

PUPT-95-1543
hep-th/9506137

Generalized Abelian S-duality and coset constructions

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Electric-magnetic duality and higher dimensional analogues are obtained as symmetries in generalized coset constructions, similar to the axial-vector duality of two-dimensional coset models described by Roček and Verlinde. We also study global aspects of duality between p -forms and $(d-p-2)$ -forms in d -manifolds. In particular, the modular duality anomaly is governed by the Euler character as in four and two dimensions. Duality transformations of Wilson line operator insertions are also considered.

1. Introduction

Recent exact results in supersymmetric gauge theories have prompted a renewed interest in S-duality, understood as a generalization of electric-magnetic weak-strong coupling duality. In addition to the classic selfduality conjecture of Montonen and Olive in $N = 4$ supersymmetric Yang-Mills [1], and its stringy generalization [2], many new examples and conjectures have been proposed recently, both in quantum field theory and string theory. In some cases [3], S-duality of a low energy abelian gauge theory plays an important role in the solution of the infrared physics of certain $N = 1$ and $N = 2$ gauge theories. A generalization of the Montonen-Olive duality to $N = 1$ Super-QCD in a non abelian Coulomb phase was also proposed in ref. [4], with the striking property that the number of gauge degrees of freedom is totally different in the two dual descriptions.

In this last example, and also in the Montonen-Olive case, the duality is supposed to operate in the full non abelian theory, and not only in a spontaneously broken phase with an abelian low energy theory. Evidence for this fact in the $N = 4$ theory has been presented in [5] and [6]. On the other hand, the dual gauge group in the Montonen-Olive sense is found by just dualizing the Cartan subalgebra. Also, dimensional reduction to two dimensions [7] apparently projects non abelian S-duality onto standard abelian T-duality.

Unlike the non abelian case, S-duality in abelian gauge theories can be described quite explicitly by different methods involving just gaussian path integrals. In view of the previous remarks, it is interesting to understand abelian duality in its full generality, and in particular the relations and analogies to the two dimensional case. Recently, discussions of global aspects in electric-magnetic duality have appeared in refs.[8] and [9]. In this paper, we present a generalization of some of the results in these papers to arbitrary dimensions. In particular, we determine the modular duality anomaly for the general duality between p -forms and $(d - p - 2)$ -forms and find the same result as in four dimensions, up to sign factors.

In order to pursue further the analogies between higher dimensional S-duality and two dimensional duality of sigma-models we generalize the Roček-Verlinde coset construction to the duality between p -form gauge theories in even dimensions. This might be interesting to address more complicated cases because coset constructions are examples of redundant gauge symmetries (the gauge fields are infinitely strongly coupled and do not propagate). The coset construction we describe is similar to the work of [10] on duality symmetric actions, although important differences are pointed out.

2. Coset constructions

2.1. The two-dimensional case

Abelian T-duality is a well known symmetry of string perturbation theory. At the world-sheet level it is a non-perturbative Kramers-Wannier transformation or, in continuum language, a Hodge duality transformation: $\partial_\alpha \theta \rightarrow \epsilon_{\alpha\beta} \partial^\beta \theta$. This mapping is non-local in terms of the field θ (it introduces winding modes and vortices) and exchanges equations of motion $\partial_\alpha \partial^\alpha \theta = 0$ with the Bianchi identity $\epsilon^{\alpha\beta} \partial_\alpha \partial_\beta \theta = 0$. Presented in this form, it is an analogue of four dimensional electric-magnetic duality which does the same in terms of the vector potential $A_\mu : dA \rightarrow *dA$. Indeed, the naive dimensional reduction of the Maxwell theory to two dimensions yields two scalar fields as the internal components of the photon, and the four dimensional S-duality is mapped into these scalars as two dimensional T-duality, global topological electric-magnetic fluxes are mapped into winding modes, and monopoles yield two dimensional vortex configurations (winding modes around a singular point).

The standard manipulation to exhibit two-dimensional Hodge duality consists in writing the path integral in first order form, changing variables from θ to $d\theta = A$. In doing so, we assume that, as a sigma-model, there is a target space isometry under constant shifts of the θ field: $\theta \rightarrow \theta + \xi$, so that the action is of the form

$$S(\theta) = \frac{1}{2\pi} \int d^2 z g_{\theta\theta} \partial\theta \bar{\partial}\theta + \dots = \frac{1}{8\pi} \int g_{\theta\theta} (d\theta)^2 + \dots \quad (2.1)$$

with $g_{\theta\theta}$ independent of θ . The change of variables can be readily implemented in a lattice regularization (see the nice discussion in [11]). There is a local constraint $*dA = 0$ which can be written as a functional integral over a Lagrange multiplier field $\tilde{\theta}$ and integrating out the one-form A completes the proof.

A simple algorithm to keep track of the functional measures in the continuum language uses a variant of the first order formalism (see [12], [13]), in which one gauges the isometry and cancels the non-propagating gauge field by means of the same local constraint $*dA = 0$.

In formulas

$$\begin{aligned} Z &= \int D\theta e^{-S(d\theta)} = \int \frac{D\theta DA}{\text{Vol}(G)} \delta(*dA) e^{-S(d\theta+A)} \\ &= \int \frac{D\theta DAD\tilde{\theta}}{\text{Vol}(G)} e^{-\frac{i}{2\pi} \int A \wedge d\tilde{\theta} - S(d\theta+A)^2} \end{aligned} \quad (2.2)$$

Since there was an isometry the action is quadratic in A ($g_{\theta\theta}$ still depends on other fields in general). Then we may gauge fix $\theta = 0$ and integrate out A to get the dual version of the model

$$Z = \int D\tilde{\theta} \exp\left(-\frac{1}{8\pi} \int \frac{1}{g_{\theta\theta}} (d\tilde{\theta})^2\right) \quad (2.3)$$

with the characteristic $g_{\theta\theta} \rightarrow 1/g_{\theta\theta}$ form. A careful treatment of the ultralocal jacobians in the integration measure yields in addition a shift in the dilaton background field $\Phi \rightarrow \Phi + \log(g_{\theta\theta})$. Formally, the local measure is regularized preserving the following structure:

$$D\theta \sim \prod_z \left(\frac{d\theta_z}{\sqrt{2\pi}} \sqrt{g_{\theta\theta}(z)} \right) \quad (2.4)$$

The coupling between the gauge field and the Lagrange multiplier is designed such that both the original field θ and its dual have the same periods. If the original current has non-trivial holonomies around homology 1-cycles

$$\oint_{\gamma} d\theta \in 2\pi\mathbf{Z} \quad (2.5)$$

then the gauged model must be invariant under large gauge transformations $A \rightarrow A - d\epsilon$ where $d\epsilon$ has $2\pi \times (\text{integer})$ periods. This condition is met if and only if $d\tilde{\theta}$ has the same periodicity, so that

$$\exp\left(\frac{i}{2\pi} \int d\epsilon \wedge d\tilde{\theta}\right) = e^{2\pi i n \cdot m} = 1$$

In fact, formula (2.5) is an abuse of notation. One should think of $d\theta$ as an exact 1-form plus a harmonic piece which is responsible for the periods.

An analogous procedure for four-dimensional electric-magnetic duality was recently used by Witten in [8]. There the ‘‘isometry’’ to be gauged is the shift of the vector potential

$$A \rightarrow A + B$$

by an arbitrary 1-form B . This is achieved introducing a 2-form gauge field G , with a 3-form field strength dG , which is required to vanish as a constraint. A convenient representation uses a one-form Lagrange multiplier

$$\delta[dG] \sim \int \frac{D\tilde{A}}{\text{Vol}(\tilde{G})} e^{\frac{i}{2\pi} \int dG \wedge \tilde{A}}$$

where the dual gauge symmetry \tilde{G} takes care of the ambiguity $\tilde{A} \rightarrow \tilde{A} + d\phi$ in the exponentiation of the delta functional. The occurrence of global electric-magnetic fluxes wrapped around homologically non-trivial two-submanifolds exactly parallels the previous two-dimensional case. In section 4 we will exploit this technique to investigate the general duality relation between p -forms and $(d - p - 2)$ -forms in an arbitrary d -manifold, as well as to compute the non-local “disorder operators” dual to generalized Wilson p -lines.

The previous formalism treats the original and dual variables (the Lagrange multipliers) in a rather asymmetric fashion. For example, the dual photon \tilde{A} only acquires a kinetic energy term after the fake gauge field has been eliminated. A more symmetric procedure exists for two dimensional T-duality, as explained by Roček and Verlinde in [13]. Roughly speaking, the method constructs both versions of the model as equivalent cosets of a single sigma model with doubled degrees of freedom and carefully chosen couplings. The duality symmetry corresponds to a discrete field redefinition in the doubled theory (in the context of WZW conformal field theories it is a Weyl transformation). To illustrate the duality between (2.1) and (2.3) consider the following sigma model with two independent fields θ_{LR}

$$S_{LR} = \frac{1}{2\pi} \int d^2z (\partial\theta_L \bar{\partial}\theta_L + \partial\theta_R \bar{\partial}\theta_R + 2B\partial\theta_R \bar{\partial}\theta_L + \dots) \quad (2.6)$$

where B is independent of θ_L, θ_R but may depend on other fields of the sigma model. This action has a $U(1)_L \times U(1)_R$ affine symmetry generated by the chiral currents

$$J^L = \partial\theta_L + B\partial\theta_R, \quad \bar{J}^R = \bar{\partial}\theta_R + B\bar{\partial}\theta_L$$

The axial-vector cosets are constructed by gauging with the minimal coupling prescriptions

$$d\theta_R \rightarrow d\theta_R + \frac{A}{2}, \quad d\theta_L \rightarrow d\theta_L \pm \frac{A}{2} \quad (2.7)$$

and the addition of a gauge invariant term

$$S' = \frac{1}{4\pi} \int d^2z (\theta_R \mp \theta_L)(\partial\bar{A} - \bar{\partial}A) \quad (2.8)$$

The gauge fields are kept non-propagating so that they simply project out the physical Hilbert space. Introducing the axial-vector combinations $\theta = \theta_R + \theta_L$, $\tilde{\theta} = \theta_R - \theta_L$ and integrating out the gauge fields we find the following semiclassical sigma models

$$S_{\text{Vector}} = \frac{1}{2\pi} \int d^2z \left(\frac{1+B}{1-B} \right) \partial\theta\bar{\partial}\theta + \dots$$

$$S_{\text{Axial}} = \frac{1}{2\pi} \int d^2z \left(\frac{1-B}{1+B} \right) \partial\tilde{\theta}\bar{\partial}\tilde{\theta} + \dots$$

defining $g_{\theta\theta} = \frac{1+B}{1-B}$ we obtain both dual sigma models by switching from vector to axial gauging, that is $B \rightarrow -B$. This transformation can be undone by a field reparametrization $\theta_L \rightarrow -\theta_L$ in the doubled action. An important point is that this transformation is not a classical symmetry of the action (2.6) as it stands, but it is always a symmetry of the path integral.

2.2. Generalization to even p forms in $2p+2$ dimensions

The previous construction readily generalizes to *even* p forms in $d = 2p+2$ dimensions. Let us define a pair of p -form potentials A_L, A_R with field strengths $F_{LR} = dA_{LR}$, and the axial-vector combinations $F = F_R + F_L, \tilde{F} = F_R - F_L$. Consider the doubled action ¹

$$4\pi \mathcal{L}_{\text{doubled}} = F_L^2 + F_R^2 + 2\mu F_R F_L + 2i\mu F_R * F_L \quad (2.9)$$

The coset construction is obtained by a direct extension of the formulas (2.7) with a $(p+1)$ -form gauge field G

$$F_R \rightarrow F_R + \frac{G}{2}, \quad F_L \rightarrow F_L \pm \frac{G}{2} \quad (2.10)$$

and adding a gauge invariant coupling

$$\int \mathcal{L}' = \frac{i}{4\pi} \int dG * (A_R \mp A_L) = \frac{i}{4\pi} \int G * d(A_R \mp A_L) \quad (2.11)$$

The vector gauging leads to the model

$$4\pi \mathcal{L}_{\text{gauged}} = \frac{1+\mu}{2} F^2 + \frac{1-\mu}{2} (\tilde{F} + G)^2 + i\mu (\tilde{F} + G) * F + iG * F$$

The gauge field G is kept non propagating (ie. extreme strong coupling), so that we can integrate it out and gauge fix $\tilde{A} = 0$ with the result

$$4\pi \mathcal{L} = \frac{1+\mu}{1-\mu} F^2 \equiv \frac{4\pi}{g^2} F^2 \quad (2.12)$$

¹ Our conventions for the product of forms in this section and the rest of the paper are $\alpha_n \beta_n \equiv \alpha_n \wedge * \beta_n = \frac{1}{n!} \alpha_{i_1 \dots i_n} \beta^{i_1 \dots i_n} d(\text{Vol})$

With this definition of the gauge coupling, duality amounts to $\mu \rightarrow -\mu$ just as in the two-dimensional case. This is precisely provided by de axial gauging

$$4\pi \tilde{\mathcal{L}}_{\text{gauged}} = \frac{1+\mu}{2}(F+G)^2 + \frac{1-\mu}{2}\tilde{F}^2 + i\mu\tilde{F} * (F+G) + iG * \tilde{F}$$

Proceeding as before, upon G integration we arrive at the dual theory

$$4\pi \tilde{\mathcal{L}} = \frac{1-\mu}{1+\mu}\tilde{F}^2 \equiv \frac{4\pi}{\tilde{g}^2}\tilde{F}^2 \quad (2.13)$$

and $\tilde{g} = 4\pi/g$ as desired (we have chosen the normalization of the coupling such that the duality is $g \rightarrow 4\pi/g$ in any dimension). In arriving at this result it is very important that the field redefinition interchanging both gaugings: $F_L \rightarrow -F_L$ or $F \leftrightarrow \tilde{F}$ is equivalent to $\mu \rightarrow -\mu$ in the doubled theory. This is the case for $p+1 = \text{odd}$ thanks to the identity

$$F_{p+1} * \tilde{F}_{p+1} = -\tilde{F}_{p+1} * F_{p+1}$$

In dimension $d = 2p+2$ with odd p the wedge product of $(p+1)$ forms is symmetric (that is the reason why we can define a θ angle). So in particular the analogue of axial-vector duality does not work in dimension four.

2.3. Odd p forms in $2d+2$ dimensions

In order to further discuss the odd p case it is useful to define the selfdual and anti-selfdual projections $F^\pm = (F \pm *F)/2$. Let us define the following doubled action:

$$4\pi \mathcal{L}_{\text{doubled}} = \frac{1+\mu}{2}(F^+)^2 + \frac{1+\bar{\mu}}{2}(F^-)^2 + \frac{1-\mu}{2}(\tilde{F}^+)^2 + \frac{1-\bar{\mu}}{2}(\tilde{F}^-)^2 + i\mu F^+ \tilde{F}^+ - i\bar{\mu} F^- \tilde{F}^- \quad (2.14)$$

where now μ is a complex number, to allow for a non zero θ angle. If we gauge the \tilde{A} field by the minimal prescription $\tilde{F}^\pm \rightarrow \tilde{F}^\pm + G^\pm$, plus the gauge invariant term

$$\int \mathcal{L}' = \frac{i}{4\pi} \int dG * A$$

we obtain, after gauge fixing and G^\pm integration

$$4\pi \mathcal{L} = \frac{1+\mu}{1-\mu}(F^+)^2 + \frac{1+\bar{\mu}}{1-\bar{\mu}}(F^-)^2 \quad (2.15)$$

so that, defining $\tau = i\frac{1+\bar{\mu}}{1-\mu}$ we get the standard lagrangian²

$$\mathcal{L} = \frac{i}{4\pi} (\bar{\tau}(F^+)^2 - \tau(F^-)^2)$$

From this manipulation it is clear that duality $\tau \rightarrow \tilde{\tau} = -1/\tau$ would again correspond to $\mu \rightarrow -\mu$ in the doubled theory. This cannot be achieved by the previous axial-vector mapping $A \leftrightarrow \tilde{A}$. However, it is easy to see that

$$\begin{pmatrix} F \\ \tilde{F} \end{pmatrix} \rightarrow \begin{pmatrix} F' \\ \tilde{F}' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} F \\ \tilde{F} \end{pmatrix}$$

does the job. Changing variables in the lagrangian (2.14) to F', \tilde{F}' yields the $\mu \rightarrow -\mu$ transformation:

$$\mathcal{L}(F, \tilde{F}, \mu) = \mathcal{L}(F', \tilde{F}', -\mu)$$

We now perform the same gauging as before: $\tilde{F}'^\pm \rightarrow \tilde{F}'^\pm + G^\pm$ and add

$$\frac{i}{4\pi} \int dG * A' = \frac{i}{4\pi} \int (G^+ F'^+ - G^- F'^-)$$

Finally we substitute back $F' = \tilde{F}, \tilde{F}' = -F$ and find

$$\begin{aligned} 4\pi \tilde{\mathcal{L}}_{\text{gauged}} &= \frac{1+\mu}{2}(F^+ - G^+)^2 + \frac{1+\bar{\mu}}{2}(F^- - G^-)^2 + \frac{1-\mu}{2}(\tilde{F}^+)^2 + \frac{1-\bar{\mu}}{2}(\tilde{F}^-)^2 \\ &\quad + i\mu(F^+ - G^+)\tilde{F}^+ - i\bar{\mu}(F^- - G^-)\tilde{F}^- + i(G^+\tilde{F}^+ - G^-\tilde{F}^-) \end{aligned} \quad (2.16)$$

And we see that we have successfully changed the gauging from \tilde{A} to A . Integrating out G^\pm and gauge fixing $A = 0$ we arrive at

$$4\pi \tilde{\mathcal{L}} = \frac{1-\mu}{1+\mu}(\tilde{F}^+)^2 + \frac{1-\bar{\mu}}{1+\bar{\mu}}(\tilde{F}^-)^2 \quad (2.17)$$

and now the corresponding coupling $\tilde{\tau} = i\frac{1-\bar{\mu}}{1+\mu}$ satisfies $\tilde{\tau} = -1/\tau$ as expected.

In summary, the field redefinition required in the doubled theory:

$$\begin{pmatrix} A \\ \tilde{A} \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 \\ (-)^p & 0 \end{pmatrix} \begin{pmatrix} A \\ \tilde{A} \end{pmatrix} \quad (2.18)$$

is the one considered in [10] in their analysis of duality invariant actions (note that for odd p the square of this transformation is -1 , but this is good enough because inverting the

² This definition is consistent with the usual normalization $\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi}$.

sign of both potentials leaves the action invariant). However, our approach differs from that in [10] in several aspects. Although we use an extended theory as an starting point, one of the fields is gauged away rather than evaluated on-shell, and we find no conflict with Lorentz invariance. The duality transformation is not a classical symmetry of the doubled lagrangian, yet the whole procedure is a simple change of variables in the path integral. From this point of view, S-duality appears as a full quantum symmetry and perhaps one should not try to implement it classically at the lagrangian level. Note that, in general, the effective actions are not explicitly duality invariant because electric and magnetic variables are mutually non-local and they never appear simultaneously in the same low energy effective lagrangian.

2.4. Global aspects

Global issues are easily dealt with in this formalism. The discrete transformations (2.18) imply that both F and \tilde{F} have the same periods around homologically non-trivial $(p+1)$ -manifolds. On the other hand, the modular anomaly in arbitrary dimension requires a more careful analysis. On general grounds, since all path integrals are gaussian, we can estimate the coupling constant dependence in the regularized theory as

$$\int \frac{DA_p}{\text{Vol}(G_p)} e^{-S(A_p, \tau)} \sim (\text{Im}\tau)^{-\frac{1}{2}\dim\mathcal{H}'_{\text{ph}}}$$

where \mathcal{H}'_{ph} is the physical Hilbert space up to zero modes (harmonic forms). That is, the space of p -forms minus the harmonic ones and gauge degrees of freedom,

$$\dim\mathcal{H}'_{\text{ph}} = B_p - b_p - (\text{gauge } e)$$

where b_p is the Betti number measuring the number of zero modes. The pure gauge p -forms are $A_p = d\lambda_{p-1}$, so we can count gauge degrees of freedom as $(p-1)$ -forms, up to harmonic or exact ones which do not really contribute to the gauge invariance of the original theory. If we continue this nested counting until we reach zero forms we get

$$\begin{aligned} \dim\mathcal{H}'_{\text{ph}} &= (B_p - b_p - (B_{p-1} - b_{p-1} - (B_{p-2} - b_{p-2} - \dots \\ &= (-)^p \sum_{j=0}^p (-)^j (B_j - b_j) \equiv \mathcal{N}_p + (-)^{p+1} \sum_{j=0}^p (-)^j b_j \end{aligned}$$

A more formal derivation of this formula, using the Fadeev-Popov procedure, will be given in the next section. Following [8] we want to get rid of the regularization dependent factor, so that we define the functional measure as

$$Z_p(\tau) \sim (\text{Im}\tau)^{\frac{1}{2}\mathcal{N}_p} \int \frac{DA_p}{\text{Vol}(G_p)} e^{-S(A_p, \tau)} \sim (\text{Im}\tau)^{\frac{(-)^p}{2} \sum_j (-)^j b_j} \quad (2.19)$$

Here the measure DA_p does not contain additional powers of $\text{Im}\tau$, and this prescription is the generalization of the sigma-model local measure of eq. (2.4). From this point of view, the modular duality anomaly is a generalization of the dilaton shift phenomenon in two dimensional T-duality. In obtaining (2.19) from the coset construction, the G^\pm integrals generate a factor

$$(1 \mp \mu)^{-\frac{1}{2}B_{p+1}^+} (1 \mp \bar{\mu})^{-\frac{1}{2}B_{p+1}^-}$$

which must be canceled (here B_{p+1}^\pm are the numbers of self dual and anti-self dual $(p+1)$ -forms). The complete coset formula for the partition function is then

$$Z_p(\tau) = (\text{Im}\tau)^{\frac{1}{2}\mathcal{N}_p} (1 - \mu)^{\frac{1}{2}B_{p+1}^+} (1 - \bar{\mu})^{\frac{1}{2}B_{p+1}^-} \int \frac{DAD\tilde{A}DG}{\text{Vol}(G_{p+1})} e^{-\int \mathcal{L}_{\text{gauged}}(\tilde{F}+G, A)}$$

and for the dual model

$$\tilde{Z}_p(\tilde{\tau}) = (\text{Im}\tilde{\tau})^{\frac{1}{2}\mathcal{N}_p} (1 + \mu)^{\frac{1}{2}B_{p+1}^+} (1 + \bar{\mu})^{\frac{1}{2}B_{p+1}^-} \int \frac{DAD\tilde{A}DG}{\text{Vol}(G_{p+1})} e^{-\int \tilde{\mathcal{L}}_{\text{gauged}}(F+G, \tilde{A})}$$

where $\tilde{\tau} = -1/\tau$. Since both path integrals are formally equal, we obtain from $\tau = i\frac{1+\bar{\mu}}{1-\mu}$, up to numerical constants:

$$Z_p(\tau) = \tau^{-\frac{1}{2}B_{p+1}^-} \bar{\tau}^{-\frac{1}{2}B_{p+1}^+} (\tau\bar{\tau})^{\frac{1}{2}\mathcal{N}_p} \tilde{Z}_p(-1/\tau)$$

Now, let us assume that the regularization procedure (a lattice for example) is self dual in the sense that $B_j = B_{d-j}$ (in two dimensions this corresponds to lattices with the same numbers of vertices and faces). We also assume that the difference $B_{p+1}^+ - B_{p+1}^-$ is equal to $b_{p+1}^+ - b_{p+1}^- = \sigma$, the generalized signature. Then we can write the Euler character as

$$\chi = \sum_{j=0}^d (-)^j b_j = \sum_{j=0}^d (-)^j B_j = 2(-)^p \mathcal{N}_p + (-)^{p+1} B_{p+1}$$

and $B_{p+1}^\pm = (B_{p+1} \pm \sigma)/2$. From here we can derive the general formula

$$Z_p(\tau) = \tau^{-\frac{1}{4}((-)^{p+1}\chi - \sigma)} \bar{\tau}^{-\frac{1}{4}((-)^{p+1}\chi + \sigma)} \tilde{Z}_p(-1/\tau) \quad (2.20)$$

for even p : ($p = 0, d = 2$), ($p = 2, d = 6$), ($p = 4, d = 10$), etc. there is no theta term and we must take τ pure imaginary. Then the anomaly equation reduces to

$$Z_p(g) = (\sqrt{4\pi}/g)^\chi \tilde{Z}_p(4\pi/g) \quad (2.21)$$

For odd p : ($p = 1, d = 4$), ($p = 3, d = 8$), etc. the resulting formula looks exactly like the $d = 4$ case derived in [8].

Regarding the possible $SL(2, \mathbf{Z})$ extension of the duality symmetry for odd p , it depends on the intersection matrix of $(p + 1)$ -forms. With our normalization of the action, an integral shift $\tau \rightarrow \tau + n$ inserts the term

$$\exp\left(\frac{in}{4\pi} \int (F_{p+1}^+)^2 - (F_{p+1}^-)^2\right) = \exp\left(\frac{in}{4\pi} \int F_{p+1} \wedge F_{p+1}\right)$$

Since F_{p+1} has periods in $2\pi\mathbf{Z}$, the Hodge decomposition has the form

$$F_{p+1} = dA_p + h_{p+1} = dA_p + \sum_I 2\pi m^I \alpha_I$$

where $m^I \in \mathbf{Z}$ and α_I are normalized harmonic forms: $\oint_{\Sigma_I} \alpha_J = \delta_J^I$. The total shift is then

$$\exp\left(2\pi i \frac{n}{2} m^I m^J \int \alpha_I \wedge \alpha_J\right)$$

Thus, as in four dimensions, we have full $SL(2, \mathbf{Z})$ invariance for even intersection forms, or just $\tau \rightarrow \tau + 2$ for the general case.

3. Generalized Partition functions

In this section we discuss the general structure of partition functions of p -form theories in d -manifolds:

$$Z_p(g, \theta, J) = g^{-\mathcal{N}_p} \sum_{h_{p+1}} \int \frac{DA_p}{\text{Vol}(G_p)} e^{-\int \mathcal{L}_p} \quad (3.1)$$

with a lagrangian

$$\mathcal{L}_p = \frac{1}{g^2} F_{p+1}^2 + \frac{i\theta}{8\pi^2} F_{p+1} * F_{p+1} - iJ_p^\epsilon A_p \quad (3.2)$$

The theta term is only present for $d = 2p + 2$, $p + 1 = \text{even}$, precisely in this case the dual form has rank $d - p - 2 = p$ and we have the electric-magnetic case. We have also included

“electric” sources which we assume conserved to preserve gauge invariance: $\delta J_e = 0$, where δ is the co-derivative defined as

$$\delta = (-)^{dp+p+1} * d *$$

acting on p forms. The presence of a source term spoils duality, but different choices of J_e are useful to study the duality transformations of operator insertions, like Wilson lines. Duality can be restored by introducing monopole sources, which appear naturally in lattice formulations as dislocations (see for example [14]). However, there is no elegant method to include them in continuum treatments, due to their singular nature.

An interesting observation is that one may restore self-duality by a suitable coupling to a smooth $(p+2)$ -form. This is suggested by the structure of the Hodge decomposition of the field strength

$$F_{p+1} = \delta \phi_{p+2} + dA_p + h_{p+1}$$

Normally one takes F to be a closed form and drops the first term. If we nevertheless keep it as an external source, it contributes to $dF \neq 0$. So we can define a “magnetic” current

$$J_{d-p-2}^m = \frac{1}{\sqrt{8\pi^2}} * d \delta \phi_{p+2} \quad (3.3)$$

This is a p form precisely when a theta term is possible, but we must stress that, since we take ϕ_{p+2} as a smooth form, J_m is not really a monopole current. Note however that the definition (3.3) makes it automatically conserved: $\delta J^m = 0$.

Since the path integral is gaussian, it has the following factorized structure

$$Z_p(g, \theta, J_e, J_m) = Z_{\text{source}}(g, \theta, J_e, J_m) Z_{\text{global}}(g, \theta) Z_{\text{non-compact}}(g)$$

We now turn to a more detailed analysis of the different factors.

3.1. Source partition function

The source dependence is easily solved in terms of the Green function for the corresponding laplacian, which is inverted in the space orthogonal to the zero modes (harmonic forms).

$$Z_{\text{source}}(g, \theta, J_e, J_m) = \exp\left(-\frac{g^2}{2} J_e' \frac{1}{\Delta} J_e'\right) \exp\left(-\frac{8\pi^2}{g^2} J_m \frac{1}{\Delta} J_m\right)$$

where

$$J'_e = J_e + \frac{\theta}{\sqrt{2\pi^2}} J_m$$

The mixing between electric and magnetic currents for $\theta \neq 0$ comes from a non vanishing cross term in the topological lagrangian $F \wedge F$. This is the well known phenomenon discovered by Witten in [15]: magnetic currents contribute to electric currents in the presence of a theta angle.

The important point about this expression is that it is formally self-dual under $g \rightarrow 4\pi/g$ provided we also exchange $J'_e \leftrightarrow J_m$. However, the analogy between J_m and a real monopole current is not complete. If J_m were really monopole currents we would have found an extra cross term coupling J_m directly to J'_e . This is the Aharonov-Bohm interaction responsible for the Dirac quantization rule of electric and magnetic charges. A more complete analysis of the physics of the source partition function in lattice models can be found in [14].

3.2. Global partition function

The fact that the $U(1)$ group on p forms is taken to be compact is reflected (in the absence of monopoles) by the presence of quantized fluxes around homologically non-trivial $(p+1)$ -submanifolds, Σ_I . This leads to the following generalized Theta function coming from the classical action evaluated on harmonic forms [9]

$$Z_{\text{global}}(g, \theta) = \sum_{h_{p+1}} e^{-S_{\text{cl}}} = \sum_{m^I} e^{-\frac{4\pi^2}{g^2} m^I G_{IJ} m^J + \frac{i\theta}{2} m^I Q_{IJ} m^J}$$

where $G_{IJ} = \int \alpha_I \wedge * \alpha_J$, $Q_{IJ} = \int \alpha_I \wedge \alpha_J$ for normalized harmonic $(p+1)$ -forms $\oint_{\Sigma_J} \alpha_I = \delta_I^J$. The interesection matrix Q_{IJ} controls the symmetry under θ -shifts. Note that Poincare duality implies

$$G_{IJ} = \int * \alpha_I \wedge * * \alpha_J, \quad Q_{IJ} = \int * \alpha_I \wedge * \alpha_J$$

So the same global partition function appears in terms of harmonic $(d-p-1)$ -forms $*\alpha_I$.

From the duality of the full partition function one can deduce *a posteriori* that the global partition function transforms under duality as a modular form of weights $(\frac{1}{2}b_{p+1}^-, \frac{1}{2}b_{p+1}^+)$:

$$Z_{\text{global}}(\tau) = \tau^{-\frac{1}{2}b_{p+1}^-} \bar{\tau}^{-\frac{1}{2}b_{p+1}^+} Z_{\text{global}}(-1/\tau)$$

3.3. Non compact partition function

The remaining term is a standard path integral which contains the explicit powers of the coupling g , and is independent of the topological angle θ . According to the heuristic arguments in section 2 we should find

$$Z_{\text{non-compact}}(g) = g^{-\mathcal{N}_p} \int \frac{DA_p}{\text{Vol}(G_p)} e^{-\frac{1}{g^2} \int (dA_p)^2} = g^{(-)^{p+1} \sum_j^p (-)^j b_j} \times (\text{determinants})$$

We can derive this scaling and compute the structure of the determinant terms by repeatedly using the Fadeev-Popov trick. Let us introduce a gauge fixing condition $f(A_p)$ in the first path integral

$$\begin{aligned} \int \frac{D'A_p}{\text{Vol}(G_p)} e^{-S(A_p)} &= \int \frac{D'A_p}{\text{Vol}(G_p)} \frac{D'\lambda_{p-1}}{\text{Vol}(G_{p-1})} \Delta_{FP}^{(p)}(A_p) f(A_p - d\lambda_{p-1}) e^{-S(A_p)} \\ &= \int D'A_p \Delta_{FP}^{(p)}(A_p) f(A_p) e^{-S(A_p)} \end{aligned}$$

Here D' means that we do not integrate over the (finite dimensional) space of zero modes. Also, the integral over the gauge degrees of freedom involves as gauge ambiguity in itself, because exact $(p-1)$ forms do not contribute to G_p . This is implicit in the formal identity

$$\text{Vol}(G_p) = \int \frac{D'\lambda_{p-1}}{\text{Vol}(G_{p-1})}$$

which we used to cancel the group volume. This means that, in evaluating the Fadeev-Popov determinant we have to gauge-fix again:

$$\Delta_{FP}^{(p)}(A_p)^{-1} = \int \frac{D'\lambda_{p-1}}{\text{Vol}(G_{p-1})} f(A_p - d\lambda_{p-1}) = \int D'\lambda_{p-1} \Delta_{FP}^{(p-1)}(\lambda_{p-1}) f(\lambda_{p-1}) f(A_p - d\lambda_{p-1})$$

where we have used $\text{Vol}(G_{p-1}) = \int \frac{D'\lambda_{p-2}}{\text{Vol}(G_{p-2})}$. The new FP functional $\Delta_{FP}^{(p-1)}(\lambda_{p-1})$ has a similar expression as a path integral over λ_{p-2} forms with G_{p-3} gauge ambiguity. In this way, the process of nested gauge fixings continues until we reach zero forms. Using the Feynman gauge at all stages:

$$f(\phi) = \int DC e^{-\frac{1}{g^2} \int C^2} \delta[\delta\phi - C] = e^{-\frac{1}{g^2} \int (\delta\phi)^2}$$

we find for the first determinant (notice that, acting on zero forms, $\delta d = \Delta_0$)

$$\Delta_{FP}^{(1)}(\lambda_1)^{-1} = \int D'\lambda_0 e^{-\frac{1}{g^2} \int (\delta\lambda_1 - \delta d\lambda_0)^2} = g^{B_0 - b_0} \det'_0(\Delta)^{-1}$$

This is the standard result for the electromagnetic Fadeev-Popov determinant. Now we can plug this in the expression for the second determinant:

$$\begin{aligned}\Delta_{FP}^{(2)}(\lambda_2)^{-1} &= \int D' \lambda_1 \Delta_{FP}^{(1)}(\lambda_1) e^{-\frac{1}{g^2} \int (\delta\lambda_1)^2 + (\delta\lambda_2 - \delta d\lambda_1)^2} \\ &= g^{b_0 - B_0 + B_1 - b_1} \det'_0(\Delta) \det'_1(d\delta + (\delta d)^2)^{-1/2} \exp\left(-\frac{1}{g^2} \int \lambda_2 K_2 \lambda_2\right)\end{aligned}$$

where K_2 is the following operator acting on the space of two forms:

$$K_2 = d\delta - d\delta d \frac{1}{d\delta + (\delta d)^2} \delta d\delta$$

It is then easy to proceed and calculate all Fadeev-Popov determinants. We find the following alternating structure:

$$\begin{aligned}\Delta_{FP}^{(n)}(\lambda_n) &= g^{(b_{n-1} - B_{n-1}) - (b_{n-2} - B_{n-2}) + \dots} \times \det'(L_{n-1})^{1/2} \det'(L_{n-2})^{-1/2} \times \dots \\ &\quad \times \exp\left(\frac{1}{g^2} \int \lambda_n K_n \lambda_n\right)\end{aligned}$$

In this expression, the operators K_n and L_n act on the space of n -forms and are defined iteratively as

$$\begin{aligned}L_n &= (\delta d)^2 + d\delta d \frac{1}{L_{n-1}} \delta d\delta \\ K_n &= d\delta + (\delta d)^2 - L_n\end{aligned}$$

with $L_0 = (\delta d)^2 = \Delta_0^2$.

Finally, combining the different pieces we obtain the final result for the partition function in the form

$$Z_{\text{non-compact}}(g) = g^{(-)^{p+1} \sum_j (-)^j b_j} \frac{\prod_{j=0}^{p-1} \det'(L_{p-j-1})^{\frac{(-)^j}{2}}}{\det'(\Delta_p - K_p)^{\frac{1}{2}}} \quad (3.4)$$

4. From p forms to $(d - p - 2)$ forms

The Lagrange multiplier method of ref.[8] is easily extended to the most general duality transformation between p -forms and $(d - p - 2)$ -forms in arbitrary dimension. We will first consider the general case without a theta term in the lagrangian, and also drop the co-exact source term (3.3). We however keep the electric source to study the order-disorder mapping.

In order to actually perform the duality transformation we need to write all terms in first order form, that is, in terms of the field strength. For the source term, this is easily accomplished if the conserved current has no harmonic piece (it is purely local). Then we can write $J_p = \delta J'_{p+1}$ and J' can be chosen to be exact $J'_{p+1} = dJ''_p$. This is only consistent if J'' is also conserved $\delta J'' = 0$ and has no harmonic piece (otherwise it would lead to a vanishing J_p). This means that, in fact, $J_p = \Delta_p J''_p$ and we can write

$$J'_{p+1} = d \frac{1}{\Delta} J_p \quad (4.1)$$

where, as always, the Green function of the laplacian $\frac{1}{\Delta}$ is defined in the space orthogonal to the harmonic forms. In terms of J' it is easy to write the source term in first order form as

$$\int J_p A_p = \int \delta J'_{p+1} A_p = \int J'_{p+1} dA_p = \int J'_{p+1} F_{p+1} \quad (4.2)$$

Now we are ready to introduce the fake gauge field G and write the partition function as a constrained gauge theory:

$$\begin{aligned} Z_p(g, J) &= g^{-\mathcal{N}_p} \sum_{h_{p+1}} \int \frac{DA_p}{\text{Vol}(G_p)} e^{-\int \mathcal{L}(F, J')} \\ &= g^{-\mathcal{N}_p} \sum_{h_{p+1}} \int \frac{DA_p DG}{\text{Vol}(G_{p+1})} \delta[dG] e^{-\int \mathcal{L}(F+G, J')} \end{aligned}$$

the local delta functional includes also a periodic delta function reducing the periods of G to $2\pi \times (\text{integer})$. In this way, solving the constraint and gauge fixing G to zero projects G_{p+1} onto the original gauge group G_p . The constraint is easily exponentiated in terms of a $(p+2)$ -form

$$\begin{aligned} \delta[dG] &= \int \frac{D\chi_{p+2}}{\text{Vol}(G_{p+2})} \sum_{n^I} \exp\left(\frac{i}{2\pi} \int (dG \chi_{p+2} + 2\pi G n^I \alpha_I)\right) \\ &= \sum_{h_{p+1}} \int \frac{D\chi_{p+2}}{\text{Vol}(G_{p+2})} \exp\left(\frac{i}{2\pi} \int G(\delta\chi_{p+2} + h_{p+1})\right) \end{aligned}$$

and the gauge group G_{p+2} appears because this representation of the delta functional has a gauge ambiguity $\chi_{p+2} \rightarrow \chi_{p+2} + \delta\psi_{p+3}$. Putting all the terms together and integrating G out we obtain

$$Z_p(g, J) = g^{-\mathcal{N}_p} g^{B_{p+1}} \sum_{h_{p+1}} \int \frac{D\chi_{p+2}}{\text{Vol}(G_{p+2})} \exp\left(-\frac{g^2}{16\pi^2} \int (\delta\chi_{p+2} + h_{p+1} + 2\pi J'_{p+1})^2\right)$$

We end up with a theory of $(p + 2)$ -forms with an “inverted” gauge symmetry, in terms of the co-derivative. A similar analysis to the one performed in the previous section would reveal that the natural definition of the gauge invariant measure in this path integral includes a prefactor

$$g^{(-)^p} \sum_{j=p+2}^d (-)^j B_j$$

so that the modular anomaly (the net power of g) is given by $g^{(-)^{p+1}\chi}$ as expected. Notice that at this point we had no need for selfduality at the level of the regulator. However, if we want to write the dual model as a theory of $(d - p - 2)$ -forms with standard gauge invariance, then we must think of the form $\tilde{A}_{d-p-2} = *\chi_{p+2}$ in the regulated theory as defined on the dual lattice. Then, the dual partition function without sources is

$$\begin{aligned} \tilde{Z}_{d-p-2}(4\pi/g) &= \left(\frac{4\pi}{g}\right)^{-\mathcal{N}_{d-p-2}^*} \sum_{\tilde{h}_{d-p-1}} \int \frac{D\tilde{A}_{d-p-2}}{\text{Vol}(\tilde{G}_{d-p-2})} e^{-\int \tilde{\mathcal{L}}} \\ \tilde{\mathcal{L}} &= \frac{g^2}{16\pi^2} (d\tilde{A}_{d-p-2} + \tilde{h}_{d-p-1})^2 \end{aligned} \quad (4.3)$$

Finally, by the definition of dual lattice, we have $B_j^* = B_{d-j}$. In this way the powers of g combine such that the modular weight depends only on the Euler character. If we include the sources, we get the general duality relation

$$Z_p(g, J) = (\sqrt{4\pi}/g)^{(-)^p\chi} \tilde{Z}_{d-p-2}(4\pi/g)_{\text{frustrated}} \quad (4.4)$$

where the frustrated partition function is the same as (4.3) with a modified lagrangian

$$\tilde{\mathcal{L}}_{\text{frustrated}} = \frac{g^2}{16\pi^2} (d\tilde{A}_{d-p-2} + \tilde{h}_{d-p-1} + 2\pi * J'_{p+1})^2$$

These formulas are easily generalized to the case where a theta term is present, $d-p-2 = p$. The integration over the fake gauge field G is easier in terms of the selfdual and anti-selfdual projections G^\pm and the result is a frustrated partition function with the modular anomaly of eq. (2.20).

The frustration means that we cannot absorb the $*J'$ term in a continuous redefinition of \tilde{A} . This is due to the fact that, according to (4.1), $*J'$ cannot be written as smooth exact differential. In fact $*J'$ is co-exact and acts as a monopole current, because it prevents the effective field strength of the dual theory from being closed.

The most interesting sources to consider are distributions localized on closed p -manifolds which lead to generalized Wilson lines. For example, for vectors, choosing J as the electromagnetic current of first quantized particles of charges Q_i

$$J = \sum_j Q_j \int_{C_j} d\tau_j \frac{dx^\alpha}{d\tau_j} \delta[x - x(\tau_j)] dx_\alpha$$

induces a term

$$\int JA = \sum_j Q_j \oint_{C_j} A$$

and the corresponding partition function is in fact a correlator of Wilson lines.

$$Z_1(g, J) = Z_1(g, J = 0) \left\langle \prod_j e^{iQ_j \oint_{C_j} A} \right\rangle$$

In the general case, since $\delta J = 0$ is conserved and has no harmonic piece, we have

$$\int_{\mathcal{M}_d} JA = Q \oint_{\Sigma_p} A_p$$

where Σ_p is the boundary of its interior $\Sigma_p = \partial\Sigma_p^0$. Then, by the Stokes theorem and equation (4.2)

$$Q \oint_{\Sigma_p} A_p = Q \int_{\Sigma_p^0} dA_p = \int_{\mathcal{M}_d} J'_{p+1} dA_p \quad (4.5)$$

so that J'_{p+1} in this case is equal to a distribution of value Q on the $(p+1)$ -dimensional ball Σ_p^0 and zero outside. We then see the geometrical meaning of the frustration: if we want to write $*J'$ as an exact differential we must introduce discontinuities across the $(p+1)$ -manifold Σ_p^0 . In general, such discontinuities are better visualized in a lattice formulation, where p -forms are functions over the p -cells of the simplicial decomposition. Then, since \tilde{A}_{d-p-2} is evaluated on $(d-p-2)$ -cells of the dual lattice, the frustration amounts to a $2\pi Q$ shift of $d\tilde{A}$ over those $(d-p-1)$ -cells dual to Σ_p^0 . For $d=4$, $p=1$ this is just the 't Hooft loop construction for compact QED.

When Σ_p is codimension two in the space-time manifold, then the dual form \tilde{A} is a scalar and a simple continuum construction can be given, which generalizes the vortex lines of two dimensional T-duality. If $p = d - 2$, then Σ_p^0 is a $(d - 1)$ -dimensional submanifold of \mathcal{M}_d such that $\partial(\mathcal{M}_d - \Sigma_p^0) = (\Sigma_p^0)^+ - (\Sigma_p^0)^-$, where $(\Sigma_p^0)^\pm$ are the ‘‘up’’ and ‘‘down’’

faces as seen from \mathcal{M}_d . Returning to (4.5) we can use the Stokes theorem on $\mathcal{M}_d - \Sigma_p^0$ to write

$$\int_{\Sigma_p^0} dA_p = \int_{(\Sigma_p^0)^+} 1 \cdot dA_p - \int_{(\Sigma_p^0)^-} 0 \cdot dA_p = \int_{\mathcal{M}_d - \Sigma_p^0} d\alpha_\Sigma \wedge dA_p$$

where α_Σ is a scalar function with a unit jump $\alpha_{\Sigma^+} - \alpha_{\Sigma^-} = 1$ across Σ_p^0 . Now we can use this discontinuous function to define $*J' = Q d\alpha_\Sigma$. This is a generalization of the two dimensional case, where Σ_p is a pair of points, Σ_p^0 is the cut joining them, and α_Σ is an angular variable with respect to the cut. In this way the dual field has the boundary conditions of a pair of oppositely charged vortices, and the partition function is a two-point correlator of winding mode operators.

5. Conclusions

We have presented a unified picture of the abelian S-duality of p -form theories in arbitrary euclidean space-times. A simple extension of the methods of two dimensional sigma-model duality addresses local and global questions in the general case, including the order-disorder mapping on gauge invariant observables, and the modular duality anomaly, which appears as a simple generalization of the dilaton shift in sigma model duality. Also, Roček-Verlinde coset constructions are easily generalized, leading to the same results as the Lagrange multiplier method.

Perhaps the most interesting open question is the existence of a non abelian version of the dual coset procedure. Such a generalization would be useful in understanding the more complicated non abelian dualities mentioned in the introduction. It is more likely that a non abelian generalization of the Lagrange multiplier method is easier to study, because the two dimensional counterpart for sigma models is known [16]. Notice, however, that the auxiliary gauge field involved in, for example, four dimensional electric-magnetic duality, is a two-form, and it is notoriously difficult to construct a non abelian three-form field strength with the right properties. In general, non abelian generalizations of higher rank gauge theories do not exist.

6. Acknowledgements

It is a pleasure to thank Y. Lozano for useful discussions. This work was supported by NSF PHY90-21984 grant.

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