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Analytical Properties of Solutions of the Schrödinger Equation and Quantization of Charge

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Abstract

The Schwinger–DeWitt expansion for the evolution operator kernel is used to investigate analytical properties of the Schrödinger equation solution in time variable. It is shown, that this expansion, which is in general asymptotic, converges for a number of potentials (widely used, in particular, in one-dimensional many-body problems), and besides, the convergence takes place only for definite discrete values of the coupling constant. For other values of charge the divergent expansion determines the functions having essential singularity at origin (beyond usual δ-function). This does not permit one to fulfil the initial condition. So, the function obtained from the Schrödinger equation cannot be the evolution operator kernel. The latter, rigorously speaking, does not exist in this case. Thus, the kernel exists only for definite potentials, and moreover, at the considered examples the charge may have only quantized values.

1 Introduction

In the quantum theory expansions in different parameters such as the coupling constant [1, 2, 3, 4], the WKB–expansion, the short-time Schwinger–DeWitt expansion [5, 6, 7, 8], the perturbation expansion in phase-space technique [9], $1/n$-expansion [10], etc. are, as a rule, asymptotic. This circumstance imposes essential restrictions on possibilities of their using, makes the theory incomplete and compels one to look for the ways of overcoming these restrictions either by summation of divergent series with special methods (see, e.g., [11]), or by constructing new convergent expansions [12, 13, 14], or by creating different approximate methods taking into consideration the so-called nonperturbative effects.

Useful information may be obtained from investigation of the character of singularity causing the divergence of expansions, as it was made, e.g., in [1] for the expansion in the coupling constant for the anharmonic oscillator. The purpose of the present paper is to investigate analytical properties of the evolution operator kernel of the Schrödinger equation in time variable in the neighborhood of origin. As usual, one works with the short-time Schwinger–DeWitt expansion [5, 6], having numerous applications mainly connected with the theories in the curved space–time [7, 15], as with asymptotic one. However, it can be helpful for examining analytical properties of the kernel in vicinity of the point $t = 0$, in particular, for searching for the potentials, for which this expansion is convergent and the point $t = 0$ is the regular one.

An important feature of the Schwinger–DeWitt expansion is that after factorization of the contribution of the free kernel ("free" case corresponds to $V \equiv 0$), having at $t = 0$ the singularity in the form of δ-function, one can concentrate attention on rest part which, according to the initial condition, should be equal to 1 when $t = 0$. The essential point of presented research is: is it possible or not to satisfy this initial condition meaning t as complex variable?

If one considers the real t only (this is usually done), then analytical properties in the vicinity of the point $t = 0$ are fully ignored. If the analytical continuation into the complex plain t is made for the kernel (this may be done only inside some sector with angle $\gamma < \pi$), then its analytical properties are masked by the singularity in the form of δ-function. But if the factorization of the free part of the kernel is made, then the rest function can be continued into the entire complex plain t and one can accurately examine its properties in the neighborhood of origin.

It is usually considered that if the function asymptotically tends to 1 at $t \to 0$ then it is quite enough to fulfil the initial condition. However, in the case of divergent expansion $t = 0$ is essential singular point, and, really, the function does not have any meaning at this point. The function tends to 1 only if its argument belongs to some sector in the complex plain. When the argument comes to zero outside this sector then one can obtain every desired meaning of the function.

In this case one is to account that, strictly speaking, the evolution operator kernel does not exist. And these potentials cannot be considered in the quantum mechanics as exact ones. They can be treated only as some approximations to unknown exact potentials for which the Schwinger–DeWitt expansion is convergent. Namely in this context one should consider the weaken Cauchy's problem, in which one does not impose the rigorous demands to behavior of the solution in the vicinity of the point $t = 0$.

In this paper some nontrivial potentials are presented for which the Schwinger–DeWitt expansion is convergent. Those are the potentials being widely used in one-dimensional many-body problems [16, 17, 18, 19]. For definite discrete values of the coupling constant the expansions for them are convergent in the entire complex plain t. For other values of the charge the expansions are asymptotic and, so, the kernels in rigorous sense do not exist. This fact can be treated as quantization of the charge. The solution of the Cauchy's problem for the Schrödinger equation exists only for some discrete values of the charge. Hence, the potentials with those charges can only be considered in the exact quantum theory.

2 The method of research

The evolution operator kernel of the Schrödinger equation in one-dimensional case is the solution of the problem

$$\frac{\partial}{\partial t}\langle q', t \mid q, 0\rangle = \frac{1}{2}\frac{\partial^2}{\partial q'^2}\langle q', t \mid q, 0\rangle - V(q')\langle q', t \mid q, 0\rangle, \tag{1}$$

$$\langle q', t = 0 \mid q, 0\rangle = \delta(q' - q). \tag{2}$$

Here and everywhere below dimensionless values, which are derived from dimension ones in an obvious way, are used for the sake of convenience. The variable t is treated as a complex one. If one means the proper Schrödinger equation then t should be equal to $i\tau$, where τ is a real variable (physical time). Because we intend to study analytical properties of the kernel in variable t, then it is convenient to use t instead of τ. We imply that $V(q)$ does not apparently depend on time.

As it is well known, in the free case ($V \equiv 0$) the solution of the problem (1), (2) is

$$\langle q', t \mid q, 0\rangle = \frac{1}{\sqrt{2\pi t}} \exp\left\{-\frac{(q'-q)^2}{2t}\right\} \equiv \phi(t; q', q). \tag{3}$$

The function ϕ has essential singularity at $t = 0$, but this singularity is such, that it provides the initial condition (2) to be fulfilled.

When interaction is present the kernel can be represented as

$$\langle q', t \mid q, 0\rangle = \frac{1}{\sqrt{2\pi t}} \exp\left\{-\frac{(q'-q)^2}{2t}\right\} F(t; q', q), \tag{4}$$

and besides, one can write for F the expansion (the short-time Schwinger–DeWitt expansion)

$$F(t; q', q) = \sum_{n=0}^{\infty} t^n a_n(q', q), \tag{5}$$

which, as a rule, is asymptotic and is usually utilized only in that quality. Particularly, in [8] it is shown that for the polynomial potentials (of an order L higher then 2) expansion (5) diverges as $\Gamma\left(\frac{L-2}{L+2}n\right)$.

The question arises if the emphasis of the δ-like function in form of the function ϕ in (4) is unique. Generally speaking, every function of the form $\Phi((q'-q)/\sqrt{t})/\sqrt{t}$ with normalization condition

$$\int_{-\infty}^{+\infty}\Phi(z)dz = 1$$

will tend to $\delta(q'-q)$ when $t \to 0$. Let us take, in this connection, the general representation

$$\langle q', t \mid q, 0\rangle = \frac{1}{\sqrt{t}}\Phi\left(\frac{q'-q}{\sqrt{t}}\right)\tilde{F}(t;q',q) \tag{6}$$

and substitute it into (1). Taking into account, that Φ depends on the variables only through the combination $z = (q'-q)/\sqrt{t}$, but in $\tilde{F}$ the variables t, q', and q vary independently with each other, one can separate the variables and get

$$\frac{d^2\Phi}{dz^2} + z\frac{d\Phi}{dz} + \Phi(z) = 0, \tag{7}$$

$$\frac{\partial\tilde{F}}{\partial t} = \frac{1}{2}\frac{\partial^2\tilde{F}}{\partial q'^2} + \frac{1}{\sqrt{t}\Phi}\frac{d\Phi}{dz}\frac{\partial\tilde{F}}{\partial q'} - V(q')\tilde{F}. \tag{8}$$

Demanding $\tilde{F}$ to be regular at $t = 0$ and at $q' = q$ one has an unambiguous definition of the function $\Phi(z)$

$$\Phi(z) = \frac{1}{\sqrt{2\pi}}\exp\{-z^2/2\}. \tag{9}$$

This exactly coincides with (4) and so, $\tilde{F} = F$ and (8) with taking into consideration of (9) gives the equation for the function F.

We shall use representation (4), (5) to test the analytical properties of the evolution operator kernel in variable t and, in particular, ascertain its behavior for $t \to 0$, which is necessary to examine whether initial condition (2) is fulfilled or not.

For this purpose let us derive from (1) (or from (8), (9)) the equation for F

$$\frac{\partial F}{\partial t} = \frac{1}{2}\frac{\partial^2 F}{\partial q'^2} - \frac{q'-q}{t}\frac{\partial F}{\partial q'} - V(q')F. \tag{10}$$

Because the factorized function ϕ yet fulfils the initial condition (2), then F should satisfy the initial condition

$$F(t=0;q',q) = 1. \tag{11}$$

It seems, at first sight, that it is possible to add to the right-hand side of (11) an arbitrary function of $q'-q$, which vanishes at $q' = q$. However, this is not true. The equation for the coefficient a_0

$$(q'-q)\frac{\partial a_0(q',q)}{\partial q'} = 0,$$

taken from general recursion relations for $a_n(q', q)$, and condition $a_0(q, q) = 1$ determines unambiguously

$$a_0(q', q) = F(0; q', q) = 1.$$

The problem (1), (2), from which we have started, has a physical sense only for the real positive t (if the heat equation and heat kernel are considered) or for $t = i\tau$, where τ is the real positive (if the quantum mechanical evolution equation is considered). The same restrictions are initially fair for equation (10) too. But we can analytically continue the function F into complex plain of the variable t using the differential equation (10) with condition (11).

Write the representation of kind (5) giving concrete form to the coordinate dependence of the coefficients a_n

$$F(t; q', q) = 1 + \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} t^n \Delta q^k b_{nk}(q). \tag{12}$$

Here $\Delta q = q' - q$. For the potential V we write the expansion in powers of Δq

$$V(q') = \sum_{k=0}^{\infty} \Delta q^k \frac{V^{(k)}(q)}{k!}. \tag{13}$$

The notation similar with

$$V^{(k)}(q) \equiv \frac{d^k V(q)}{dq^k}$$

will be used further everywhere. We mean that the point q is a regular point of the function $V(q)$ because we are interested in the regular points only when calculating the kernel.

It is obvious that

$$\sum_{k=0}^{\infty} \Delta q^k b_{nk}(q) = a_n(q', q) = -Y_n(q', q),$$

where Y_n are the functions introduced in [20]. The behavior of Y_n was studied in [8] using representation adduced in [20].

Substitution of (12), (13) into (10) leads to recurrent relations for the coefficients b_{nk}

$$b_{nk} = \frac{1}{n+k} \left[\frac{(k+1)(k+2)}{2} b_{n-1,k+2} - \sum_{m=0}^{k} \frac{V^{(m)}(q)}{m!} b_{n-1,k-m} \right] \tag{14}$$

with condition $b_{0k} = \delta_{k0}$. Specifically,

$$b_{1k} = -\frac{V^{(k)}(q)}{(k+1)!}. \tag{15}$$

Expressions (12), (14) determine a formal solution of problem (10), (11). As to the expansion in powers of Δq in (12), one may expect that its convergence range is equal to the one for expansion (13) of the potential. The series in t in (12) is usually treated as the divergent one. At first sight, it is really so always. Let us estimate the convergence of the series in (12).

At the beginning let n be fixed and $k \to \infty$. Expressing $b_{n-1,k+2}$ from (14) via the coefficients with smaller k we shall come to some linear combination of the coefficients of type $b_{n_0,0}$ and $b_{n_1,1}$ with any indexes n_0, n_1 (for the sake of brevity we shall write further only the terms with $b_{n_0,0}$, implying that the same statements are concerned with the terms with $b_{n_1,1}$). The main growth for large k takes place if the second index of b_{nk} is diminished: a) with using the term $V^{(k)} b_{n-1,0}/k!$, b) with using the expression on the left-hand side of (14).

In the case a) we get for $k \to \infty$

$$|b^{(a)}_{n-1,k+2}| \sim \frac{2}{(k+1)(k+2)} \frac{|V^{(k)}|}{k!} |b_{n-1,0}|. \tag{16}$$

Because series (13) converges at some circle with radius $R(q)$ the estimate

$$\frac{|V^{(k)}|}{k!} \sim \frac{1}{R^k(q)}$$

for $k \to \infty$ is fair. So, for every fixed n and for $k \to \infty$ we have

$$|b^{(a)}_{nk}| \sim \frac{|b_{n0}|}{R^k(q)}. \tag{17}$$

The contributions of type (17) correspond to the expansion in Δq, which is convergent for every fixed n with convergence range $R(q)$.

In the case b) for $k \to \infty$ we get

$$|b^{(b)}_{n-1,k+2}| \sim \frac{2^{k/2+1}(n+k)!}{k!(n+k/2-1)!} |b_{n+k/2,0}|.$$

Behavior of $b^{(b)}_{nk}$ for $k \to \infty$ depends on the behavior of b_{n0} for $n \to \infty$. If b_{n0} decreases when $n \to \infty$ or increases more slowly than $\Gamma(\alpha n)$ (α is any positive number), then

$$|b^{(b)}_{nk}| \sim \frac{|b_{n0}|}{\Gamma(k/2)}$$

for $k \to \infty$, i.e., these contributions will disappear at large k. If b_{n0} increases as $\Gamma(\alpha n)$ (here $0 < \alpha \leq 1$, in [8] showed that α cannot be larger then 1), then for $k \to \infty$ and $\alpha < 1$

$$|b^{(b)}_{nk}| \sim \frac{|b_{n0}|}{\Gamma\left(\frac{1-\alpha}{2} k\right)},$$

so, these contributions will disappear too with the growth of k. If $\alpha = 1$, then the following estimate will take place (n is fixed, $k \to \infty$)

$$|b^{(b)}_{nk}| \sim |b_{n0}| k^c \rho^k. \tag{18}$$

In this case the expansion in Δq in 12) will have the finite convergence range too, but it will be equal to minimum from two values $R(q)$ and ρ.

Now let us examine the behavior of $|b_{n0}|$ (the same will be also correct for $|b_{n1}|$) when $n \to \infty$. Consider the decreasing of n till 1 by means of the first term on the right-hand side of (14)

$$|b_{n0}| \sim \frac{|b_{n-1,2}|}{n} \sim \cdots \sim \frac{(n-1)!}{2^{n-1}}|b_{1,2n-2}| = \frac{(n-1)!}{2^{n-1}} \frac{|V^{(2n-2)}|}{(2n-1)!}. \tag{19}$$

Because $|V^{(k)}| \sim k!/R^k(q)$ for $k \to \infty$, then for $n \to \infty$ we get

$$|b_{n0}| \sim \frac{(n-1)!}{2^{n-1}(2n-1)} \sim n!. \tag{20}$$

Really, the contributions taken into account in (19) provide the main growth only for the potentials, for which $R(q) < \infty$. If the potentials with $R(q) = \infty$ are considered (e.g., polynomial ones), then, at first sight, one can conclude from (19) that $|b_{n0}| \sim 1/n!$. But it is not so, in fact. As it was shown in [8], the combination of contributions of the first term and terms of sum over m in (14) leads to the estimate of type $|b_{n0}| \sim \Gamma(\alpha n)$.

So, for arbitrary potentials the series in t in (12) is divergent. But in our estimates, in fact, absolute values of all contributions to every coefficient b_{nk} were summed. Nevertheless, for some potentials the cancellation of different terms may occur. It can lead to convergence of the expansion in (12). For the potentials considered in Secs. 3–5 this cancellation takes place only for definite values of the coupling constant.

Note that we, really, test expansion (12) for the absolute convergence. So, it is enough for the convergence of double series that (12), in which instead of b_{nk} absolute values $|b_{nk}|$ taken, would converge for any order of summation. Our consideration corresponds to the following order: at first the series over k for every fixed n are summed and then summation over n is made. If one assumes that there is convergence of the series in index n then, as it was shown before, the convergence in index k will take place at every fixed n, and to establish the convergence of the series in index n it is enough to determine the behavior of the coefficients b_{n0}, b_{n1} only (but not all b_{nk}) at $n \to \infty$.

3 Potential $V(q) = g/q^2$

Let us introduce standard notation for the coupling constant $g = \lambda(\lambda-1)/2$ ($\lambda > 0$) and investigate the potential

$$V(q) = \frac{\lambda(\lambda-1)}{2} \frac{1}{q^2} \tag{21}$$

for the convergence of expansion (12).

Expansion (13) for potential (21) has the finite convergence range $R(q) = q$ which is connected with singularity of $V(q)$ at the point $q = 0$. The derivatives $V^{(k)}$ may be easily calculated

$$V^{(k)}(q) = (-1)^k \frac{\lambda(\lambda-1)}{2} \frac{(k+1)!}{q^{k+2}}. \tag{22}$$

From (14) with account of (22) we take

$$b_{nk} = \frac{1}{n+k}\left[\frac{(k+1)(k+2)}{2}b_{n-1,k+2} + \frac{\lambda(\lambda-1)}{2}\sum_{m=0}^{k}(-1)^{m+1}\frac{m+1}{q^{m+2}}b_{n-1,k-m}\right]. \qquad (23)$$

If we shall diminish n times the first index of b_{nk} by means of (23), then we get

$$b_{nk} = \frac{(-1)^{n+k}}{q^{2n+k}}\frac{(k+n-1)!}{n!(n-1)!k!}\prod_{j=1}^{n}\left(\frac{\lambda(\lambda-1)}{2} - \frac{j(j-1)}{2}\right). \qquad (24)$$

For noninteger λ the main growth at $n\to\infty$ of the polynomial of the order n in the variable $g = \lambda(\lambda-1)/2$ which we denote as

$$\Lambda_n(g) = \prod_{j=1}^{n}\left(g - \frac{j(j-1)}{2}\right) \qquad (25)$$

will be provided by the coefficient in front of the first power of g. This coefficient is equal to $n!(n-1)!/2^{n-1}$. So, for $n\to\infty$ the estimate

$$|b_{nk}| \sim \frac{(n+k-1)!}{2^{n-1}|q|^{2n+k}k!} \sim n! \qquad (26)$$

is true. It means that for noninteger λ expansion (12) for potential (21) is divergent.

Consider now integer λ ($\lambda > 1$, because cases $\lambda = 0,\ \lambda = 1$ are trivial). In this case $\Lambda_n = 0$ for $n \geq \lambda$. Hence, only the coefficients b_{nk} for $n < \lambda$ are different from zero, and in (12) the series in powers of t is really the polynomial of finite degree $\lambda - 1$.

Let us substitute (24) (for $n < \lambda$) into (12) and make summation over k. Then we get finally

$$F(t;q',q) = 1 + \sum_{n=1}^{\lambda-1}\frac{(-1)^n t^n}{q'^n q^n}\frac{1}{n!}\prod_{j=1}^{n}\left(\frac{\lambda(\lambda-1)}{2} - \frac{j(j-1)}{2}\right). \qquad (27)$$

Rigorously speaking, the derivation of (27) is made in supposition that $|\Delta q| < |q|$. If this condition is not satisfied, then calculations should be made with the expansion about the point q' in powers of $q - q'$. But because F is symmetric in $q',\ q$, then it is clear that the answer in this case would be the same as in (27). These reasonings are valid for the case when q and q' have the same sign. But the potential (21) at $q \to 0$ tends to ∞, and the transition through the point $q = 0$ is impossible. Hence, the kernel is equal to zero if q and q' have different signs.

So, we established that for integer λ expansion (4), (12) for the evolution operator kernel is not asymptotic, but it is the convergent one. The function F is presented by the polynomial of degree $\lambda - 1$ in powers of t and inverse powers of q' and q. It is single-valued and analytic in the entire complex plain of the variable t function. There is the pole of the order $\lambda - 1$ in the infinite point. Further consideration for noninteger λ will be made in Sec. 6.

4 Modified Pöschl–Teller's potential $V(q) = -g/\cosh^2 q$

Another example of convergent series (12) we shall get considering the potential

$$V(q) = -\frac{\lambda(\lambda-1)}{2}\frac{\beta^2}{\cosh^2(\beta q)}. \tag{28}$$

Because the constant β is connected with the choice of length scale one can put $\beta = 1$ without the restriction of generality. Further, for the sake of brevity we shall denote

$$f(q) = -\frac{1}{\cosh^2 q}. \tag{29}$$

Then the potential reads briefly $V(q) = gf(q)$.

The potential (28) has the expansion of type (13) about every real point q. Its convergence range is equal to $R(q) = \sqrt{(\pi/2)^2 + q^2}$ and is determined by the distance to the nearest singularities of the function $1/\cosh^2 q$ placed at the points $q = \pm i\pi/2$. The derivatives can be calculated as follows

$$V^{(k)}(q) = gf^{(k)}(q), \tag{30}$$

where $f^{(k)}$ are represented as expansions in powers of f

$$f^{(2n)}(q) = \sum_{l=1}^{n+1} a_l^{(2n)} f^l(q), \tag{31}$$

$$f^{(2n+1)}(q) = \sum_{l=1}^{n+1} l a_l^{(2n)} f^{l-1} f^{(1)} = \sum_{l=1}^{n+1} a_l^{(2n+1)} f^{l-1} f^{(1)}. \tag{32}$$

To obtain all coefficients $a_l^{(k)}$ it is enough to put $a_l^{(0)} = \delta_{l1}$ and take into account

$$(f^{(1)})^2 = 4f^3 + 4f^2.$$

For $a_l^{(2n)}$ one has the recursion relations

$$a_l^{(2n)} = 4l^2 a_l^{(2n-2)} + 4(l-1)(l-1/2)a_{l-1}^{(2n-2)}. \tag{33}$$

So, every derivative of the function $f(q)$ is represented as a polynomial in powers of this function.

From (14) one gets for potential (28)

$$b_{nk} = \frac{1}{n+k}\left[\frac{(k+1)(k+2)}{2}b_{n-1,k+2} - \frac{\lambda(\lambda-1)}{2}\sum_{m=0}^{k}\frac{f^{(m)}}{m!}b_{n-1,k-m}\right], \tag{34}$$

where the derivatives $f^{(m)}$ are calculated via (31)–(33).

According to the note at the end of Sec. 2, it is enough for testing the convergence of series (12) to examine the behavior at $n \to \infty$ of the coefficients $b_{n0},\ b_{n1}$ only. Introduce in this connection the functions

$$B_k(t,q) = \sum_{n=0}^{\infty} t^n b_{nk}(q) \tag{35}$$

and consider them for $k = 0,\ 1$.

The analysis of relations (34) with taking into account of (31)–(33) shows that $B_0,\ B_1$ can be represented in the form

$$B_0(t,q) = 1 + \sum_{n=1}^{\infty} t^n \sum_{l=1}^{n} \frac{(-1)^l}{l!} f^l(q)\beta_{nl} \prod_{j=1}^{l} \left(\frac{\lambda(\lambda-1)}{2} - \frac{j(j-1)}{2} \right), \tag{36}$$

$$B_1(t,q) = \sum_{n=1}^{\infty} t^n \sum_{l=1}^{n} \frac{(-1)^l}{l!} \frac{l}{2} f^{l-1}(q) f^{(1)}(q)\beta_{nl} \prod_{j=1}^{l} \left(\frac{\lambda(\lambda-1)}{2} - \frac{j(j-1)}{2} \right), \tag{37}$$

where

$$\beta_{nl} = \frac{1}{2^{n-l}} \frac{(n-1)!}{(l-1)!} \frac{a_l^{(2n-2)}}{(2n-1)!}. \tag{38}$$

To estimate the behavior of β_{nl} when $n \to \infty$ we probe the asymptotics of $a_l^{(2n-2)}$. Let us take $a_l^{(2n-2)}$ for sufficiently large n and begin to express $a_l^{(2n-2)}$ with the help of (33) via the coefficients with smaller n and l so that to come to $a_1^{(0)} = 1$ at the end. Maximal contribution arises in this procedure when, at the beginning, n will be diminished at fixed l by means of the first term on the right-hand side of (33), and then, when the equality $n = l-1$ becomes valid, n and l will start to be decreased simultaneously by unit at every step by means of the second term in (33). For large n this gives the estimate

$$a_l^{(2n-2)} \sim 4^{n-l} l^{2(n-l)} (2l-1)!.$$

Then β_{nl} behaves itself as

$$\beta_{nl} \sim 2^{n-l} l^{2(n-l)} \frac{(n-1)!}{(l-1)!} \frac{(2l-1)!}{(2n-1)!}. \tag{39}$$

Now one can evaluate the asymptotics at $n \to \infty$ of the coefficients of series (36), (37). For noninteger λ at sufficiently large l the polynomial $\Lambda_l(g)$ (see (25)) behaves itself as $\Lambda_l \sim g l!(l-1)!/2^{l-1}$. Taking in (36), (37) in the sum over l the term with $l = n$ we shall obtain that the coefficients in front of t^n growth in (36) as

$$\frac{(n-1)! f^n}{2^{n-1}},$$

and in (37) as

$$\frac{n! f^{n-1} f(1)}{2^n}.$$

So, for noninteger λ series (36), (37), and, hence, (12) are asymptotic ones.

Let now λ be integer ($\lambda > 1$). Then the polynomial $\Lambda_l(g)$ is different from zero only if $l < \lambda$. So, in (36), (37) in the sums over l only the terms with $l < \lambda$ are different from zero, and, in fact, one should take instead of $\sum_{l=1}^{n}$ the sum $\sum_{l=1}^{\min\{n,\lambda-1\}}$. For $n \geq \lambda - 1$ the sum over l will always contain the same number of terms, equal to $\lambda - 1$, and its dependence on n will be determined only by the dependence on n of the coefficients β_{nl}. And the dependence of the latter on n, as it is clear from estimate (39), at fixed $l \leq \lambda - 1$ and at $n \to \infty$ is determined by the factor

$$\beta_{nl} \sim \left(2(\lambda-1)^2\right)^n \frac{(n-1)!}{(2n-1)!}.$$

So, the coefficients in front of t^n in (36), (37) behave themselves at large n as

$$\frac{C^n(n-1)!}{(2n-1)!},$$

i.e., the series will be convergent at the circle of infinite range.

To obtain finally the function $F(t;q',q)$ it is necessary either to take the coefficients b_{n0}, b_{n1} from (36), (37) to calculate other b_{nk} using (34), or starting from B_0, B_1 to calculate other functions $B_k(t,q)$ from the equations

$$B_{k+2} = \frac{2}{(k+1)(k+2)} \left(\frac{\partial B_k}{\partial t} + \frac{k}{t} B_k + \sum_{m=0}^{k} \frac{f^{(m)}}{m!} B_{k-m} \right), \tag{40}$$

which in an obvious way are derived from (10) after substitution

$$F(t;q',q) = \sum_{k=0}^{\infty} \Delta q^k B_k(t,q), \tag{41}$$

and substitute them into (41).

Particularly, for $\lambda = 2$ ($g = 1$) we have the potential $V(q) = -1/\cosh^2 q$, for which

$$B_0(t,q) = 1 - f(q) \sum_{n=1}^{\infty} \frac{t^n}{(2n-1)!!} = 1 - f(q)\sqrt{\frac{\pi t}{2}} e^{t/2}\, erf(\sqrt{t/2}), \tag{42}$$

$$B_1(t,q) = -\frac{1}{2} f^{(1)}(q) \sum_{n=1}^{\infty} \frac{t^n}{(2n-1)!!} = -\frac{1}{2} f^{(1)}(q)\sqrt{\frac{\pi t}{2}} e^{t/2}\, erf(\sqrt{t/2}). \tag{43}$$

With the help of (40) one is able to determine all coefficient functions B_k starting from (42), (43) and then to substitute them into (41). In this manner the function F will be found.

We established, that for integer λ expansion (12) was convergent if $|\Delta q| < R(q)$ and the representation (4), (12) for the evolution operator kernel was not asymptotic. The function F is single-valued analytic in the entire complex plain of the variable t function and it has an essential singularity at the infinite ($t = \infty$) point.

5 Other samples of potentials

Calculations made in Secs. 3, 4 may be easily repeated for some similar potentials which are often used in one-dimensional many-body problems [16, 17, 18, 19].

5.1 Potential $V(q) = g/\sinh^2 q$

Analogously to (28) one can consider the potential

$$V(q) = \frac{\lambda(\lambda-1)}{2}\frac{1}{\sinh^2 q}. \tag{44}$$

Denote

$$f(q) = \frac{1}{\sinh^2 q}, \tag{45}$$

and notice, that

$$(f^{(1)})^2 = 4f^3 + 4f^2,$$

i.e., it exactly coincides with the corresponding expression for the function $f(q)$ defined by (29) in Sec. 4. So, all relations for the derivatives of f obtained there and, hence, expressions for b_{nk}, B_k, and F remain right in this case. There exists only one difference: function f is defined now by (45), but not by (29).

One can conclude that the kernel for potential (44) for integer λ exists and is determined by equations (4), (36)–(38), (40), (41), (45). The function F is single-valued and analytic function in the entire complex plain of the variable t.

5.2 Potential $V(q) = g/\sin^2 q$

The potential

$$V(q) = \frac{\lambda(\lambda-1)}{2}\frac{1}{\sin^2 q} \tag{46}$$

can be also considered in similar way. Denote

$$f(q) = \frac{1}{\sin^2 q} \tag{47}$$

and take into account, that

$$(f^{(1)})^2 = 4f^3 - 4f^2.$$

This expression differs from analogous ones for potentials (28), (44) only by the sign of the second term. So, we are able to reconstruct the expressions from Sec. 4 with small changes only: in (36), (37) will appear an additional multiplier $(-1)^{n+l}$, and the function f will be defined by (47). The conclusion about the existence of the kernel for integer λ remains right for the potential (47).

6 Divergent expansions. Quantization of charge

Now we shall consider the potentials discussed above, but take noninteger λ. For this we, at first, establish the character of singularity of the function F at $t = 0$, which causes the divergence of expansion (12). In general case the following variants are possible: 1) F is the single-valued function of t and the point $t = 0$ is the singular point of one-valued character; 2) F is the many-valued function and the point $t = 0$ is the branching point of finite order $N-1$; 3) F is the many-valued function and the point $t = 0$ is the logarithmic branching point.

Show, at first, that the case 3) is really excluded, i.e., that the solution of (10) cannot have the logarithmic branching point at $t = 0$. Assume the contrary statement: the point $t = 0$ is the logarithmic branching point of F, where F is the solution of (10). Then in any neighborhood of zero $G = \{t : 0 < |t| < \rho\}$ the representation

$$F(t; q', q) = \sum_{n=0}^{\infty} \left(\frac{Ln\, t - \alpha}{Ln\, t + \alpha} \right)^n A_n(q', q) \tag{48}$$

takes place. Here α is some number, satisfying, in general case, the condition $Re\, \alpha < \ln \rho$; $Ln\, t$ means the full many-valued logarithmic function. Note, that (48) is the simplified version of general representation, in which α is taken real and $\rho = 1$.

Let us introduce the new variable

$$w = \frac{Ln\, t - \alpha}{Ln\, t + \alpha}. \tag{49}$$

This transformation transfers the region G into the unit circle in the complex plain w. The point $w = 1$ corresponds to $t = 0$. Expansion (48) takes the form

$$F(w; q', q) = \sum_{n=0}^{\infty} w^n A_n(q', q), \tag{50}$$

and according to the assumption, should be convergent in the unit circle. We substitute (50) into (10) taking into account (49) and expand all arising expressions in powers of w. This leads to the equation

$$\begin{aligned}
&\frac{1}{2\alpha} \sum_{n=1}^{\infty} n A_n \left[\left(1 - \alpha + \frac{\alpha^2}{2}\right) w^{n-1} - (2 - \alpha^2) w^n + \left(1 + \alpha + \frac{\alpha^2}{2}\right) w^{n+1} \right. \\
&\left. + \sum_{k=0}^{\infty} \sum_{l=1}^{\infty} \sum_{m=0}^{2l+2} \frac{(-\alpha)^{l+2}}{(l+2)!} \binom{2l+2}{m} w^{2lk+m+n-1} \right] \\
&= \sum_{n=0}^{\infty} \left(\frac{1}{2} \frac{\partial^2 A_n}{\partial q'^2} - V(q') A_n - \Delta q \frac{\partial A_n}{\partial q'} \right) w^n \\
&- \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=1}^{\infty} \sum_{m=0}^{2l} \Delta q \frac{\partial A_n}{\partial q'} \frac{(-\alpha)^l}{l!} \binom{2l}{m} w^{2lk+m+n}.
\end{aligned} \tag{51}$$

After equating the coefficients in front of the same powers of w the function A_{n+1} can be expressed via A_i with $i \le n$ and their derivatives with respect to q'. For example,

$$A_1 = 2\alpha e^{\alpha} \left(\frac{1}{2} \frac{\partial^2 A_0}{\partial q'^2} - V(q') A_0 - e^{-\alpha} \Delta q \frac{\partial A_0}{\partial q'} \right). \quad (52)$$

Let us take sufficiently large n and calculate the coefficients A_{n+1} through A_i. We shall not write the relations between A_i in the general form, but write only the terms, which will be considered. From (51) we get

$$\frac{e^{-\alpha}}{2\alpha} \{(n+1)A_{n+1} - 2(1+\alpha)nA_n\} = \{\textit{other terms depending on } A_i,\ i \le n\}. \quad (53)$$

The contributions into A_{n+1}, arising at the transition $A_{i+1} \to A_i$ by means of the terms displayed in (53), will be considered now

$$|A_{n+1}| \sim \frac{2|1+\alpha|n}{n+1} |A_n| \sim \cdots \sim \frac{2^n |1+\alpha|^n}{n+1} |A_0|. \quad (54)$$

The existence of these contributions means that expansion (50) can converge at the circle of radius not larger then

$$R_w = \frac{1}{2|1+\alpha|}.$$

Because α is an arbitrary negative number, it is clear that R_w may be done by choosing the corresponding α as small as one whishes. But if the point $t = 0$ were the logarithmic branching point of F then expansion (50) would be convergent at the circle of unit radius. We have a contradiction, so the solution of (10) cannot have the logarithmic branching point at $t = 0$.

Let us consider now case 1). According to known theorem [21], the point $t = 0$, if the series (12) (or (5)) is divergent, is an essential singular point of the function F. And what's more, expansion (12) approximates F only in some sector of the complex plain t with angle $\gamma < 2\pi$. The function F does not have any meaning at the point $t = 0$.

Case 2). The point $t = 0$ is the branching point of F of the order $N - 1$. Then in some neighborhood of zero it can be represented by the convergent series

$$F(t; q', q) = \sum_{n=-\infty}^{+\infty} t^{n/N} d_n(q', q). \quad (55)$$

Now we exclude the variant, when the function F has any definite meaning at $t = 0$ and, at the same time, is represented by divergent series (12). Suppose that instead of (55) the expansion

$$F(t; q', q) = \sum_{n=0}^{\infty} t^{n/N} d_n(q', q) \quad (56)$$

takes place (the function having the branching point at zero will have definite meaning at this point only if (56) is convergent). Substitution of (56) into (10) will give the recurrent relations for d_n. There will be N independent series of the coefficients d_n having the following forms: $d_{Nl},\ d_{Nl+1}, \ldots, d_{Nl+N-1},\ l = 0, 1, 2, \ldots$ The coefficients of every series are expressed through each other only, but not through the coefficients of other series.

Let us take the series with numbers of the form $n = Nl$. The equations for its coefficients read

$$l d_{Nl} + (q' - q)\frac{\partial d_{Nl}}{\partial q'} = \frac{1}{2}\frac{\partial^2 d_{N(l-1)}}{\partial q'^2} - V(q') d_{N(l-1)}. \tag{57}$$

If one compares $d_{Nl}(q', q)$ with $a_l(q', q)$, where a_l are the coefficients from (5), then one can see that the equations for d_{Nl} are the same as the ones for a_l. So, if expansion (5) is divergent, then (56) is divergent too.

We can state now, that the divergence of series (5), (12) means that F may be either single-valued or N-valued with the branching point at origin, but in both cases the point $t = 0$ is the essential singular point (to be exact, in many-valued case F is the single-valued function of the variable $t^{1/N}$ and the point $t^{1/N} = 0$ is the essential singular point of this function). So, the function F does not have any definite meaning at $t = 0$ when series (5), (12) are divergent.

Because the asymptotic representation (5) for F is correct at some sector of the complex plain t, then if one approaches zero in bounds of this sector the function F tends to 1. But it does not mean, that $F(t = 0) = 1$, because if one approaches zero in other ways, lying beyond this sector, one can obtain every initially chosen meaning.

Note, that in [22] the solutions of the complex Schrödinger equation for the potential (21) are presented. And they have in general case the form of the Loran series in powers of t.

As a rule, solving the Schrödinger equation (1), one ignores the fact, that for a number of potentials the solution has more strong singularity at $t = 0$, than $\delta(q' - q)$, and so, the initial condition (2) for these solutions is not satisfied. Practically, it means, that the solution of problem (1), (2) does not exist, although the solution of equation (1) exists. So, the evolution operator kernel, strictly speaking, does not exist for the potentials, for which expansion (5) is asymptotic. Hence, if we wish to remain in the framework of conventional formalism of the quantum mechanics, then we should take into account that these potentials are not acceptable in the quantum theory.

It is noteworthy, that such strong statement is true only with respect to the exact theory. If one considers some model which is meant initially only as approximate, then there are no reasons to demand that Cauchy's problem would have the solution in the rigorous sense. One can consider for that model potential the weaken Cauchy's problem, in which the strict requirements are not imposed on the behavior of the solution at $t \to 0$. Nevertheless, one should understand that this theory can be treated only as an approximation according to other, usually unknown, exact theory, for which the evolution operator kernel exists in the rigorous sense, and one should be ready to operate with asymptotic expansions.

From the physical viewpoint the fact of existence of the kernel only for discrete values of the coupling constant one can treat as natural quantization of the charge in the quantum theory. Thus, the potentials of kind (21), (28), (44), (46) can be considered in the quantum mechanics as exact ones only if at the coupling constant $g = \lambda(\lambda - 1)/2$ the number λ is integer. Of course, these potentials are the model ones and they have not direct relation to quantization of, e.g., electric charge. Nevertheless, the mechanism of appearance of charge discreteness from the requirement that the solution of Cauchy's problem for the quantum equation of motion exists can work in application to more realistic models.

At first sight, the statement about unacceptability in the quantum theory the potentials with arbitrary coupling constants (for which the Hamiltonian is self-adjoint) seems unexpected. Nevertheless, if one takes into account, that the charge in the nature is discrete and, on the other hand, that the charge is a parameter of the potential, then it becomes clear that a correct theory must be constructed so as only the potentials with quantized values of the coupling constants may be acceptable.

Thus, thorough analysis of analytical structure of the solutions of the Schrödinger equation allows us to discover the potentials, for which the solution of the rigorous Cauchy's problem exists and which may be considered as exact ones. For other potentials only weaken Cauchy's problem may be formulated and these potentials may be used only as approximate ones. Moreover, it occurs that for the potentials considered the kernel exists not for all values of the coupling constant, but only for some discrete values, i.e., quantization of the charge takes place.

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