On Matrix KP and Super-KP Hierarchies in the Homogeneous Grading.

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Abstract

Constrained KP and super-KP hierarchies of integrable equations (generalized NLS hierarchies) are systematically produced through a Lie algebraic AKS-matrix framework associated to the homogeneous grading. The role played by different regular elements to define the corresponding hierarchies is analyzed as well as the symmetry properties under the Weyl group transformations. The coset structure of higher order hamiltonian densities is proven.

For a generic Lie algebra the hierarchies here considered are integrable and essentially dependent on continuous free parameters. The bosonic hierarchies studied in [1, 2] are obtained as special limit restrictions on hermitian symmetric-spaces.

In the supersymmetric case the homogeneous grading is introduced consistently by using alternating sums of bosons and fermions in the spectral parameter power series.

The bosonic hierarchies obtained from $\hat{sl(3)}$ and the supersymmetric ones derived from the $N = 1$ affinization of $sl(2)$, $sl(3)$ and $osp(1|2)$ are explicitly constructed.

An unexpected result is found: only a restricted subclass of the $sl(3)$ bosonic hierarchies can be supersymmetrically extended while preserving integrability.

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1 Introduction.

Recently a lot of research has been devoted to the hierarchies of integrable differential equations due to their connection with the discretized version (matrix-model formulation) of the 2-dimensional gravity (see e.g. [3] for a review). It is in fact by now clear that constrained KP flows [4, 5] define the partition functions of single and multi-matrix models.

Moreover it has been suggested [6] that supersymmetric hierarchies describe the 2-dimensional supergravity, even if no matrix-model formulation is at present available in this case.

The problem of classifying all possible (both bosonic and supersymmetric) hierarchies is therefore quite a crucial one.

In the bosonic case a strategy, based on a generalized Drinfeld-Sokolov approach, has been developed in many papers [7, 8, 9]. Basically to produce integrable hierarchies the following ingredients are needed: a matrix-type Lax operator valued on a Lie algebra $\mathcal{G}$; a suitable Z-grading for $\mathcal{G}$ and the existence of constant non-zero graded regular elements for the algebra (for details see [9]). The problem of classifying hierarchies is therefore reduced to the Lie-algebraic problem of determining the acceptable gradings and the corresponding regular elements. This problem has been solved for non-exceptional Lie algebras as well as for (at least some of) the exceptional ones. Therefore in the bosonic case the situation seems completely satisfactory apart perhaps some questions like e.g. do different Lie algebras and different gradings always produce different hierarchies? Which is the role played by different regular elements once chosen a given Lie algebra and a given grading? This second question will be addressed in this paper for the special case of the homogeneous grading and it will be shown that indeed different regular elements induce different hierarchies; moreover, in the general case, the integrability is preserved even in presence of an essential dependence on continuous free parameters.

In the supersymmetric case the situation is much less satisfactory: supersymmetric versions of matrix super-KP hierarchies [10] have been constructed only for Lax operators which take values on superalgebras and are associated to the principal grading (generalized super-KdV hierarchies); moreover, since the Lax operator is in this case a fermionic object, in its turn the constant regular element must be fermionic; as a consequence only superalgebras which admit a presentation in terms of fermionic simple roots can produce super-hierarchies [10].

On the other hand integrable super-hierarchies which do not fit in the above scheme have actually been constructed (see [11, 12, 13]). The lack of a clear Lie algebraic understanding of such hierarchies makes difficult to find their generalizations; moreover the recognition of their integrability (i.e. the construction of their Lax operators) is left to an ad hoc procedure. In [13, 14, 15] it has been recognized that (bosonic and supersymmetric) Non-Linear Schrödinger-type hierarchies can be obtained from coset algebra structures. Here it will be shown a method to systematically produce such kind of hierarchies and their Lax operators, in terms of the above mentioned AKS matrix (super)-KP framework within the homogeneous grading. In particular the supersymmetric case allows producing generalized super-NLS hierarchies from any given starting Lie or super-Lie algebra (the Poisson brackets structure being expressed by the corresponding $N = 1$ affinizations).

Now the regular elements belong to the Cartan sector and are bosonic (while the full Lax operators must be fermionic). The consistency of this procedure is guaranteed once introduced the notion of "twisted" bosonic and fermionic power series in the spectral parameter λ as alternating sums of bosonic-even powers of λ and fermionic odd powers (and conversely); the derived equations of motion and hamiltonians are of course standard supersymmetric theories expressed in a manifestly supersymmetric formalism. At first sight this construction looks strange, but it is algebraically perfectly well-defined: it should be also noticed that constructions which present similar features have already been encountered in supersymmetric theories, think for instance to the GSO projection and to the supersymmetric Witten index [16].

This paper therefore contains the extension of the [10] method to the homogeneous grading case.

It should be noticed that, even if a manifestly $N = 1$ super-formalism only has been considered here, $N = 2$ supersymmetric hierarchies can be obtained from the above picture by starting from algebras (and regular elements) which admit an anti-involution J compatible with the supersymmetry (see [17, 18]); this is however a sufficient but not necessary condition: it has been pointed out in [19, 15] that already the standard super-NLS equation, obtained from the $N = 1$ affinization of the $sl(2)$ algebra, admits an $N = 2$ structure.

Besides the above construction, the following points will also be analyzed here:
i) the arising of possible symmetries under the finite Weyl group or the outer Lie algebra automorphisms transformations.
ii) The field reductions which can be consistently imposed once a finite symmetry is present (this is always the case for the positive versus negative root symmetry).
iii) The iterative prove of the coset structure for the higher order hamiltonian densities, i.e. their vanishing Poisson brackets with respect to some affine Lie subalgebra.
iv) The already mentioned role played by different regular elements to produce different integrable hierarchies. It is applied in particular to obtain more general hierarchies, containing free parameters, than those studied in [1, 2] (this new situation appears already from the $sl(3)$ algebra); the integrable structure is preserved even in presence of these free parameters. The generalization with respect to [1, 2] is due to the fact that the extra-restriction that the diagonal-transformed Lax operator belongs to a symmetric space is no longer imposed here (translated into a geometrical language, this implies not imposing the vanishing of the torsion).
v) a heuristic derivation of the constrained KP-scalar Lax operators from the matrix ones is also presented.

Besides the standard NLS and super-NLS hierarchies obtained from the affine (and respectively $N = 1$ super-affine) $sl(2)$ algebra, the bosonic hierarchies derived from the affine $sl(3)$ algebra, as well as the supersymmetric hierarchies obtained from the $N = 1$ affinization of the $sl(3)$ algebra and the $osp(1|2)$ superalgebra are also explicitly presented.

A rather surprising result is found: in the $sl(3)$ case, only the bosonic hierarchies depending on a restrict class of values for the free continuous parameter can be supersymmetrically extended in such a way to lead to an integrable supersymmetric hierarchy.

The scheme of this paper is the following:

the bosonic AKS approach to matrix-Lax operators is at first recalled. The coset-structure property of the homogeneous-grading hierarchies is derived. Then the symmetry properties under Weyl group and outer automorphisms are analyzed. The $sl(3)$-algebra case will be carried out completely and the whole set of associated hierarchies will be written down. Next, the supersymmetric AKS approach for the homogeneous grading will be introduced. In particular this scheme will be applied to explicitly compute the supersymmetric hierarchies associated to $sl(3)$ and $osp(1|2)$.

2 Reviewing the AKS framework.

In this section I will shortly review the AKS-matrix Lax operator approach to bosonic integrable hierarchies. For a more complete account see e.g. [9].

As a starting point a matrix-type Lax operator $\mathcal{L}$ is assumed, defined through

$$\mathcal{L} = \frac{\partial}{\partial x} + J(x) + \Lambda \tag{1}$$

Here x is a space coordinate (it can be assumed either $x \in R$ or $x \in S^1$).

$J(x)$ denotes a set of currents valued in the semisimple finite Lie algebra $\mathcal{G}$:

$$J(x) = \sum_i J_i(x) g_i \tag{2}$$

where g_i's are the $\mathcal{G}$-Lie algebra generators

$$[g_i, g_j] = \sum_k f^k{}_{ij} g_k \tag{3}$$

and $f^k{}_{ij}$ are the $\mathcal{G}$-structure constants.

The Lie algebra $\mathcal{G}$ is naturally extended into a loop algebra $\tilde{\mathcal{G}}$ defined through

$$\tilde{\mathcal{G}} = \mathcal{G} \otimes C(\lambda, \lambda^{-1}) \tag{4}$$

The elements in $\tilde{\mathcal{G}}$ are Laurent expansions in the spectral parameter λ. The brackets for $\tilde{\mathcal{G}}$ are given by

$$[g_i \cdot \lambda^m, g_j \cdot \lambda^n] = \sum_k f^k{}_{ij} g_k \cdot \lambda^{m+n} \tag{5}$$

for any integers n, m.

The adjoint operator ad_Y is defined through

$$ad_Y(X) = [Y, X] \tag{6}$$

where we can assume both $X, Y \in \mathcal{G}$ or $X, Y \in \tilde{\mathcal{G}}$.

Λ in (1) is a constant (i.e. not depending on x) regular element of $\tilde{\mathcal{G}}$.

The regularity has the following meaning: $\tilde{\mathcal{G}}$ is decomposed as a direct sum

$$\tilde{\mathcal{G}} = \tilde{\mathcal{K}} \oplus \tilde{\mathcal{M}} \tag{7}$$

where

$$\begin{aligned} \tilde{\mathcal{K}} &=_{def} Ker(ad_\Lambda) \\ \tilde{\mathcal{M}} &=_{def} Im(ad_\Lambda) \end{aligned} \quad (8)$$

It is furthermore assumed $\tilde{\mathcal{K}}$ to be abelian; symbolically

$$[\tilde{\mathcal{K}}, \tilde{\mathcal{K}}] = 0 \quad (9)$$

In our case the following commutator is satisfied, too:

$$[\tilde{\mathcal{K}}, \tilde{\mathcal{M}}] \subset \tilde{\mathcal{M}} \quad (10)$$

If, moreover, the following condition is satisfied

$$[\tilde{\mathcal{M}}, \tilde{\mathcal{M}}] \subset \tilde{\mathcal{K}} \quad (11)$$

then $\frac{\tilde{\mathcal{G}}}{\tilde{\mathcal{K}}}$ is a symmetric space.

This case has been studied in [1, 2], and a full classification of symmetric spaces is available (for an account see [20]).

In this paper the most general case, obtained by dropping the condition (11), will be considered. As a consequence generalizations of the results in [1, 2] will be obtained.

To produce integrable hierarchies the concept of Z-grading for the Lie algebra $\tilde{\mathcal{G}}$ must be introduced. A grading deg is a linear operator of the form

$$deg = N\lambda\frac{d}{d\lambda} + ad_Z \quad (12)$$

(where N is a non-zero integer and Z is a suitable element in the Cartan subalgebra) such that the elements in $\tilde{\mathcal{G}}$ are eigenvectors of deg having integer eigenvalues.

Λ in (1) must be an eigenvector of deg having non-zero eigenvalue.

I will leave to [9] the discussion about which are the admissible gradings for any given Lie algebra. Here I will just remember that any Lie algebra always admits two extremal gradings (plus, possibly, a series of intermediate ones): the principal grading and the homogeneous one. The former associates grade-one to the simple roots of the algebra. The integrable hierarchies produced from this grading are generalizations of the KdV equation (the standard KdV is obtained from the $sl(2)$ algebra, the Boussinesque from $sl(3)$ and so on).

The homogeneous grading is defined through

$$deg = \lambda\frac{d}{d\lambda} \quad (13)$$

(it counts the powers in λ).

The grade-one regular elements Λ have in this case the form

$$\Lambda = \lambda H \quad (14)$$

where H is a given, generic element in the Cartan subalgebra of $\mathcal{G}$ (such that all its eigenvalues in the adjoint representation of $\mathcal{G}$ are different).

For this particular grading the decomposition

$$\mathcal{G} = \mathcal{K} \oplus \mathcal{M} \tag{15}$$

holds, where

$$\begin{aligned} \mathcal{K} &= Ker(ad_H) \\ \mathcal{M} &= Im(ad_H) \end{aligned} \tag{16}$$

We have now

$$\begin{aligned} \tilde{\mathcal{K}} &= \mathcal{K} \otimes C(\lambda, \lambda^{-1}) \\ \tilde{\mathcal{M}} &= \mathcal{M} \otimes C(\lambda, \lambda^{-1}) \end{aligned} \tag{17}$$

The crucial feature of the AKS approach consists in the fact that the Lax operator $\mathcal{L}$ provides (1+1)-dimensional integrable hamiltonian systems through the following procedure: at first it should be noticed that $\mathcal{L}$ can be diagonalized via a similarity transformation

$$\mathcal{L} \mapsto \hat{\mathcal{L}} \tag{18}$$

defined by

$$\hat{\mathcal{L}} = exp(ad_M) \cdot \mathcal{L} = \sum_{n=0}^{\infty} \frac{1}{n!}(ad_M)^n(\mathcal{L}) \tag{19}$$

where M is a *uniquely* defined expansion of negative-graded elements of $\tilde{\mathcal{M}}$ which can be iteratively computed.

It turns out that $\hat{\mathcal{L}}$ is expanded as a sum of negative-graded diagonal elements of the Lie algebra $\mathcal{G}$; they provide an infinite series of mutually commuting (i.e. having vanishing Poisson brackets), local in the $J_i(x)$ fields, hamiltonian densities.

At least two compatible Poisson brackets structures can be defined for such systems. Throughout this paper we will be interested only in the second one, which is given by the affine-Lie Poisson brackets algebra, defined as the central extension of the $\tilde{\mathcal{G}}$-loop algebra.

Explicitly we have

$$\{J_i(x), J_j(y)\} = \sum_k f^k{}_{ij} J_k(y)\delta(x-y) + K_{ij}\partial_y\delta(x-y) \tag{20}$$

where in the above formula

$$K_{ij} = Tr(g_i g_j) \tag{21}$$

in the adjoint representation for $\mathcal{G}$.

In the specific case of the homogeneous grading the element M in (19) is given by

$$M = \sum_{k=1}^{\infty} \lambda^{-k} M_k \tag{22}$$

with

$$M_k \in \mathcal{M} = Im(ad_H) \tag{23}$$

The transformed Lax operator $\hat{\mathcal{L}}$ is expanded in powers of λ as

$$\hat{\mathcal{L}} = \lambda H + \partial_x + J_\alpha h_\alpha + \sum_{k=1}^{\infty} \lambda^{-k} R_{k,\alpha} h_\alpha \tag{24}$$

where H is given by (14) and h_α denote the Cartan generators of $\mathcal{G}$ (the sum over α in the above formula is understood).

$J_\alpha, R_{k,\alpha}$ for any k, α are mutually commuting hamiltonian densities which provide the compatible flows associated to the given hierarchy.

3 The coset property of the homogeneous hierarchies.

In this section it will be proven the coset structure of the hierarchies associated to the homogeneous grading. More precisely, the following property is satisfied for the hierarchies determined by a regular element

$$\Lambda = \lambda \sum_\alpha c_\alpha h_\alpha \tag{25}$$

where h_α are the Cartan generators of a given Lie algebra $\mathcal{G}$ ($\alpha = 1, 2, ..., r$, with r the rank of the algebra) and c_α are generic constants.

It turns out that with respect to the second Poisson brackets structure (the affine-Lie Poisson brackets given in (20)), the whole set of higher hamiltonian densities $R_{k,\alpha}$ of eq. (24) have vanishing Poisson brackets with respect to the $J_\alpha(x)$ currents associated to the Cartan generators.

Since the J_α's generate independent $\hat{U(1)}$ Kac-Moody subalgebras, the above hamiltonian densities are elements of the $\hat{\mathcal{G}}$-enveloping algebra which belong to the $\hat{U(1)}^r = \hat{U(1)} \otimes ... \otimes \hat{U(1)}$ (r times) coset subsector.

The above property is satisfied for generic c_α in (25). For some specific values of c_α the coset algebra can be bigger and coincide with some non-abelian Kac-Moody subalgebra of $\hat{\mathcal{G}}$.

In order to prove the above theorem it is convenient to introduce in full generality [21, 14, 13] the notion of charged fields and covariant derivative with respect to the $\hat{U(1)}$ Kac-Moody algebra defined by the brackets

$$\{J_0(x), J_0(y)\} = \frac{\partial}{\partial y}\delta(x-y) \tag{26}$$

A q-charged field V_q is defined to satisfy

$$\{J_0(x), V_q(y)\} = qV_q(y)\delta(x-y) \tag{27}$$

while a covariant derivative $\mathcal{D}$ acting on V_q can be introduced through the position

$$\mathcal{D}V_q(x) =_{def} (\partial + qJ_0(x))V_q(x) \tag{28}$$

Notice that the covariant derivative maps q-charged fields into new fields of definite charge having the same value q.

The coset property of the $R_{k,\alpha}$ hamiltonian densities implies that they are chargeless differential polynomials constructed with the subset of $J_i(x)$'s given by charged fields and covariant derivatives acting on them.

Such a theorem can be easily proven by using an iterative procedure. For simplicity it will be given for the $sl(2)$ case (which allows also to introduce the standard Non-Linear-Schrödinger equation), the generalization to generic Lie algebras $\mathcal{G}$ is straightforward.

The $sl(2)$ algebra is generated by $H, E_{\pm}$, satisfying the commutation relations:

$$\begin{aligned}[H, E_{\pm}] &= \pm 2E_{\pm} \\ [E_+, E_-] &= H \end{aligned} \tag{29}$$

The associated second (affine-Lie) Poisson brackets structure is expressed by

$$\begin{aligned}\{J_0(x), J_0(y)\} &= \partial_y \delta(x-y) \\ \{J_0(x), J_{\pm}(y)\} &= \pm 2J_{\pm}(y)\delta(x-y) \\ \{J_+(x), J_-(y)\} &= 2\mathcal{D}_y\delta(x-y) = 2(\partial_y\delta(x-y) - 2J_0(y)\delta(x-y)) \end{aligned} \tag{30}$$

Here $J_{\pm}(x)$ have charge ± 2 with respect to the $U\hat{(1)}$ subalgebra generated by $J_0(x)$.

Any other Poisson bracket is vanishing.

The Lax operator $\mathcal{L}$ is given by

$$\mathcal{L} = \partial + J_o(x)H + J_+(x)E_+ + J_-(x)E_- + \lambda H \tag{31}$$

We can diagonalize, order by order in λ, the above Lax operator into $\hat{\mathcal{L}}$ such that

$$\hat{\mathcal{L}} = exp(ad_M)(\mathcal{L}) = \partial + J_0(x)H + \sum_{k=1}^{\infty} \lambda^{-k} R_k H \tag{32}$$

where the diagonalizing matrix has the form

$$M = \sum_{i=1}^{\infty} \lambda^{-i}(M_{i,+}E_+ + M_{i,-}E_-) \tag{33}$$

At the lowest orders we find

$$\begin{aligned} M_{1\pm} &= \pm J_{\pm} \\ M_{2,\pm} &= -\mathcal{D}J_{\pm} \\ M_{3,\pm} &= \pm(\mathcal{D}^2 J_{\pm} - \tfrac{4}{3}(J_+J_-)J_{\pm} \end{aligned} \tag{34}$$

while the hamiltonian densities are given by

$$\begin{aligned} R_1 &= J_+J_- \\ R_2 &= \tfrac{1}{2}(J_+\mathcal{D}J_- - J_-\mathcal{D}J_+) \end{aligned} \tag{35}$$

At the lowest orders $M_{i,\pm}$ are differential polynomials with definite charge ± 2 respectively, while the hamiltonian densities are chargeless differential polynomials.

It is immediately shown, due to the properties of the adjoint action $exp(ad_M)$ acting on $\mathcal{L}$, that assuming $M_{i,\pm}$ having charge ± 2 for $i = 1, ..., N$ and R_i being chargeless for $i = 1, ..., N$, necessarily follows that $M_{N+1,\pm}$ have charges ± 2 and R_{N+1} is chargeless. The theorem is therefore proven by induction. Its generalization to arbitrary Lie algebras is straightforward.

The different hamiltonian flows for integral values $k = 1, 2, ...$ are defined through the following equation, for any given field $\phi(x)$:

$$\frac{\partial}{\partial t_k}\phi(x) = \frac{1}{2}\{\phi(x), \int dy R_k(y)\} \quad (36)$$

(the factor 2 is introduced for normalization convenience).

Since $J_0(x)$ has vanishing Poisson brackets with respect to any R_k, we get for any flow

$$\frac{\partial}{\partial t_k} J_0(x) = 0 \quad (37)$$

It follows that it is consistent with the equations of motion to set

$$J_0 \equiv 0 \quad (38)$$

In literature the above position is in general set as a Dirac constraint. In our approach it is recovered as a consequence of the equations of motion, which implies a simplified analysis (in particular it avoids computing Dirac's brackets to obtain the flows, which is of great help in many cases, see e.g. [15]).

The first two flows for the fields $J_\pm(x)$ are respectively given by

$$\frac{\partial}{\partial t_1} J_\pm(x) = -\mathcal{D} J_\pm(x) \quad (39)$$

and

$$\frac{\partial}{\partial t_2} J_\pm(x) = \pm(\mathcal{D}^2 J_\pm(x) + 2(J_+ J_-) J_\pm(x)) \quad (40)$$

The second flow is precisely the two-components Non-Linear-Schrödinger equation.

The covariant derivative in the above formulas can be replaced by the standard derivative, once setting the (38) solution to the equations of motion.

4 From matrix to scalar Lax operators: a heuristic derivation.

Before going ahead, let me just point out that a connection exists between matrix-type Lax operators and consistent field-restrictions of the scalar KP operator. A detailed analysis has been given in [2]. Here I wish just furnish a simple heuristic argument to understand such a connection. For simplicity I will treat the $sl(2)$ case in the homogeneous grading; the extension to generic algebras can also be given along the same lines.

Let us consider as a starting point the equation

$$\mathcal{L} \cdot \Psi = 0 \quad (41)$$

(where $\mathcal{L}$ is the matrix Lax operator (31)) in some given representation of the $sl(2)$ algebra.

The $\lambda \equiv 0$ component of the above equation in the fundamental (spin$\frac{1}{2}$) representation for $sl(2)$ gives us:

$$\left(\partial + \begin{pmatrix} J_0 & J_+ \\ J_- & -J_0 \end{pmatrix}\right)\begin{pmatrix} \Psi_+ \\ \Psi_- \end{pmatrix} \quad = \quad 0 \tag{42}$$

If we solve the above equation for, let's say, the Ψ_- component and allow formally inverting the derivative operator, then we can plug the result into the equation for the Ψ_+ component, obtaining:

$$(\mathcal{D} + J_-\mathcal{D}^{-1}J_+)\Psi_+ = 0 \qquad \equiv \qquad L \cdot \Psi_+ = 0 \tag{43}$$

The scalar operator

$$L \quad = \quad \mathcal{D} + J_-\mathcal{D}^{-1}J_+ \tag{44}$$

(It also turns out $L \equiv \partial + J_-\partial^{-1}J_+$ when inserting the constraint, compatible with the equations of motion, $J_0 = 0$) provides the consistent field reduction of the scalar KP operator associated to the 2-component NLS equation (see [14]).

It should be noticed that to a given matrix-type Lax operator one can associate different but equivalent scalar KP restrictions, according to which representation of the algebra has been chosen. For instance, if instead of starting with the fundamental representation of $sl(2)$ we proceed from the triplet representation (acting on the vector $(\Psi_1, \Psi_0, \Psi_{-1})$) we are led, after solving the equations for the $\Psi_{\pm 1}$ components, to the following relation:

$$(\mathcal{D} + J_-\mathcal{D}^{-1}J_+ + J_+\mathcal{D}^{-1}J_-)\Psi_0 \quad = \quad 0 \tag{45}$$

The new scalar Lax operator L'

$$L' \quad = \quad \mathcal{D} + J_-\mathcal{D}^{-1}J_+ + J_+\mathcal{D}^{-1}J_- \tag{46}$$

is equivalent to the L Lax operator (45) since their hamiltonian densities differ by total derivatives. L' is basically the symmetrized form of L under the exchange $J_- \leftrightarrow J_+$.

5 Field reductions: the positive-negative roots exchange.

In the following sections the symmetries under Weyl group transformations and outer automorphisms for generic Lie algebras will be analyzed. Here I will study the simplest such kind of symmetries, already appearing for the $sl(2)$ algebra: the algebra automorphism which exchanges positive and negative roots. Such Z_2 symmetry is always present for any Lie algebra; in general it will be provided by a combination of a Weyl transformation and an outer automorphism; for the $sl(2)$ algebra, which does not admit outer automorphisms, it coincides with the (unique) Weyl transformation.

Explicitly we have

$$E_+ \quad \leftrightarrow \quad E_-; \qquad H \mapsto -H \tag{47}$$

Such transformation can be extended to the affine-Lie automorphism

$$J_+(x) \quad \leftrightarrow \quad J_-(x); \qquad J_0(x) \mapsto -J_0(x) \tag{48}$$

where $J_0(x), J_\pm(x)$ generates the $\hat{sl(2)}$ algebra given in (30).

The $\mathcal{L} = \partial + J_0(x)H + J_+(x)E_+ + J_-(x)E_- + \lambda H$
Lax operator of (31) is invariant under the above transformation, provided that the spectral parameter λ being transformed according to

$$\lambda \quad \mapsto \quad -\lambda \tag{49}$$

Moreover, it is easily realized that under the above transformation the diagonalizing matrix M in (33) is left invariant. As a consequence the diagonalized Lax operator $\hat{\mathcal{L}}$ itself is invariant.

Due to the λ-transformation property (49) the odd hamiltonian densities R_k (k odd) are left invariant, while the even ones (R_k with k even) are transformed into their opposite:

$$R_k \mapsto (-1)^{k+1} R_k \tag{50}$$

Each time we have a symmetry we can perform a field reduction, identifying the fields which are related by the symmmetry transformation. Such a step can be performed for our Z_2-symmetry, which allows us passing from the 2-component NLS equation to the standard-form single component Non-Linear-Schrödinger equation. However it is worth to notice the following point: the hamiltonian which generates the NLS equation is the second one (R_2) which is not invariant under the symmetry, but it is transformed into its opposite. For that reason it is not possible to identify J_+ with J_-, instead we have to assume the time t_2 being imaginary ($t_2 = it$) and

$$J_+(x) \quad = \quad u(x) = {J_-}^\star(x) \tag{51}$$

The situation here is parallel to what happens in quantum-mechanics when disposing of a time reversal transformation which flips the sign of the hamiltonian: in that case the symmetry is recovered in terms of an antiunitary transformation which involves complex conjugation.

The final result for the single-component NLS equation is the following:

$$i\dot{u} \quad = \quad u'' + 2u|u|^2 \tag{52}$$

(here the standard convention of denoting time and spatial derivatives with respectively a dot or a prime has been adopted).

6 The dependence on the regular element and its symmetry properties.

In this section I wish to analyze a new feature, not touched by our previous discussion, that is the dependence of the integrable hierarchies from the choice of the regular element and its symmetry properties. For the $sl(2)$ algebra case this problem can not be posed since the grade-one regular element with respect to the homogeneous grading (that is λH of (14)) is essentially unique (apart an overall normalization factor which can be reabsorbed by rescaling the spectral parameter).

For general Lie algebras the problem of determining which different hierarchies are produced from different regular elements is a very interesting one. To be definite here we analyze in full detail the $sl(3)$ algebra case. This is indeed a very fundamental case because it already contains all the features (namely a non-trivial Weyl group and the presence of an outer automorphism) which are found in more complicated examples for generic Lie algebras. The generalization of the approach here developed to such cases is immediate, it is only technically more involved.

Before introducing my conventions concerning the $sl(3)$ algebra let me just recall (see [22] for a complete account) that the Weyl group associated to a given Lie algebra is a finite group of reflections which leave invariant the root systems of the algebra. It coincides with a subgroup of the inner automorphisms of the Lie algebra.

Besides the inner automorphisms a generic Lie algebra admits also a group of outer automorphisms (i.e. they can not be obtained as an Adjoint action
$x \mapsto x' = exp(ad_y)(x)$, with $x, y, x' \in \mathcal{G}$)
which coincides with the group of symmetries of its Dynkin diagram.

The $sl(3)$ algebra admits 8 generators. 2 generators, denoted as H_1 and H_2, belong to the Cartan sector (rank 2); the simple (positive and negative) roots will be denoted as $E_{\pm 1}$, $E_{\pm 2}$ respectively. The extra (maximal) root will be represented as $E_{\pm 3}$.

It is convenient to introduce the 3×3 matrices e_{ij}, for $i, j = 1, 2, 3$, defined as follows: e_{ij} has all zero entries apart 1 in the i-th raw, j-th column position.

The fundamental 3×3 representation of $sl(3)$ is obtained by setting

$$H_1 \ = \ e_{11} - e_{22}, \qquad H_2 = e_{22} - e_{33}, \tag{53}$$

for the Cartan generators,

$$E_{+1} \ = \ e_{12}, \qquad E_{+2} = e_{23}, \qquad E_{+3} = e_{13}, \tag{54}$$

for the positive roots and

$$E_{-1} \ = \ e_{21}, \qquad E_{-2} = e_{32}, \qquad E_{-3} = e_{31}, \tag{55}$$

for the negative ones.

The full commutation relations of the $sl(3)$ algebra can be easily computed from the above positions.

The Weyl group for $sl(3)$ coincides with the S_3 permutation group (of order 6) of three elements denoted as e_1, e_2, e_3.

The positive roots can be associated to the following combinations of e_i's (see [22]):

$$E_{+1} \equiv e_1 - e_2; \qquad E_{+2} \equiv e_2 - e_3; \qquad E_{+3} \equiv e_1 - e_3 \tag{56}$$

The Weyl group admits 3 distinct Z_2 subsymmetries s_i, $i = 1,2,3$, given by the corresponding reflections along the e_i element in S_3, acting as:

$$\begin{aligned} s_1 &: E_{\pm1} \leftrightarrow E_{\pm3}; \quad E_{+2} \leftrightarrow E_{-2}; \quad H_1 \mapsto H_1 + H_2; \quad H_2 \mapsto -H_2. \\ s_2 &: E_{\pm1} \leftrightarrow E_{\mp2}; \quad E_{+3} \leftrightarrow E_{-3}; \quad H_1 \leftrightarrow -H_2. \\ s_3 &: E_{+1} \leftrightarrow E_{-1}; \quad E_{\pm2} \leftrightarrow E_{\pm3}; \quad H_1 \mapsto -H_1; \quad H_2 \mapsto H_1 + H_2. \end{aligned} \tag{57}$$

The Z_3 subsymmetry obtained by sending $1 \mapsto 2 \mapsto 3 \mapsto 1$ leads to

$$\begin{aligned} &E_{\pm1} \mapsto E_{\pm2} \mapsto E_{\mp3} \mapsto E_{\pm1}; \\ &H_1 \mapsto H_2 \mapsto -(H_1 + H_2). \end{aligned} \tag{58}$$

Besides the above Weyl transformation, an extra Z_2 symmetry is present: it is realized by the outer automorphism σ which exchanges the two simple roots. It is explicitly given by the following relations

$$\sigma : E_{\pm1} \leftrightarrow E_{\pm2}; \qquad E_{\pm3} \mapsto -E_{\pm3}; \qquad H_1 \leftrightarrow H_2. \tag{59}$$

It should be noticed that the automorphism $s_\pm$ which exchanges positive and negative roots is in this case given by the combination of the s_2 Weyl transformation and the outer automorphism σ:

$$s_\pm = s_2 \cdot \sigma \tag{60}$$

Explicitly we have

$$s_\pm : E_{+1} \leftrightarrow E_{-1}; \qquad E_{+2} \leftrightarrow E_{-2}; \qquad E_{+3} \leftrightarrow -E_{-3}; \qquad H_1 \leftrightarrow -H_1; \qquad H_2 \leftrightarrow -H_2. \tag{61}$$

Both the Weyl transformations and the σ outer automorphism can be extended to be automorphisms for the full affine $\hat{sl(3)}$ algebra, in precise analogy to what discussed in the previous section.

The generic grade-one $sl(3)$ regular element Λ for the homogeneous grading has the following form

$$\Lambda = \lambda H = \lambda(cos^2\theta H_1 + sin^2\theta H_2) \tag{62}$$

Therefore it will depend on an arbitrary angle θ (as in the $sl(2)$ case an overall normalization factor can be reabsorbed in the definition of λ).

It will be explained in the next section that the integrable hierarchies have an essential dependence on θ, that is θ can not be rescaled at will.

As already recalled, the AKS framework works if the eigenvalues of the regular element are all distinct. In the 3×3 fundamental representation the diagonal for H in (62) is given by

$$(t, 1-2t, t-1) \tag{63}$$

(where for simplicity we have set $t = cos^2\theta$, $t \in [0,1]$).

It follows that two values exist

$$t = \tfrac{1}{3}; \qquad t = \tfrac{2}{3} \tag{64}$$

which must be excluded.

However it will be shown later that both the (64) conditions still produce admissible integrable hierarchies (of degenerate type); they have been discussed in [2].

Let us discuss now the symmetry properties under Weyl transformations and outer automorphism σ for the matrix Lax operator $\mathcal{L}$ (and therefore of its associated hierarchies) according to the choice of the regular element Λ.

We have for $sl(3)$

$$\mathcal{L} \;=\; \partial + \sum_i J_i(x) g_i + \lambda(tH_1 + (1-t)H_2) \tag{65}$$

The term $\sum_i J_i(x) g_i$ (the sum is over the $sl(3)$ generators) is invariant under both the Weyl and the σ transformations due to the combined transformation properties of $g_i, J_i(x)$; obviously the derivative ∂ is invariant too.

For what concerns Λ, the transformations act as follows:

i) σ maps $\mathcal{L}(t)$ into $\mathcal{L}(1-t)$. As a consequence t and $1-t$ produce the same set of hamiltonian densities and therefore the same hierarchies. There is only one symmetric point

$$t \;=\; \tfrac{1}{2} \tag{66}$$

which leaves $\mathcal{L}$ invariant.

It corresponds to the choice

$$\Lambda \;\equiv\; \lambda(\tfrac{1}{2}, 0, -\tfrac{1}{2}) \tag{67}$$

on the diagonal. This value of t allows the folding procedure (see [23]) which will be discussed in more detail in the next section.

ii) the s_2 transformation is a symmetry for $\mathcal{L}$ only for $t = \frac{1}{2}$ and assuming λ to be mapped into its opposite ($\lambda \mapsto -\lambda$).

iii) The combined $s_\pm = s_2 \cdot \sigma$ transformation (positive-negative roots exchange) is a symmetry of $\mathcal{L}$ for any value of t, provided that $\lambda \mapsto -\lambda$ under $s_\pm$. As a consequence, for any value of t (or of the θ-angle), the reduction from the 2-component fields hierarchy to the single(complex)-component fields hierarchy can be performed. The same remarks concerning the alternate parity of the hamiltonian in the $sl(2)$ case hold here as well.

iv) For what concerns the s_1 transformation, it can act as a symmetry for $\mathcal{L}$ if one of the two conditions below is satisfied:

either λ is unchanged ($\lambda \mapsto \lambda$) under s_1; in this case t must assume the degenerate value $t = \frac{2}{3}$ so that

$$\Lambda \;\equiv\; \lambda(\tfrac{2}{3}, -\tfrac{1}{3}, -\tfrac{1}{3}) \tag{68}$$

on the diagonal,

or λ is mapped into its opposite ($\lambda \mapsto -\lambda$) and $t = 0$; therefore

$$\Lambda \;\equiv\; \lambda(0, 1, -1) \tag{69}$$

on the diagonal.
v) The case concerning the s_3 symmetry is specular to the previous one, to which it can be reduced after performing a σ transformation. $\mathcal{L}$ is symmetric under s_3 if either

$s_3 : \lambda \mapsto \lambda$ and $t = \frac{1}{3}$, or

$s_3 : \lambda \mapsto -\lambda$ and $t = 1$.

It is not necessary to analyze the transformation properties for other elements of the Weyl group since the latter, being a permutation group, is recovered from the application of two generators (which can be assumed to be e.g. s_1, s_2).

7 The $sl(3)$ hierarchies in the homogeneous grading.

In this section the results previously obtained will be applied to construct the whole set of integrable hierarchies associated to the $sl(3)$ algebra in the homogeneous grading.

At first it is convenient to explicitly introduce the affine $\hat{sl(3)}$ algebra which provides the second Poisson brackets structure.

The two currents $J_{0,1}(x), J_{0,2}(x)$ are associated with the two generators in the Cartan subalgebra, while the positive (negative) roots correspond to the currents $J_{\pm i}$, $i = 1, 2, 3$.

The Cartan subalgebra reads as follows

$$\begin{aligned} \{J_{0,1}(x), J_{0,1}(y)\} &= 2\partial_y \delta(x-y) \\ \{J_{0,1}(x), J_{0,2}(y)\} &= -\partial_y \delta(x-y) \\ \{J_{0,2}(x), J_{0,2}(y)\} &= 2\partial_y \delta(x-y) \end{aligned} \tag{70}$$

The currents $J_{\pm i}(x)$ are charged fields

$$\{J_{0,j}(x), J_{\pm i}(y)\} = q_{\pm i,j} J_{\pm i}(y)\delta(x-y) \tag{71}$$

with charges $q_{\pm i} \equiv (q_{\pm i,1}, q_{\pm i,2})$ given by

$$\begin{aligned} q_{\pm 1} &= \pm(2,-1) \\ q_{\pm 2} &= \pm(-1,2) \\ q_{\pm 3} &= \pm(1,1) \end{aligned} \tag{72}$$

The covariant derivatives turn out to be

$$\begin{aligned} \mathcal{D}J_{\pm 1} &= \partial J_{\pm 1} \mp J_{0,1} J_{\pm 1} \\ \mathcal{D}J_{\pm 2} &= \partial J_{\pm 2} \mp J_{0,2} J_{\pm 2} \\ \mathcal{D}J_{\pm 3} &= \partial J_{\pm 3} \mp (J_{0,1} + J_{0,2}) J_{\pm 3} \end{aligned} \tag{73}$$

The algebra is completed by the following relations

$$\begin{aligned} \{J_{+i}(x), J_{-i}(y)\} &= \mathcal{D}_y \delta(x-y) \qquad &&for \quad i = 1,2,3. \\ \{J_{+1}(x), J_{-3}(y)\} &= -J_{-2}(y)\delta(x-y) &&\{J_{+2}(x), J_{-3}(y)\} = J_{-1}(y)\delta(x-y) \\ \{J_{+3}(x), J_{-1}(y)\} &= -J_{+2}(y)\delta(x-y) &&\{J_{+3}(x), J_{-2}(y)\} = J_{+1}(y)\delta(x-y) \\ \{J_{\pm 1}(x), J_{\pm 2}(y)\} &= \pm J_{\pm 3}(y)\delta(x-y) \end{aligned} \tag{74}$$

Any other Poisson bracket is vanishing.

We have now all the ingredients to compute the integrable hierarchies.

The Lax operator $\mathcal{L}$ is given in (65) and depends on the parameter $t = cos^2\theta$. It is diagonalized with a similarity transformation into $\hat{\mathcal{L}}$:

$$\begin{aligned} \hat{\mathcal{L}} &= exp(ad_M)(\mathcal{L} = \\ &= \lambda(tH_1 + (1-t)H_2) + \partial + J_{0,1}(x)H_1 + J_{0,2}(x)H_2 + \sum_{k,\alpha} \lambda^{-k} R_{k,\alpha}(x) H_\alpha \end{aligned} \quad (75)$$

where $k = 1, 2, ...$ denotes positive integers and $\alpha = 1, 2$.

The diagonalizing matrix M can be expanded as

$$M = \sum_{i=1}^{\infty} (\lambda^{-i} \cdot \sum_j M_{i,\pm j} E_{\pm j}) \quad (76)$$

for $j = 1, 2, 3$.

The compatible hamiltonian densities $R_{k,\alpha}$ can be computed with straightforward techniques. We get at the lowest orders:

$$\begin{aligned} R_{1,1}(x) &= (J_{+3}J_{-3} + \tfrac{1}{(3t-1)} J_{+1}J_{-1})(x) \\ R_{1,2}(x) &= (J_{+3}J_{-3} - \tfrac{1}{(3t-2)} J_{+2}J_{-2})(x) \end{aligned} \quad (77)$$

and

$$\begin{aligned} R_{2,1}(x) &= \tfrac{1}{2}(\mathcal{D}J_{+3}J_{-3} - J_{+3}\mathcal{D}J_{-3}) + \tfrac{1}{2(3t-1)^2}(\mathcal{D}J_{+1}J_{-1} - J_{-1}\mathcal{D}J_{+1}) \\ &\quad + \tfrac{1}{(3t-1)}(J_{+3}J_{-1}J_{-2} + J_{-3}J_{+1}J_{+2}) \\ R_{2,2}(x) &= \tfrac{1}{2}(\mathcal{D}J_{+3}J_{-3} - J_{+3}\mathcal{D}J_{-3}) + \tfrac{1}{2(3t-2)^2}(\mathcal{D}J_{+2}J_{-2} - J_{+2}\mathcal{D}J_{-2}) \\ &\quad + \tfrac{1}{(3t-2)}(J_{+3}J_{-1}J_{-2} + J_{-3}J_{+1}J_{+2}) \end{aligned} \quad (78)$$

Notice that the hamiltonian densities $R_{k,2}$ are just obtained by applying a σ transformation (59) to $R_{k,1}$ (and conversely); here

$$\sigma \ : \ J_{\pm 1} \leftrightarrow J_{\pm 2}; \qquad J_{\pm 3} \mapsto -J_{\pm 3}; \qquad t \mapsto (1-t). \quad (79)$$

The hamiltonians $H_{k,\alpha}$ are defined as the integrals

$$H_{k,\alpha} = \int dy R_{k,\alpha}(y) \quad (80)$$

and the corresponding flows, for a generic field $\phi(x)$, are given by

$$\frac{\partial}{\partial t_{k,\alpha}} \phi(x) = \{\phi(x), H_{k,\alpha}\} \quad (81)$$

where in the right hand side we have the affine Lie Poisson brackets (70,71,74).

In the above formulas (77,78) t can assume any value $t \in [0,1]$ apart $t = \frac{1}{3}, \frac{2}{3}$.

Neverthless even in such degenerate cases we obtain integrable hierarchies. Indeed for $t = \frac{1}{3}$ the integrals $H_{k,1}$ are not defined. However, the subset of integrals $H_{k,2}$, $k = 1, 2, ...$, is well-defined; they provide the mutually commuting hamiltonians with respect to the second Poisson brackets structure (70,71,74).

It can be easily shown that in this case it is consistent with the whole set of $t_{k,2}$ flows not only to put the constraint

$$J_{0,1}(x) \quad = \quad J_{0,2}(x) = 0 \tag{82}$$

(see the discussion in section 3), but also to set

$$J_{\pm 1}(x) \quad = \quad 0 \tag{83}$$

(the consistency of this position is due to the fact that, in the equations of motion for $J_{\pm 1}(x)$, the right hand side is proportional to $J_{\pm 1}$).

The hierarchy so derived will depend on the fields $J_{\pm 2}(x), J_{\pm 3}(x)$ only.

The hamiltonians $H_{k,2}$ belong to the symmetric coset space $\frac{sl(3)}{sl(2)\times U(1)}$ (it should be noticed however that the hamiltonian densities $R_{k,2}$ do not belong to the full affine coset subspace $sl\hat{(}2) \times U\hat{(}1)$).

Similar considerations hold for $t = \frac{2}{3}$, but now we have to replace $1 \leftrightarrow 2$ in the discussion above.

Another value of t which must be singled out is $t = \frac{1}{2}$; it corresponds to the symmetric point which leaves $\mathcal{L}$ invariant under the outer automorphism σ.

In this case it is convenient to reexpress the fields in terms of the σ-eigenvectors: we have as eigenvectors corresponding to the $(+1)$ eigenvalue:

$$J_{\pm up} \quad = \quad J_{\pm 1} + J_{\pm 2} \tag{84}$$

while the (-1) eigenvectors are

$$J_{\pm down} \quad = \quad J_{\pm 1} - J_{\pm 2}; \qquad J_{\pm 3}. \tag{85}$$

A consistent reduction of the $(t = \frac{1}{2})$ symmetric hierarchy can be obtained by considering the subset (for positive integers k) of

$$H_{k,up} \quad =_{def} \quad H_{k,1} + H_{k,2} \tag{86}$$

$(+1)$ eigenvectors hamiltonians and setting all the fields corresponding to the (-1) eigenvalue equal to zero:

$$J_{\pm down} \quad = \quad J_{\pm 3} = 0 \tag{87}$$

The reduced hierarchy is just the NLS-hierarchy of section 3.

The above procedure is nothing else than a simple example of folding, that is the reduction associated to a symmetry of the Lie algebra Dynkin diagram. For general folding constructions see [23].

Let us come back now to the general case (corresponding to generic values of t).

The hierarchies will depend on the whole set of fields $J_{\pm i}$, $i = 1, 2, 3$ while, as before, we can consistently set

$$J_{0,1} \quad = \quad J_{0,2} = 0 \tag{88}$$

at the level of the equations of motion (i.e. after computing the Poisson brackets).

From the hamiltonian $H_{1,1}$ we obtain the flow

$$
\begin{aligned}
\dot{J}_{\pm1} &= \mp J_{\pm3}J_{\mp2} - \tfrac{1}{(3t-1)}J_{\pm1}{}' \\
\dot{J}_{\pm2} &= \pm(1-\tfrac{1}{(3t-1)})J_{\pm3}J_{\mp1} \\
\dot{J}_{\pm3} &= \mp\tfrac{1}{(3t-1)}J_{\pm1}J_{\pm2} - J_{\pm3}{}'
\end{aligned} \tag{89}
$$

The flow associated to $H_{1,2}$ is obtained from (89) by replacing $t \mapsto (1-t)$ and $1 \leftrightarrow 2$ (for any couple of k-th order hamiltonians $H_{k,1}$, $H_{k,2}$ such a replacement obviously holds).

From the second order hamiltonian $H_{2,1}$ we get the flow

$$
\begin{aligned}
\dot{J}_{\pm1} &= -\gamma J_{\pm3}J'_{\mp2} - (1+\gamma)J'_{\pm3}J_{\mp2} \mp \gamma^2 J''_{\pm1} + \\
&\quad \pm J_{\pm1}[2\gamma^2 J_{+1}J_{-1} - \gamma J_{+2}J_{-2} + (1+\gamma)J_{+3}J_{-3}] \\
\\
\dot{J}_{\pm2} &= \gamma(\gamma-1)J_{\pm3}J'_{\mp1} + (1-\gamma)J'_{\pm3}J_{\mp1} + \\
&\quad \pm J_{\pm2}[\gamma(1-\gamma)J_{+1}J_{-1} + (1-\gamma)J_{+3}J_{-3}] \\
\\
\dot{J}_{\pm3} &= \mp J''_{\pm3} - \gamma(\gamma+1)J'_{\pm1}J_{\pm2} - \gamma J_{\pm1}J'_{\pm2} + \\
&\quad \pm J_{\pm3}[\gamma(\gamma+1)J_{+1}J_{-1} - \gamma J_{+2}J_{-2} + 2J_{+3}J_{-3}]
\end{aligned} \tag{90}
$$

where, in order to simplify our notation, we have set

$$
\gamma =_{def} \frac{1}{(3t-1)} \tag{91}
$$

The above (90) relations provide the $sl(3)$ generalization of the 2-component fields NLS equation.

Due to the results of the previous section concerning the $\pm$ roots exchange symmetry $s_\pm$, the above relations can be consistently reduced to the equations of motion for single-component complex fields; for the first flow the identification implies

$$
\begin{aligned}
\phi_1(x) &= J_{+1} = J_{-1} \\
\phi_2(x) &= J_{+2} = J_{-2} \\
\phi_3(x) &= J_{+3} = -J_{-3}
\end{aligned} \tag{92}
$$

We obtain as a consequence

$$
\begin{aligned}
\dot{\phi}_1 &= -\phi_2\phi_3 - \gamma\phi'_1 \\
\dot{\phi}_2 &= (1-\gamma)\phi_1\phi_3 \\
\dot{\phi}_3 &= -\gamma\phi_1\phi_2 - \phi'_3
\end{aligned} \tag{93}
$$

To get the second flow restriction we must let the time being imaginary and set:

$$
\begin{aligned}
\phi_1(x) &= J_{+1} = J^\star_{-1}; \\
\phi_2(x) &= J_{+2} = J^\star_{-2}; \\
\phi_3(x) &= J_{+3} = -J^\star_{-3}
\end{aligned} \tag{94}
$$

The second flow provides the generalization of the single-component NLS equation:

$$
\begin{aligned}
i\dot{\phi}_1 &= -\gamma\phi_3{\phi_2}^{\star\prime} - (1+\gamma)\phi_3'{\phi_2}^{\star} - \gamma^2\phi_1'' + \\
&\quad +\phi_1[2\gamma^2|\phi_1|^2 - \gamma|\phi_2|^2 - (1+\gamma)|\phi_3|^2] \\
i\dot{\phi}_2 &= \gamma(\gamma-1)\phi_3{\phi_1}^{\star\prime} + (1-\gamma)\phi_3'{\phi_1}^{\star} + \\
&\quad +\phi_2[\gamma(1-\gamma)|\phi_1|^2 + (\gamma-1)|\phi_3|^2] \\
i\dot{\phi}_3 &= -\phi_3'' - \gamma(\gamma+1)\phi_1'\phi_2 - \gamma\phi_1\phi_2' + \\
&\quad +\phi_3[\gamma(\gamma+1)|\phi_1|^2 - \gamma|\phi_2|^2 - 2|\phi_3|^2]
\end{aligned}
\tag{95}
$$

Since the $s_\pm$ symmetry commutes with the s_1, s_3 symmetries (see the previous section), the reduction from 2-component fields to single-component fields can be performed also in the case of "degenerate" hierarchies for $t = \frac{1}{3}, \frac{2}{3}$.

Let us make now some comments about the (90) hierarchy. The parameter γ can assume the real values

$$
\gamma \;\geq\; \tfrac{1}{2} \qquad for \quad \tfrac{1}{3} < t \leq 1 \tag{96}
$$

and

$$
\gamma \;\leq\; -1 \qquad for \quad 0 \leq t < \tfrac{1}{3} \tag{97}
$$

Under the σ transformation γ is mapped into $\tilde{\gamma}$:

$$
\sigma \;:\; \gamma \mapsto \tilde{\gamma} = \frac{\gamma}{(\gamma-1)} \tag{98}
$$

A fundamental domain for γ is therefore given by

$$
\tfrac{1}{2} \;\leq\; \gamma \leq 2 \tag{99}
$$

The special point $\gamma = 1$ corresponds to the degenerate hierarchy obtained from $t = \frac{2}{3}$.

$\gamma = 2$ corresponds to the symmetric point $t = \frac{1}{2}$, while $\gamma = \frac{1}{2}$ is obtained from the extremal value $t = 1$.

Clearly different values of γ correspond to hamiltonians having different symmetry properties: for instance $\gamma = 2$ corresponds to the σ-symmetric hamiltonians, but such a symmetry is broken for $\gamma \neq 2$.

Anyway the equations of motion may have a bigger symmetry property than the corresponding hamiltonians and a natural question one can ask is the following: is γ a fake or a genuine parameter in our theory? Stated otherwise, is it possible to redefine the variables in our theory in such a way that γ could be rescaled to a given fixed value of reference? A simple inspection shows that this is not the case: it is not possible, under the combined action of linear transformations for time and space variables and linear mappings of the fields $\phi_i(x) \mapsto \tilde{\phi}_i = A_{ij}\phi_j$ (with $det(A_{ij} \neq 0)$ to recast our equations (90) in a way that γ assumes a fixed, specified value.

Therefore γ is a genuine free parameter in our theory which labels a continuous class of inequivalent integrable hierarchies. The integrability of (95) is guaranteed for any value of γ satisfying (96) or (97). The restricted (99) interval for γ corresponds to the class of

$sl(3)$ fundamental hierarchies (the remaining hierarchies are obtained by σ-transforming this fundamental class).

$sl(3)$ is the simplest algebra admitting such a structure. Applying the same considerations here developed to the $sl(n)$ algebra, we expect that in this case there exists a continuous class of inequivalent integrable hierarchies specified by $n-2$ real parameters.

It will be shown later that only the hierarchies corresponding to the restricted class of values for γ,

$$\gamma \geq 1$$

can be supersymmetrically extended while mantaining the integrability property.

8 The supersymmetric AKS framework for the homogeneous grading.

In the previous sections we have analyzed the matrix AKS framework with respect to the homogeneous grading for bosonic hierarchies.

In this section I will define and show the general procedure which allows to extend the AKS construction to the supersymmetric hierarchies for the homogeneous grading.

This approach furnishes a method to systematically construct a vast class of super-hierarchies and to automatically prove their integrability. A manifest $N=1$ supersymmetric formalism will be used; by no means this implies restriction to $N=1$ super-hierarchies only. It is indeed true (see [17, 18] for general considerations and [19, 15] for an actual construction) that some of the hierarchies here considered admit an $N=2$ supersymmetry.

Before introducing the basic ingredients of such a framework, let me recall what already stated in the introduction: the matrix AKS framework for super-hierarchies has already been considered in [10], but only for the principal grading case. The super-hierarchies derived in such a case are of super-KdV type and form a rather restricted class (the hierarchies are put in correspondence with the subclass of super-Lie algebras which admit a presentation in terms of fermionic simple roots only). On the contrary, the super-hierarchies derived within our homogeneous-grading procedure are those of super-NLS type; apparently they form a "wider" class since any bosonic Lie algebra and any super-Lie algebra can be used to produce their corresponding hierarchies. More comments on that will be given later.

Let us fix at first our conventions concerning the superspace. We denote with capital letters the $N=1$ supercoordinate ($X \equiv x, \theta$, with x and θ real, respectively bosonic and grassmann, variables).

The supersymmetric spinor derivative is given by

$$D \equiv D_X \;=\; \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial x} \qquad (100)$$

With the above definition ${D_X}^2 = \frac{\partial}{\partial x}$.

The supersymmetric delta-function $\Delta(X, Y)$ is a fermionic object

$$\Delta(X, Y) \;=\; \delta(x-y)(\theta-\eta) \qquad (101)$$

(here $Y \equiv y, \eta$).

It satisfies the relations

$$\Delta(X,Y) = -\Delta(Y,X) \qquad D_X\Delta(X,Y) = -D_Y\Delta(X,Y) \tag{102}$$

Our convention for the integration over the grassmann variable is

$$\int d\theta \cdot \theta = -1 \tag{103}$$

For any given superfield $F(X)$ we get then

$$\int dY \Delta(X,Y)F(Y) = F(X) \tag{104}$$

As in the bosonic case, the (super)-line integral over a total derivative gives a vanishing result.

At this point it is quite natural to introduce $\mathcal{L}$ as the supersymmetrized version of the matrix Lax operator through the following position:

$$\mathcal{L} = D_X + \sum_i \Psi_i(X)g_i + \Lambda \tag{105}$$

where in the above relation g_i's denote either the generators of a semisimple Lie algebra or the generators of a super-Lie algebra; in the latter case the g_i's can be either bosonic (even elements ${g_i}^{(0)}$), or fermionic (odd elements ${g_i}^{(1)}$), see [24] for an account on super-Lie algebras.

Ψ_i denotes the set of $N=1$ currents associated to the (super)-Lie algebra; they have opposite parity with respect to that of g_i, i.e. the current associated to a bosonic generator ${g_i}^{(0)}$ is fermionic and conversely to a fermionic generator ${g_i}^{(1)}$ corresponds a bosonic current. In particular the whole set of $N=1$ currents of a standard Lie algebra which admits bosonic generators only is given by purely fermionic superfields.

With the above assumption the second term in the right hand side of (105) is fermionic, just like the first term (the spinor derivative D).

In the following bosonic and fermionic superfields will be distinguished by conventionally denoting as $\Phi_j(X)$ the bosonic superfields and $\Psi_j(X)$ the fermionic ones.

The $N=1$ supercurrents are the generators of the $N=1$ $\hat{\mathcal{G}}$ affinization of the $\mathcal{G}$ Lie or super-Lie algebra (see [25] and [26] for the affinization of respectively bosonic algebras and superalgebras). This means they satisfy the following supersymmetric Kac-Moody Poisson brackets algebra

$$\{\Psi_i(X), \Psi_j(Y)\} = \sum_k {f^k}_{ij}\Psi_k(Y)\Delta(X,Y) + K_{ij}D_Y\Delta(X,Y) \tag{106}$$

The above is the $N=1$ extension of the (20) formula.

Here ${f^k}_{ij}$ are the (super)structure constants of $\mathcal{G}$ and $K_{ij} = Str(g_ig_j)$ is the (super)trace in the adjoint representation of $\mathcal{G}$. The brackets are either symmetric or antisymmetric according to the grading of the supercurrents.

Just as in the bosonic case the above algebra will furnish the Poisson brackets structure for the derived integrable hierarchies.

We still need to specify what is Λ in (105). It must be a constant regular element as in the previously studied case. Here however an extra-condition appears: due to the fermionic character of D it seems unavoidable to assume Λ fermionic too in order to keep a definite statistics for $\mathcal{L}$; since in the principal grading case $\mathcal{L}$ is a sum over a simple-roots set of the $\mathcal{G}$ (super)algebra, it follows that the above construction works only if superalgebras are considered which, moreover, admits a presentation in terms of fermionic simple roots only. This case has been analyzed in [10].

When the homogeneous grading is concerned, then Λ should assume the form

$$\Lambda \equiv \lambda H \tag{107}$$

with λ a bosonic spectral parameter and H a generic element in the Cartan sector of the $\mathcal{G}$ (super)algebra.

Since the Cartan sector is always bosonic, even for super-Lie algebras, the above consideration seems to rule out the possibility of introducing integrable hierarchies in connection with the homogeneous grading. However, supersymmetric generalizations of the NLS hierarchies have been produced ([11, 12, 13]) and furthermore it has been shown that at least some of them satisfy a coset property [13, 15], which puts them on the same foot as the corresponding bosonic hierarchies. It seems therefore rather puzzling that they cannot be accomodated in an AKS framework.

There is however a key point which allows us to overcome the previous argument: the presence of the λ bosonic spectral parameter makes possible to introduce a sort of "twisted" statistics for Laurent series in λ. We can indeed assume that the Laurent expansion $B(\lambda)$ is a "twisted" boson if it is given by an alternating sum of bosonic and fermionic power series in λ, such that

$$B(\lambda) = b(\lambda^2) + \lambda \cdot f(\lambda^2) \tag{108}$$

with $b(\lambda^2)$, $f(\lambda^2)$ respectively ordinary bosonic and fermionic Laurent expansions in λ^2.

Conversely $F(\lambda)$ is a "twisted" fermion if

$$F(\lambda) = \xi(\lambda^2) + \lambda \cdot \phi(\lambda^2) \tag{109}$$

where $\xi(\lambda^2)$ ($\phi(\lambda^2)$) is an ordinary fermion (boson).

"Twisted" bosons and fermions are closed under multiplication with the same rules as the ordinary bosons and fermions.

As already pointed out alternating sums of bosons and fermions already appeared in the context of GSO projection and the supersymmetric Witten index.

In this respect $\mathcal{L}$ in (105) must be considered as a twisted fermion.

The crucial feature in the bosonic AKS picture is the existence of an *uniquely* defined adjoint action which allows diagonalizing the Lax operator. The same property holds in the supersymmetric case, but since now we must respect the twisted fermionic character of $\mathcal{L}$, the adjoint action should be defined with respect to a twisted boson.

For two generic twisted boson and fermion given by (108,109) respectively, we can define the $ad_B(F)$ adjoint action as follows

$$ad_B(F) =_{def} [b, \xi] + \lambda^2 [f, \phi] + \lambda \cdot (\{f', \xi\} + [b, \phi]) \tag{110}$$

where the brackets denote the standard commutator and the curly braces the anticommutator (in consistence with the statistics of the component fields b, f, f', ξ, ϕ). In the above formula

$$f' \equiv t(f) \tag{111}$$

is fermionic and t is a linear transformation such that $t^2 = 1$.

This transformation will be explicitly defined in the next section.

The map

$$F \mapsto \tilde{F} = \tilde{\xi} + \lambda \cdot \tilde{\phi} = ad_B(F) \tag{112}$$

sends F into a new twisted fermion since $\tilde{\xi}, \tilde{\phi}$

$$\begin{aligned} \tilde{\xi} &= [b, \xi] + \lambda^2 [f, \phi] \\ \tilde{\phi} &= \{f', \xi\} + [b, \phi] \end{aligned} \tag{113}$$

are respectively bosonic and fermionic.

In the following we will need to introduce the $Ad_B(F)$ action through:

$$Ad_B(F) =_{def} exp(ad_B)(F) = \sum_{n=0}^{\infty} (ad_B)^n (F) \tag{114}$$

It turns out there exists an uniquely defined twisted boson M expanded in non-positive powers in λ, with the boundary condition $M \to 0$ for $\lambda \to \infty$,

$$M = \sum_{k=1}^{\infty} \lambda^{-2k} (b_k + \lambda \cdot f_k) \tag{115}$$

(the b_k's are bosons and the f_k's fermions), which diagonalizes $\mathcal{L}$ under its Ad_M action:

$$\hat{\mathcal{L}} = Ad_M(\mathcal{L}) = \lambda H + D + \Psi_\alpha h_\alpha + \sum_{k=1}^{\infty} \lambda^{-k} R_{k,\alpha} h_\alpha \tag{116}$$

h_α are the Cartan generators. For even (odd) values of k, $R_{k,\alpha}$ are fermions (bosons).

In particular the $R_{2k,\alpha}$ fermionic quantities provide the infinite set of hamiltonian densities for a given hierarchy. We recall that the integration over the superspace is fermionic, so that

$$H_{k,\alpha} = \int dX R_{2k,\alpha}(X) \tag{117}$$

are the bosonic hamiltonians, mutually in involution under the (106) Poisson brackets structure.

The compatible flows are defined, for any given superfield $\Xi(X)$, as

$$\frac{\partial \Xi(X)}{\partial t_{k,\alpha}} = \{\Xi(X), H_{k,\alpha}\} \tag{118}$$

9 The simplest example: the super-NLS hierarchy from the $N=1$ $s\hat{l(2)}$ algebra.

In the previous section the basic ingredients underlining the supersymmetric AKS framework for the homogeneous grading have been introduced. Here a more detailed analysis will be given by illustrating the simplest example of such a construction which arises from the $N=1$ affinization of the $sl(2)$ algebra. The associated hierarchy is the super-NonLinear Schrödinger hierarchy already discussed in [13]. With respect to that paper the method here developed can be immediately extended to more complicated hierarchies.

The $sl(2)$ algebra has been introduced in (29). Its $N=1$ affinization [13] is generated by the 3 fermionic superfields $\Psi_\epsilon(X)$ (with $\epsilon \equiv 0, \pm$) and is determined by the following relations:

$$
\begin{aligned}
\{\Psi_0(X), \Psi_0(Y)\} &= D_Y \Delta(X,Y) \\
\{\Psi_0(X), \Psi_\pm(Y)\} &= \pm 2\Delta(X,Y)\Psi_\pm(Y) \\
\{\Psi_+(X), \Psi_-(Y)\} &= D_Y \Delta(X,Y) + 2\Delta(X,Y)\Psi_0(Y) \equiv \mathcal{D}_Y \Delta(X,Y)
\end{aligned}
\tag{119}
$$

(any other Poisson bracket is vanishing).

The superfields $\Psi_\epsilon(X)$ are decomposed in terms of their component fields as

$$
\Psi_\epsilon(X) = \psi_\epsilon(x) + \theta J_\epsilon(x) \tag{120}
$$

where ψ_ϵ (J_ϵ) are respectively fermionic (bosonic).

The Poisson brackets for the component fields can be directly read from (119).

The upper line in (119) specifies the $N=1$ $\hat{U(1)}$ Kac-Moody subalgebra generated by $\Psi_0(X)$. In general, as in the bosonic case, we can introduce charged superfields $V_q(X)$ (q is the charge), assumed to satisfy the following relation with respect to the $N=1$ $\hat{U(1)}$ Kac-Moody generator:

$$
\{\Psi_0(X), V_q(Y)\} = q\Delta(X,Y)V_q(Y) \tag{121}
$$

In the above formula $V_q(X)$ can either be a bosonic or a fermionic superfield.

The notion of a (supersymmetric) covariant derivative $\mathcal{D}$ can be introduced as in the bosonic case through the following position:

$$
\mathcal{D}V_q(X) = DV_q(X) + q\Psi_0(X)V_q(X) \tag{122}
$$

The covariant derivative is now fermionic and maps a q-charged superfield $V_q(X)$ into a new q-charged superfield (of opposite statistics).

It follows immediately from (119) that $\Psi_\pm(X)$ are superfields of definite charge ± 2. For that reason the right hand side of the last equation in (119) can also be reexpressed by making use of the fermionic covariant derivative, as shown above.

It is worth to mention that the higher order fermionic hamiltonian densities $R_{2k}(X)$ ($k = 1, 2, \ldots$) which will be introduced later, belong to the $N=1$ $\hat{U(1)}$ coset, that is they have vanishing brackets with respect to $\Psi_0(X)$ and the (119) Poisson structure:

$$
\{\Psi_0(X), R_{2k}(Y)\} = 0 \tag{123}
$$

It follows that the whole set of R_{2k}'s hamiltonian densities is provided by chargeless differential polinomials in the superfields $\Psi_\pm(X)$ and fermionic covariant derivatives acting on them. The actual demonstration of this property plainly follows the one already discussed for the bosonic case and for that reason it will be omitted here.

At this point we are not yet entitled to plug the above formulas concerning the $sl(2)$ algebra in the the previous section construction and derive the associated hierarchy, because we have still to define the t linear transformation in (111) and explain its origin.

The discussion concerning the role of the t-transformation has been postponed until now in order to introduce at first the needed algebraic setup, but it will be immediately clear that no peculiar features of $sl(2)$ appear in the following reasoning, which admits a trivial generalizion to any given Lie and super-Lie algebra.

It has been already remarked that in the bosonic case, no matter which Lie algebra and which regular element in its Cartan sector are chosen, the $\pm$-roots exchange symmetry always allows to perform a reduction from a two-component fields hierarchy into a single-component hierarchy. The same feature we wish to preserve in the supersymmetric case as well.

In the particular case of the $sl(2)$ algebra, the $s_\pm$-roots exchange provides the following automorphisms:

$$\begin{array}{cc} E_+ \leftrightarrow E_- & H \mapsto -H \\ \Psi_+(X) \leftrightarrow \Psi_-(X) & \Psi_0(X) \mapsto -\Psi_0(X) \end{array} \tag{124}$$

The hamiltonian densities of the associated hierarchy turn out to have a well-defined transformation property with respect to the above $s_\pm$ exchange (i.e. they are eigenfunctions with eigenvalue ± 1), if the component superfields ξ_k, ϕ_k of the twisted fermionic $F(\lambda)$ series:

$$F(\lambda) \;=\; \sum_{k=-\infty}^{+\infty} \lambda^{2k}(\xi_k + \lambda \cdot \phi_k) \tag{125}$$

satisfy the following transformation properties:

$$s_\pm : \qquad \xi_k \mapsto (-1)^k \xi_k; \qquad\qquad \phi_k \mapsto (-1)^{k+1}\phi_k. \tag{126}$$

Notice in particular that the original Lax operator $\mathcal{L}$ given by

$$\mathcal{L} \;=\; D + \Psi_0(X)H + \Psi_+(X)E_+ + \Psi_-(X)E_- + \lambda H \tag{127}$$

satisfies the above relations for $k = 0$.

The twisted boson $M(\lambda)$ which diagonalizes $\mathcal{L}$ can be decomposed as follows:

$$M(\lambda) \;=\; M(\lambda)_> + M(\lambda)_< \tag{128}$$

where the underscript $\star_>$ ($\star_<$) denotes the projection over the positive (and respectively negative) roots sector of any given Lie or super-Lie algebra. In the $sl(2)$ case this is just the projection over E_+ (E_-). If the bosonic (fermionic) components b_k (f_k) in $M(\lambda)$ given in (115) are mapped into

$$s_\pm : \qquad b_k \mapsto (-1)^k b_k; \qquad\qquad f_k \mapsto (-1)^k f_k \tag{129}$$

then, the adjoint action $ad_M(F)$ as defined in (110), sends $F(\lambda)$ in (109) into a new twisted fermion $ad_M(F)$ whose components satisfy the same $s_{\pm}$-transformation properties as (126), provided that in (110) the linear transformation

$$f(\lambda^2)' \;\;=\;\; t(f(\lambda^2)) =_{def} f(\lambda^2)_> - f(\lambda^2)_< \tag{130}$$

is taken into account.

Such a t-transformation is therefore necessary in order to mantain a well-defined transformation property under the $s_{\pm}$ symmetry in the supersymmetric case too. As a result the diagonalized $\hat{\mathcal{L}}$ Lax operator turns out to be expressed as

$$\hat{\mathcal{L}} \;\;=\;\; Ad_M(\mathcal{L}) = \lambda H + (D + \Psi_0(X)H) + \sum_{k=1}^{\infty} \lambda^{-k}(R_k(X)H) \tag{131}$$

and, under the $s_{\pm}$ transformation, the fermionic hamiltonian densities $R_{2k}(X)$ behave as follows:

$$s_{\pm}: \qquad R_{2k}(X) \mapsto (-1)^{k+1} R_{2k}(X) \tag{132}$$

Apparently it seems that a certain degree of arbitrariness is involved in choosing the parity for the transformation properties of the ξ_k, ϕ_k components of $F(\lambda)$ in (125); this is however not true: the parity of ξ_k, ϕ_k (and as a consequence, that of b_k, f_k) is uniquely fixed by the following two requirements:

i) the supersymmetric fermionic hamiltonian densities are eigenfunctions under the $s_{\pm}$ transformation and

ii) the set of first hamiltonian densities (in the general case denoted as $R_{2k,\alpha}$ with $k = 1$, just R_2 in the specific case of $sl(2)$), should have parity $+1$ (so that $R_2 \mapsto R_2$).

This second requirement is due to the fact that the first hamiltonian densities should guarantee a non-trivial flow (in the $sl(2)$ case R_2 provides chiral equations of motion). This requirement can be understood also as follows: it implies the supersymmetric first flow to coincide with the bosonic first flow when all the fermionic fields are set equal to zero.

We already know indeed that the first hamiltonian density for the $sl(2)$ hierarchy can be expressed as

$$\mathcal{D}\Psi_+ \cdot \Psi_- + \mathcal{D}\Psi_- \cdot \Psi_+ \tag{133}$$

and has $s_{\pm}$-parity $+1$.

The corresponding term of same dimension

$$\mathcal{D}\Psi_+ \cdot \Psi_- - \mathcal{D}\Psi_- \cdot \Psi_+ \tag{134}$$

which has -1 parity is a fermionic total derivative and gives trivial equations of motion.

Similarly in the $osp(1|2)$ case we obtain non-trivial equations of motion if the first hamiltonian density is given by

$$\mathcal{D}\Phi_+ \cdot \Phi_- - \mathcal{D}\Phi_- \cdot \Phi_+ \tag{135}$$

which has parity +1 (here $\Phi_\pm$ are bosonic superfields and $s_\pm : \Phi_- \mapsto \Phi_+ \mapsto -\Phi_-$ as it will be discussed in the next section).

Here again the -1 $s_\pm$-parity term

$$\mathcal{D}\Phi_+\Phi_- + \mathcal{D}\Phi_-\Phi_+ \tag{136}$$

is a total derivative.

It can be explicitly checked with an iterative proof that at any order in the expansion over the λ spectral parameter, the b_k, f_k coefficients of the diagonalizing operator $M(\lambda)$ satisfy the (129) $s_\pm$-transformation properties, which guarantees the consistency of our procedure.

Therefore the whole set of algebraic rules is specified to compute the infinite tower of hamiltonians, mutually in involution, for any given Lie or super-Lie algebra.

For the specific case of the $sl(2)$ algebra we obtain as $M(\lambda)$ diagonalizing operator, at the lowest orders:

$$\begin{aligned} f_1 &= \tfrac{1}{2}(\Psi_+E_+ - \Psi_-E_-) \\ b_1 &= \tfrac{1}{4}(\mathcal{D}\Psi_+E_+ - \mathcal{D}\Psi_-E_-) \end{aligned} \tag{137}$$

The diagonalized Lax operators $\hat{\mathcal{L}}$ reads

$$\begin{aligned} R_1 &= \tfrac{1}{2}\Psi_+\Psi_- \\ R_2 &= \tfrac{1}{8}(\mathcal{D}\Psi_+ \cdot \Psi_- + \Psi_+ \cdot \mathcal{D}\Psi_-) \end{aligned} \tag{138}$$

The second fermionic hamiltonian density R_4 is proportional to

$$R_4 \;\propto\; \mathcal{D}^3\Psi_+ \cdot \Psi_- - \Psi_+\mathcal{D}^3\Psi_- \tag{139}$$

The following flows are obtained, in a convenient normalization:

$$\frac{\partial\Psi_\pm}{\partial t_1} = \mathcal{D}^2\Psi_\pm \tag{140}$$

and

$$\frac{\partial\Psi_\pm}{\partial t_2} = \pm\mathcal{D}^4\Psi_\pm \mp 4\Psi_\pm\mathcal{D}(\Psi_\mp\mathcal{D}\Psi_\pm) \tag{141}$$

while for any flow, due to the above specified coset property of the hamiltonian densities, we get

$$\frac{\partial\Psi_0}{\partial t_k} = 0 \tag{142}$$

In particular the second flow coincides (apart a normalization factor) with the two-component super-NLS equation of ref. [13] once set the constraint, compatible with the equations of motion

$$\Psi_0 \equiv 0$$

The single-component superfield super-NLS equation is recovered by setting the time being imaginary ($t = -it_2$) and

$$\Psi_+ = {\Psi_-}^\star = \Psi$$

The final result is

$$i\dot{\Psi} = D^4\Psi - 4\Psi D(\Psi^\star \cdot D\Psi) \tag{143}$$

10 The integrable super-hierarchy associated to the $osp(1|2)$ superalgebra.

In this section it will be shown that the supersymmetric AKS framework for the homogeneous grading can be worked out not only for bosonic Lie algebras (as it is the case for the super-NLS equation), but also for super-Lie algebras. It will be analyzed in detail the simplest example of such kind of construction, namely the hierarchy derived from the $osp(1|2)$ superalgebra.

This superalgebra admits H as a bosonic generator and $F_\pm$ as fermionic ones and is given by the following relations:

$$
\begin{aligned}
[H, F_\pm] &= \pm 2F_\pm \\
\{F_+, F_-\} &= H
\end{aligned}
\tag{144}
$$

Its $N=1$ affinization is realized by the 3 superfields $\Psi_0(X)$ (fermionic) and $\Phi_\pm(X)$ (bosonic). Formally it is given by the same relations as (119) with the replacement $\Psi_\pm \mapsto \Phi_\pm$, but now we have to take into account that the last Poisson bracket in (119) is antisymmetric due to the bosonic character of $\Phi_\pm$.

The $s_\pm$ algebra automorphism associated with the positive versus negative roots exchange is now a Z_4 symmetry. In a chosen normalization we can define it to be

$$
\begin{aligned}
s_\pm : \qquad & H \mapsto -H; \quad F_+ \mapsto -F_-; \quad F_- \mapsto F_+. \\
& \Psi_0 \mapsto -\Psi_0; \quad \Phi_+ \mapsto -\Phi_-; \quad \Phi_- \mapsto \Phi_+.
\end{aligned}
\tag{145}
$$

One can easily check that in this case the same transformation properties (126,129) under $s_\pm$ for the twisted fermion $F(\lambda)$, the Lax operator $\mathcal{L}$ and its diagonalizing matrix $M(\lambda)$ which hold for bosonic algebras are verified too.

Moreover here again we find that the hamiltonian fermionic densities belong to the $N=1$ $\hat{U}(1)$ subalgebra coset generated by Ψ_0.

We find explicitly, at the lowest orders in the λ expansion, for the diagonalizing matrix $M(\lambda)$:

$$
\begin{aligned}
f_1 &= \tfrac{1}{2}(\Phi_+ F_+ - \Phi_- F_-) \\
b_1 &= \tfrac{1}{4}(\mathcal{D}\Phi_+ F_+ - \mathcal{D}\Phi_- F_-)
\end{aligned}
\tag{146}
$$

while the diagonalized $\hat{\mathcal{L}}$ Lax operator is given by

$$
\hat{\mathcal{L}} = \lambda H + (D + \Psi_0 H) + \lambda^{-1}\cdot\tfrac{1}{2}\Phi_+\Phi_- H + \lambda^{-2}\cdot\tfrac{1}{8}(\mathcal{D}\Phi_+\cdot\Phi_- - \mathcal{D}\Phi_-\cdot\Phi_+)H + O(\lambda^{-3})
\tag{147}
$$

Up to an overall normalization the first fermionic hamiltonian density is

$$
R_2 = \mathcal{D}\Phi_+\cdot\Phi_- - \mathcal{D}\Phi_-\cdot\Phi_+ \tag{148}
$$

which has $s_\pm$-parity $+1$.

Due to the considerations developed in the previous section, the second hamiltonian density R_4 has parity -1.

In principle there exists two independent chargeless terms $X_{1,2}$ having the right dimensions and parity -1 which can contribute to R_4:

$$\begin{aligned} X_1 &= \mathcal{D}^3\Phi_+ \cdot \Phi_- + \mathcal{D}^3\Phi_- \cdot \Phi_+ \\ X_2 &= \Phi_+\Phi_-(\mathcal{D}\Phi_+ \cdot \Phi_- - \mathcal{D}\Phi_- \cdot \Phi_+) \end{aligned} \tag{149}$$

The integrability condition for the hierarchy requires precisely $R_4 \propto X_2$ (notice that the corresponding term associated to the $sl(2)$ algebra is vanishing due to the fermionic character of $\Psi_\pm$).

In a convenient normalization we obtain for the first flow

$$\frac{\partial \Phi_\pm}{\partial t_1} = \mathcal{D}^2\Phi_\pm \pm 4\Phi_\pm(\Phi_+ \cdot \Phi_-) \tag{150}$$

and for the second flow

$$\frac{\partial \Phi_\pm}{\partial t_2} = \mathcal{D}(\mathcal{D}\Phi_\pm \cdot \Phi_+\Phi_-) \pm 2\Phi_\pm(\Phi_+\Phi_-)^2 \tag{151}$$

Here again we can consistently set $\Psi_0 \equiv 0$.

The single-component superfields hierarchies are recovered by setting

$$\Phi_+ = -i{\Phi_-}^\star = \Phi \tag{152}$$

We get for the first flow the equation

$$\dot{\Phi} = \Phi' + 4i\Phi|\Phi|^2 \tag{153}$$

while the second flow, obtained by letting the time imaginary, is

$$\dot{\Phi} = \Phi'|\Phi|^2 - \Phi D\Phi \cdot D\Phi^\star + 2i\Phi|\Phi|^4 \tag{154}$$

(the prime denotes the ordinary spatial derivative).

In terms of the component fields we have

$$\Phi(X) = \phi(x) + \theta\psi(x) \tag{155}$$

with $\phi(x)$ bosonic and $\psi(x)$ fermionic.

We get respectively

$$\begin{aligned} \dot{\phi} &= \phi' + 4i\phi|\phi|^2 \\ \dot{\psi} &= \psi' + 8i\psi|\phi|^2 + 4i\phi^2\psi^\star \end{aligned} \tag{156}$$

for the first flow and

$$\begin{aligned} \dot{\phi} &= \phi'|\phi|^2 + 2i\phi|\phi|^4 - \phi|\psi|^2 \\ \dot{\psi} &= \psi'|\phi|^2 + \phi'(\psi\phi^\star + 2\psi^\star\phi) - \phi^{\star\prime}\phi\psi + \\ &\quad 2i\psi|\phi|^4 + 4i\phi(|\phi|^2 + \psi\phi^\star + \phi^\star\psi) \end{aligned} \tag{157}$$

for the second one.

Notice in the right hand side of the equation of motion for the bosonic component the presence of the fermionic field, which implies a non-trivial coupling.

11 The $N=1$ $\hat{sl(3)}$ hierarchies.

In this last section I will construct the $N=1$ supersymmetric extensions of the $\hat{sl(3)}$ hierarchies introduced in section (7).

It is a rather unexpected result that integrable supersymmetric extensions can be produced only for the hierarchies labelled by the γ real parameter belonging to the range

$$\gamma \geq 1 \tag{158}$$

The approach here followed has been already illustrated in detail in the previous sections, so that here I will limit myself to furnish the results.

Since in order to reach the above conclusion it is sufficient to explicitly compute the first flow only, just this case will be presented in this paper.

It should be noticed that the computations in the supersymmetric case are much more involved than in the bosonic case basically because to get the k-th ordered flow we have to perform a double number (equal to $2k$) of diagonalizations. It soon appears that computer is needed to explicitly obtain even the next simplest flows.

The $N=1$ $\hat{sl(3)}$ algebra is generated by the fermionic superfields $\Psi_{0,1}, \Psi_{0,2}$ and $\Psi_{\pm i}$ with $i=1,2,3$.

The following Poisson brackets are verified

$$\begin{aligned}
\{\Psi_{0,1}(X), \Psi_{0,1}(Y)\} &= -2D_Y\Delta(X,Y)\\
\{\Psi_{0,1}(X), \Psi_{0,2}(Y)\} &= D_Y\Delta(X,Y)\\
\{\Psi_{0,2}(X), \Psi_{0,2}(Y)\} &= -2D_Y\Delta(X,Y)
\end{aligned} \tag{159}$$

and

$$\begin{aligned}
\{\Psi_{+i},(X), \Psi_{-i}(Y)\} &= \mathcal{D}_Y\Delta(X,Y) \qquad for \quad i=1,2,3.\\
\{\Psi_{+1}(X), \Psi_{-3}(Y)\} &= -\Delta(X,Y)\Psi_{-2}(Y) \qquad \{\Psi_{+2}(X), \Psi_{-3}(Y)\} = \Delta(X,Y)\Psi_{-1}(Y)\\
\{\Psi_{+3}(X), \Psi_{-1}(Y)\} &= -\Delta(X,Y)\Psi_{+2}(Y) \qquad \{\Psi_{+3}(X), \Psi_{-2}(Y)\} = \Delta(X,Y)\Psi_{+1}(Y)\\
\{\Psi_{\pm 1}(X), \Psi_{\pm 2}(Y)\} &= \pm\Delta(X,Y)\Psi_{\pm 3}(Y)
\end{aligned} \tag{160}$$

The covariant derivative $\mathcal{D}$ acts as

$$\begin{aligned}
\mathcal{D}\Psi_{\pm 1} &= D\Psi_{\pm 1} \mp \Psi_{0,1}\Psi_{\pm 1}\\
\mathcal{D}\Psi_{\pm 2} &= D\Psi_{\pm 2} \mp \Psi_{0,2}\Psi_{\pm 2}\\
\mathcal{D}\Psi_{\pm 3} &= D\Psi_{\pm 3} \mp (\Psi_{0,1}+\Psi_{0,2})\Psi_{\pm 3}
\end{aligned} \tag{161}$$

As in section (7) the regular Cartan element is chosen to be

$$\lambda H = \lambda(tH_1 + (1-t)H_2) \tag{162}$$

with $0 \leq t \leq 1$.

We will also make use of the variables

$$\begin{aligned}\gamma &= \tfrac{1}{3t-1}\\ \tilde{\gamma} &= \tfrac{1}{2-3t}\end{aligned} \qquad (163)$$

The first (bosonic) densities for the diagonalized Lax operator are

$$\begin{aligned}R_{1,1} &= \Psi_{+3}\Psi_{-3}+\gamma\Psi_{+1}\Psi_{-1}\\ R_{1,2} &= \Psi_{+3}\Psi_{-3}+\tilde{\gamma}\Psi_{+2}\Psi_{-2}\end{aligned} \qquad (164)$$

which are antisymmetric under the $\pm$-roots exchange

$$\Psi_{+1}\leftrightarrow\Psi_{-1};\qquad \Psi_{+2}\leftrightarrow\Psi_{-2};\qquad \Psi_{+3}\leftrightarrow-\Psi_{-3}. \qquad (165)$$

and mutually transforms under the σ outer automorphism (see section (7)).

The first hamiltonian density $R_{2,1}$ is fermionic. It is given by

$$\begin{aligned}R_{2,1} &= \tfrac{1}{2}\gamma^2(\mathcal{D}\Psi_{+1}\cdot\Psi_{-1}+\mathcal{D}\Psi_{-1}\cdot\Psi_{+1})\\ &\quad+\tfrac{1}{2}(\mathcal{D}\Psi_{+3}\cdot\Psi_{-3}+\mathcal{D}\Psi_{-3}\cdot\Psi_{+3})\\ &\quad+C(\Psi_{+1}\Psi_{+2}\Psi_{-3}-\Psi_{-1}\Psi_{-2}\Psi_{+3})\end{aligned} \qquad (166)$$

with

$$C = \tfrac{1}{12}(4\gamma\tilde{\gamma}+5\gamma+4\tilde{\gamma}+3) \qquad (167)$$

$R_{2,1}$ is invariant under the $\pm$-roots exchange; the hamiltonian density $R_{2,2}$ is obtained from the previous expression by replacing $1\leftrightarrow 2$ and $\gamma\leftrightarrow\tilde{\gamma}$.

Notice that, while the relative coefficient of the first two terms in the right hand side of (166) is fixed by requiring that the correct bosonic limit would be reproduced when setting equal to zero all the fermionic fields, the coefficient $C$of the third term cannot be recovered from the bosonic limit. As far as the supersymmetrization only is concerned C is a free parameter. However, when the integrability property is taken into account, C must be restricted to be the particular value (167).

From the above hamiltonian, together with the (159,160) Poisson brackets structure, the following set of equations of motion is derived:

$$\begin{aligned}\dot{\Psi}_{\pm1} &= \gamma^2\mathcal{D}^2\Psi_{\pm1}\mp(C\mathcal{D}\Psi_{\mp2}\cdot\Psi_{\pm3}+(1-C)\Psi_{\mp2}\mathcal{D}\Psi_{\pm3})\\ &\quad\pm\Psi_{\pm1}((1+C)\Psi_{+3}\Psi_{-3}-C\Psi_{+2}\Psi_{-2})\\ \dot{\Psi}_{\pm2} &= \pm((1-C)\Psi_{\mp1}\mathcal{D}\Psi_{\pm3}-(\gamma^2-C)\Psi_{\pm3}\mathcal{D}\Psi_{\mp1})\\ &\quad\pm\Psi_{\pm2}((1+C)\Psi_{+3}\Psi_{-3}-\gamma^2\Psi_{+1}\Psi_{-1})\\ \dot{\Psi}_{\pm3} &= \pm((C-\gamma^2)\mathcal{D}\Psi_{\pm1}\cdot\Psi_{\pm2}-C\Psi_{\pm1}\mathcal{D}\Psi_{\pm2})\\ &\quad\pm\Psi_{\pm3}((\gamma^2+C)\Psi_{+1}\Psi_{-1}+C\Psi_{+2}\Psi_{-2})\end{aligned} \qquad (168)$$

Pewrforming the single-component reduction

$$\begin{aligned}\Psi_{+1} &= \Psi_{-1}\equiv\Psi_1\\ \Psi_{+2} &= \Psi_{-2}\equiv\Psi_2\\ \Psi_{+3} &= -\Psi_{-3}\equiv\Psi_3\end{aligned} \qquad (169)$$

and setting $\Psi_{0,1} \equiv \Psi_{0,2} \equiv 0$, we obtain

$$\begin{aligned}
\dot{\Psi}_1 &= \gamma^2 \Psi_1{}' - C D\Psi_2 \cdot \Psi_3 - (1-C)\Psi_2 D\Psi_3 \\
\dot{\Psi}_2 &= (1-C)\Psi_1 D\Psi_3 - (\gamma^2 - C)\Psi_3 D\Psi_1 \\
\dot{\Psi}_3 &= \Psi_3{}' + (C-\gamma^2) D\Psi_1 \cdot \Psi_2 - C\Psi_1 D\Psi_2
\end{aligned} \tag{170}$$

In terms of the component fields

$$\Psi_i(X) = \psi_i(x) + \theta\phi_i(x) \tag{171}$$

with $\psi_i(x)$ fermionic and $\phi_i(x)$ bosonic, we get the following set of equations

$$\begin{aligned}
\dot{\psi}_1 &= \gamma^2 \psi_1{}' - C\phi_2\psi_3 - (1-C)\psi_2\phi_3 \\
\dot{\psi}_2 &= (1-C)\psi_1\phi_3 - (\gamma^2 - C)\psi_3\phi_1 \\
\dot{\psi}_3 &= \psi_3{}' + (C-\gamma^2)\psi_2\phi_1 - C\psi_1\phi_2
\end{aligned} \tag{172}$$

for the fermionic components and

$$\begin{aligned}
\dot{\phi}_1 &= \gamma^2 \phi_1{}' - \phi_2\phi_3 - C\psi_2{}'\psi_3 + (1-C)\psi_2\psi_3{}' \\
\dot{\phi}_2 &= (1-\gamma^2)\phi_1\phi_3 - (1-C)\psi_1\psi_3{}' + (\gamma^2 - C)\psi_3\psi_1{}' \\
\dot{\phi}_3 &= \phi_3{}' - \gamma^2\phi_1\phi_2 + C\psi_1\psi_2{}' + (\gamma^2 - C)\psi_2\psi_1{}'
\end{aligned} \tag{173}$$

for the bosonic ones.

Notice in particular that, when setting equal to zero the fermionic ψ_i fields, we recover precisely the (93) equations of motion with γ replaced by γ^2 (and the spatial coordinate x mapped to $x \mapsto -x$ due to normalization conventions). Therefore only the bosonic hierarchies associated to non-negative values of γ can be supersymmetrically extended while preserving integrability. More than that, since the σ automorphism sends $\gamma \mapsto \tilde{\gamma}$ (see (98)), and the corresponding transformed hamiltonian all belong to the integrable hierarchy, it turns out that only for

$$\gamma \geq 1$$

we have a supersymmetric integrable extension, which is the result stated above.

Conclusions

In this paper an analysis of several aspects of the Lie algebraic approach towards integrable hierarchies have been furnished. The final aim consists in arriving at a complete classification of the whole class of inequivalent hierarchies.

Two points have been raised: the role of the regular element in determining the corresponding hierarchy and the possible relation between different hierarchies associated to different gradings.

The first problem has been here addressed in the particular case of the homogeneous-graded hierarchies. It has been shown that indeed inequivalent hierarchies have been obtained as a consequence and that, for a generic Lie algebra, they are labelled by continuous values of some real parameters.

Moreover their mutual transformations under Weyl group or outer automorphisms actions have been investigated, as well as their coset property. The possible fields reductions when a symmetry is present have also been considered.

For what concerns the supersymmetric integrable hierarchies, the "trick" of introducing alternating series of bosons and fermions in the spectral parameter expansion, considerably allowed us to enlarge the class of Lie-algebraic-derived hierarchies, which until now was rather restricted (associated to a very specific principal-graded construction). As a simple byproduct of our method we were able to explicitly produce new supersymmetric hierarchies, not yet investigated so far.

The investigation concerning supersymmetric hierarchies is rather important if we wish to arrive at a consistent formulation of discretized 2-dimensional gravity. The fact that until now no direct supersymmetric matrix model formulation is available, obliges us to bypass this step and to directly formulate such models in terms of super-$\mathcal{W}$ constraints, associated to some integrable hierarchy, on the partition function.

However there exists a large amount of arbitrariness in performing such supersymmetrizations and a definite criterium should be found to extract the "meaningful" hierarchies. A very good example of that is one of the features discussed in this paper, namely that there exists a class of bosonic integrable hierarchies which seem do not admit an integrable supersymmetric extension. In the $sl(3)$ case we proved that the standard procedure to obtain integrable supersymmetric hierarchies defines supersymmetric extensions only for a restricted class of the bosonic hierarchies.

Another example is associated to KdV (see [27]): there exists a continuous class of $N = 2$ super-KdV equations, but only for 3 specific values of the parameter we have integrability. More than that, only one of these values corresponds to a nice Lie algebraic setting and the associated hierarchy seems, in some sort, more fundamental than the others.

It is surprising that such a hierarchy is obtained, through a non-local Darboux transformation, from the super-NLS hierarchy we have here discussed. The existence of a Darboux transformation in this very specific case naturally leads us to ask about the second point mentioned in this conclusion, that is: which hierarchies, associated to different gradings, are truly independent and which, on the contrary, are related to each other through a Darboux transformation. This point needs to be clarified in order to have a complete understanding of the integrable hierarchy picture and deserves investigation.

Besides this major question, let us list some of the topics which can be easily addressed in the future:

Is it possible to use the same "trick" of introducing twisted fermionic and bosonic power series to define supersymmetric integrable hierarchies for the intermediate grading (i.e. different from the principal and the homogeneous) case?

Next, a rather technical problem: how to formulate $N = 2$ hierarchies in terms of a manifestly $N = 2$ formalism. This is not a fundamental problem because, as already explained, our constructions accomodates $N = 2$ supersymmetric hierarchies in an $N = 1$ manifestly superfield formalism.

Furthermore, there exists a certain degree of parallelism between the integrable hierarchy formulation on one side and the WZNW reductions (leading to Toda and coset models) on the other. It is likely that at least some of the ideas here discussed can find applications to investigate and generalize the Witten's black hole construction.

As a final point let me recall that the algebraic machinery here developed allows, with the help of computer, to explicitly produce in a systematic way more general hierarchies than those here presented, associated e.g. to the superalgebra $sl(2|1)$ and so on.

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