algebraic relation (4.10), and hence provides an involution on $\mathcal{A}'_p$. Let us select instead the representatives $\mathcal{O} \equiv \{\hat{1}, (\hat{c}_\theta + \hat{t}\hat{H}), \hat{s}_\theta, \hat{p}_\theta\}$. From Eqs. (2.7) and (4.7) (and the commutator of $\hat{t}$ and $\hat{H}$), we get

$$\hat{1}^\star = \hat{1}, \qquad (\hat{c}_\theta + \hat{t}\hat{H})^\star = (\hat{c}_\theta + \hat{t}\hat{H}) - i\hbar\hat{1}, \qquad \hat{s}_\theta^\star = \hat{s}_\theta, \qquad \hat{p}_\theta^\star = \hat{p}_\theta. \tag{4.11}$$

These reality conditions are of the type (3.7). Thus, we can apply the results of Section III to arrive at an involution on $\mathcal{B}'$ which is defined through the $\star$-relations (4.11), but imposed on equivalence classes in $\mathcal{O}'$. However, such a $\star$-operation is incompatible with the algebraic relation (4.10), because

$$\left((\hat{c}'_\theta)^2 + (\hat{s}'_\theta)^2 - \hat{1}'\right)^\star = \left(\hat{c}'_\theta - i\hbar\hat{1}'\right)^2 + (\hat{s}'_\theta)^2 - \hat{1}' \neq 0\ . \tag{4.12}$$

So, the involution introduced on $\mathcal{B}'$ does not supply a well-defined involution on the algebra $\mathcal{A}'_p$ of physical observables. This example shows that the freedom in choosing representatives of the complete set of physical observables is in general restricted by the consistency of the algebraic structures with the $\star$-operation constructed on $\mathcal{A}'_p$.

V. Conclusions and Further Comments

We have shown that, in systems with first-class constraints, the involution defined on the algebra $\mathcal{A}^{(\star)}$ of quantum operators does never project unambiguously to the algebra $\mathcal{A}'_p$ of physical observables. The reason for this is that the $\star$-conjugates of all the representatives of any class of observables never belong to the same equivalence class in $\mathcal{A}'_p$.

We have also proved that, under sufficiently general circumstances, it is nevertheless possible to obtain a well-defined involution on $\mathcal{A}'_p$ via reality conditions by making a particular choice of representatives for the equivalence classes of physical observables. The procedure to arrive at this involution is the following. One must first find a complete set of physical observables $\{\hat{U}'_a\}$ in $\mathcal{A}'_p$, and select representatives $\{\hat{U}_a\}$ of them such that their $\star$-conjugates $\{\hat{U}^\star_a\}$ satisfy requirements (3.2) and (3.6), namely, such that every $\hat{U}^\star_a$ belongs to the free associative algebra generated by $\{\hat{U}_a\}$ up to an operator which, as well as its $\star$-conjugate, vanish modulo quantum constraints. One can then introduce an involution $\star$ in the free associative algebra $\mathcal{B}'$ by defining $(\hat{U}'_a)^\star$ as the equivalence class of the observable $\hat{U}^\star_a$ [see Eqs. (3.2,3)]. This involution on $\mathcal{B}'$ straightforwardly supplies an involution on $\mathcal{A}'_p$, provided that the constructed $\star$-operation is compatible with the commutation and algebraic relations which exist between the physical observables in the complete set $\{\hat{U}'_a\}$.

The involution obtained in this way on $\mathcal{A}'_p$ depends on the selection of a complete set of physical observables and of specific representatives for them. While these choices are severely restricted by the consistency conditions explained above, there is in general some freedom left, so that, by adopting different choices, one may arrive at non-equivalent involutions on the algebra of physical observables. This introduces an ambiguity in the quantization method suggested by Ashtekar which has to be added to that existing in other steps of the programme.[2] However, such an extra ambiguity, rather than being a supplementary complication, may become an additional help when attempting to complete the quantization. This is due to the fact that, given an involution $\star$ on the algebra $\mathcal{A}'_p$ and a certain representation for $\mathcal{A}'_p$ on a vector space V_p of quantum states, there is a priori no guarantee that there exists an inner product on V_p with respect to which the $\star$-relations on physical observables are realized as Hermitian adjoint relations in the resulting Hilbert space. Thus, if such an inner product does not exist for a particular involution on $\mathcal{A}'_p$, one can always try to induce a different involution on this algebra via reality conditions, and see whether it is possible to find then an inner product with the desired properties.

We notice, on the other hand, that the introduction of an involution on $\mathcal{A}'_p$ amounts essentially to determine the $\star$-conjugate to a complete set of physical observables. When one expects that a set of this kind, or at least some of its elements, correspond classically to real observables of the system, it is reasonable to assume that they should be represented by self-adjoint operators. The involution defined on $\mathcal{A}'_p$ should therefore ensure that these operators coincide with their $\star$-conjugates. These requirements clearly restrict the admissible involutions on physical observables. Moreover, in the case that this type of physical arguments would apply to a complete set in $\mathcal{A}'_p$, one would fully specify the involution on this algebra. In this way, one can use physical intuition to remove (either partially or totally) the ambiguity encountered when inducing an involution on the algebra of physical observables from reality conditions.

Finally, an alternative strategy to rule out such an ambiguity could consist in adopting a specific procedure to induce the involution $\star$ on $\mathcal{A}'_p$. A procedure of this type might be, eg, the following.[11] Let us denote by $\mathcal{A}_s \subset \mathcal{A}_p$ the subalgebra formed by all the strong quantum observables of the theory (that is, the operators which commute strongly with all the quantum constraints $\{\hat{C}_i\}$), and define $\mathcal{I}_s \equiv \mathcal{I}_C \bigcap \mathcal{A}_s$. It is immediate to check that $\mathcal{I}_s$ is an ideal of $\mathcal{A}_s$. Suppose then that, in the system under consideration, the involution $\star$ defined on $\mathcal{A}^{(\star)}$ and the representation constructed for the algebra $\mathcal{A}$ and for the constraints $\{C_i\}$ are such that:

a) The complex vector space spanned by the quantum constraints $\{\hat{C}_i\}$ is closed under reality conditions, ie, $\hat{C}_i^{\star} = \lambda_i^j \hat{C}_j$, where the λ_i^j's are complex numbers.

b) The algebra $\mathcal{A}'_s \equiv \mathcal{A}_s/\mathcal{I}_s$ is isomorphic to $\mathcal{A}'_p$.

c) The ideal $\mathcal{I}_s$ is invariant under the $\star$-operation.

Notice that hypothesis c) is in principle compatible with the fact that $\mathcal{I}_C$ is not a $\star$-ideal of $\mathcal{A}_p$. Requirement b), on the other hand, guarantees that each physical observable in $\mathcal{A}'_p$ possesses (at least) one representative which is a strong observable.

Using condition a), it is possible to prove that the $\star$-operation leaves $\mathcal{A}_s$ invariant. Assumption c) ensures then that the $\star$-relations project unambiguously to $\mathcal{A}'_s$. One hence obtains a well-defined involution on $\mathcal{A}'_s$ which, given condition b), supplies a unique involution on $\mathcal{A}'_p$ through the existing isomorphism between these two algebras. So, provided that hypotheses a)-c) are satisfied, the above strategy actually allows one to induce an unambiguous involution on $\mathcal{A}'_p$ from reality conditions.

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[11] This procedure implements a suggestion by A. Ashtekar (private communication). See also Section II of A. Ashtekar, J. Lewandowski, D. Marolf, J. Mourão and T. Thiemann, gr-qc/9504018.