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Anti-gravitating BPS monopoles and dyons

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ABSTRACT

We show that the exact static, i.e. 'anti-gravitating', magnetic multi monopole solutions of the Einstein/Maxwell/dilaton-YM/Higgs equations found by Kastor, London, Traschen, and the authors, for arbitrary non-zero dilaton coupling constant a, are equivalent to the string theory BPS magnetic monopole solutions of Harvey and Liu when $a = \sqrt{3}$. For this value of a, the monopole solutions also solve the equations of five-dimensional supergravity/YM theory. We also discuss some features of the dyon solutions obtained by boosting in the fifth dimension and some features of the moduli space of anti-gravitating multi-monopoles.

It has been known for some time that certain non-abelian Yang-Mills/Higgs theories in flat spacetime admit multi-monopole solutions in which the magnetic repulsion is balanced by the attractive forces due to Higgs exchange. More recently it has been shown that this equilibrium continues to be possible in the presence of additional attractive forces due to gravitation and a massless scalar field [1,2]. In [2] this result was obtained directly in four dimensions by the inclusion of an additional abelian vector potential, A_μ, having a non-renormalizable coupling to the Yang-Mills magnetic charge density. Remarkably, it is then possible to find exact analytic solutions for the metric, dilaton and abelian vector fields entirely in terms of solutions of the *flat space* Bogomol'nyi equations in the Yang-Mills/Higgs sector. These results were shown to hold for all non-zero values of the 'dilaton coupling constant', a, defined by the coupling of σ to the Maxwell field strength, $F_{\mu\nu}$, provided that the scalar field, σ, has particular couplings to the Yang-Mills gauge potential, $\mathbf{B}_\mu$ through its field strength tensor $\mathbf{G}_{\mu\nu}$, and to the Higgs field $\mathbf{\Phi}$. These interaction terms might appear to be artificial but they are precisely those required by local supersymmetry (at least for certain values of a) and therefore arise naturally in supergravity and superstring theories. The action of [2] that was shown to admit these static self-gravitating solitons is

$$\begin{aligned} S =& \frac{1}{4}\int d^4x \left\{\sqrt{-g}[R - 2(\partial\sigma)^2 - e^{-2a\sigma}F^2 - e^{-\frac{(1-a^2)}{a}\sigma}|\mathbf{G}|^2 - 2e^{\frac{1+a^2}{a}\sigma}|\mathcal{D}\mathbf{\Phi}|^2]\right. \\ & \left. - 2\sqrt{1+a^2}\, A_\mu\, \epsilon^{\mu\nu\lambda\rho}\mathbf{G}_{\nu\lambda}\cdot\mathcal{D}_\rho\mathbf{\Phi}\right\} \end{aligned} \tag{1}$$

where $\mathcal{D}$ is the YM covariant derivative, and we have set $4\pi G$ and the Yang-Mills (YM) coupling constant to unity. The dilaton coupling constant a is related to the constant b used in [2] by $a = -b$. We may choose $a \geq 0$ without loss of generality, and we shall assume this in what follows.

The spacetime metric, Maxwell one-form, and scalar field in the self-gravitating monopole solution of [2] have the form

$$
\begin{aligned}
ds^2 &= -U^{\frac{-2}{1+a^2}}dt^2 + U^{\frac{2}{1+a^2}}d\mathbf{x}^2 \\
A &= \frac{dt}{U}\frac{1}{\sqrt{1+a^2}} \\
\sigma &= -\frac{a}{1+a^2}\ln U \ .
\end{aligned}
\tag{2}
$$

The function U satisfies

$$
\nabla^2 U = -(1+a^2)\sum_{i=1}^{3}|\mathcal{D}_i\mathbf{\Phi}|^2 \ , \tag{3}
$$

where $\mathbf{\Phi}$ is a solution of the *flat space* Bogomol'nyi equations:

$$
\mathbf{G}_{ij} = \delta_{il}\delta_{jm}\varepsilon^{lmk}\mathcal{D}_k\mathbf{\Phi} \ . \tag{4}
$$

Since the Bogomol'nyi equations imply that

$$
2\sum_{i=1}^{3}|\mathcal{D}_i\mathbf{\Phi}|^2 = \nabla^2\big(|\mathbf{\Phi}|^2\big) \ , \tag{5}
$$

and we require that $U \to 1$ at spatial infinity, we have that

$$
U = 1 + \frac{1}{2}(1+a^2)\big(\eta^2 - |\mathbf{\Phi}|^2\big) \ , \tag{6}
$$

where η is the value at infinity of $\sqrt{|\mathbf{\Phi}|^2}$, so the solution is entirely, and explicitly, determined in terms of $\mathbf{\Phi}$. For example, for the $SO(3)$ BPS monopole we have

$$
\mathbf{\Phi} = \frac{\eta\mathbf{r}}{r}\Big[\frac{1}{\eta r} - \coth(\eta r)\Big] \tag{7}
$$

from which we compute

$$
\big(\eta^2 - |\mathbf{\Phi}|^2\big) = \frac{1}{r^2}\Big[2(\eta r)\coth(\eta r) - (\eta r)^2\mathrm{cosech}^2(\eta r) - 1\Big] \ , \tag{8}
$$

and hence the function U. Note that we get an asymptotically flat solution with the scalar field σ tending to zero for all values of the integration constant η and hence for all values of the length of the Higgs field $\mathbf{\Phi}$ at infinity.

The construction of [1] can be be viewed [3] (see also [4]) as a dimensional reduction of a fivebrane solution of the field theory limit of the ten-dimensional heterotic string. This ten-dimensional supergravity/YM theory can be reduced to five-dimensions and the resulting action can be consistently truncated such that the only surviving fields are the 5-metric, the dilaton, ϕ, the two-form potential, with 3-form field strength H, and the Lie-algebra valued Yang-Mills gauge potential, $\mathbf{Y}$, with two-form YM field strength $\mathbf{M}$. The five-dimensional action for these fields is

$$S = \int d^5x \, \sqrt{-g} e^{-2\phi} \Big(R + 4(\partial\phi)^2 - \frac{1}{3} H^2 - |\mathbf{M}|^2 \Big) \ , \tag{9}$$

where the three-form H satisfies the modified Bianchi identity

$$\partial_{[A} H_{BCD]} = \frac{3}{2} \mathbf{M}_{[AB} \cdot \mathbf{M}_{CD]} \ . \tag{10}$$

Note that (9) is not the bosonic sector of a five-dimensional supergravity because it lacks the five-dimensional Higgs field and an abelian gauge potential to partner the scalar ϕ. It is however, a consistent truncation of five dimensional supergravity coupled to both a YM supermultiplet and an abelian vector multiplet, and in this context the coefficient $\frac{3}{2}$ in (10) is fixed by five-dimensional supersymmetry, as we shall explain later. The field equations of (9) can be solved with Kaluza-Klein type boundary conditions to give the soliton solution of [1] of the dimensionally reduced four-dimensional field theory.

The purpose of this paper is to clarify the relation between the results of [1] and those of [2]. Note first that these multi-monopoles exist for *all* values of the dimensionless parameter $4\pi G\eta^2$ governing the relative strength of (super)gravitational versus YM/Higgs forces. That is, regardless of the ratio of the Higgs mass to the Planck mass, BPS monopoles do not undergo gravitational collapse to form black holes. In [3], this feature was attributed principally to the dilaton but since forces due to scalar fields are *attractive* it seems unlikely that this is the explanation. For the solutions of [2] this feature seems to be a consequence of the electrostatic *repulsion* brought about by the vector field, which also allows the solutions to saturate a Bogomolnyi-type energy bound. This interpretation is less clear in the context of the solutions of [1] because the string inspired five-dimensional action (9) has a two-form potential rather than a vector potential. However, in five dimensions a two-form potential can be exchanged for a vector potential by a duality transformation. This may be accomplished by imposing the constraint (10) by a Lagrange multiplier one-form potential V, and then promoting H to the status of an independent field (a procedure that is consistent with supersymmetry [5]). One introduces a new vector potential V as a Lagrange multipler field and adds to the action (9) the Lagrange multipler term

$$S_L = \frac{2}{3}\int d^5x \; V_E \, \epsilon^{EABCD}\Big(\partial_A H_{BCD} - \frac{3}{2}\mathbf{M}_{AB}\cdot\mathbf{M}_{CD}\Big) \; . \tag{11}$$

Variation of the combined action with respect to H_{ABC} reveals that

$$H_{ABC} = \frac{1}{2}e^{2\phi} \; \epsilon_{ABCDE}F^{DE} \; , \tag{12}$$

where $F_5 = dV$ is the two-form field strength of V. One may now back substitute into the action (9) augmented by (11) to obtain the new, dual, action

$$\tilde{S} = \int d^5x \; \Big\{ e^{-2\phi}\sqrt{-g}\Big[R + 4(\partial\phi)^2 - F_5^2 - |\mathbf{M}|^2\Big] - V_A \, \epsilon^{ABCDE}\mathbf{M}_{BC}\cdot\mathbf{M}_{DE}\Big\} \tag{13}$$

Here we pause to remark that the unit coefficient of the last, topological, term in this action is determined by the 3/2 coefficient in (10). It is also precisely what is required by supersymmetry. To see this, one needs to compare (13) with the results of [6] for the coupling of five-dimensional supergravity to vector multiplets. To do this it is convenient to rescale the metric by $g_{AB} \to e^{\frac{4}{3}\phi} \; g_{AB}$. Discarding a surface term, one then obtains the dual action in Einstein conformal gauge:

$$\tilde{S} = \int d^5x \; \Big\{ \sqrt{-g}[R - \frac{4}{3}(\partial\phi)^2 - e^{\frac{8}{3}\phi} \; F_5^2 - e^{-\frac{4}{3}\phi} \; |\mathbf{M}|^2] - V_A \, \epsilon^{ABCDE}\mathbf{M}_{BC}\cdot\mathbf{M}_{DE}\Big\} \; . \tag{14}$$

By choosing the YM group to be $U(1)$ we can compare with the bosonic sector of the Maxwell/Einstein supergravity action which we review in an appendix. One finds agreement provided that the coefficient of the topological term is as given above.

We now have a form of the five-dimensional string-related action in which a vector potential replaces the two-form potential. To relate this to the action (1) we must dimensionally reduce it to four-dimensions and then truncate to the fields of (1). The dimensional reduction can be done by setting

$$\begin{aligned} (ds_5)^2 &= e^{2\rho}(dx^5 - 2K)^2 + e^{2\psi}(ds_4)^2 \\ V &= v\,(dx^5 - 2K) + A \\ \phi &= \psi + \frac{1}{2}\rho \\ \mathbf{Y} &= \mathbf{\Phi}\,(dx^5 - 2K) + \mathbf{B} \end{aligned} \tag{15}$$

where ψ, ρ, v and $\mathbf{\Phi}$ are four-dimensional scalar fields, K, A and $\mathbf{B}$ are one-forms on four-dimensional spacetime and the 5-metric is in string conformal gauge. The particular choice of four-dimensional fields in (15) ensures that A and $\mathbf{B}$ are invariant under the KK gauge transformation $K \to K + df$ induced by the coordinate transformation $x^5 \to x^5 + f(x^\mu)$. It is convenient to define the 'modified' four-dimensional field strength two-forms

$$F' = F - 2vdK \qquad , \qquad \mathbf{G}' = \mathbf{G} - 2\mathbf{\Phi} dK \ , \tag{16}$$

where $F = dA$ and $\mathbf{G}$ is the four-dimensional YM field strength for $\mathbf{B}$. On substitution of the ansatz (15) into the five-dimensional action (13) one obtains, up to a surface term, the four-dimensional action

$$\begin{aligned} S = \int d^4x \Big\{ & \sqrt{-g}\big[R - 2(\partial\psi)^2 - (\partial\rho)^2 - 2e^{4\psi}(\partial v)^2 - e^{-2\psi+2\rho}(L)^2 \\ & - e^{2\psi+2\rho}(F')^2 - 2e^{-2\rho}|\mathcal{D}\mathbf{\Phi}|^2 - e^{-2\psi}|\mathbf{G}'|^2\big] \\ & - v\,\epsilon^{\mu\nu\lambda\rho}\mathbf{G}'_{\mu\nu}\cdot\mathbf{G}'_{\lambda\rho} - 4A_\mu\,\epsilon^{\mu\nu\lambda\rho}\mathbf{G}'_{\nu\lambda}\cdot\mathcal{D}_\rho\mathbf{\Phi} \Big\} \ , \end{aligned} \tag{17}$$

where $L_{\mu\nu} = 2\partial_{[\mu}K_{\nu]}$.

In order to obtain the action (1) by a truncation of (17) we must choose

$$a = \sqrt{3} \tag{18}$$

and set

$$v = 0 \ , \qquad K = 0 \ , \qquad \rho = -\frac{2}{\sqrt{3}}\sigma \ , \qquad \psi = -\frac{1}{\sqrt{3}}\sigma \ . \tag{19}$$

However, *this truncation is not a consistent one*, in the sense that solutions of the equations of motion of the truncated theory are not automatically solutions of those of the untruncated theory but will be so only if the untruncated fields satisfy constraints. This is the principal complicating factor in relating the results of [2] to those of [1]. These constraints are

$$\begin{aligned}
0 &= \varepsilon^{\mu\nu\rho\sigma}\mathbf{G}_{\mu\nu}\cdot\mathbf{G}_{\rho\sigma} \\
0 &= \partial_\mu\left[|\mathbf{\Phi}|^2\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} + 2\sqrt{-g}\,e^{\frac{2}{\sqrt{3}}\sigma}\mathbf{G}^{\mu\nu}\cdot\mathbf{\Phi}\right] \\
0 &= 2e^{\frac{2}{\sqrt{3}}\sigma}|\mathcal{D}\mathbf{\Phi}|^2 - |G|^2 \ .
\end{aligned} \tag{20}$$

It is straightforward to verify that the solution (2) of the field equations of (1) satisfies these constraints. We deduce from this that (2) is also a solution of the field equations of the untruncated four-dimensional action (17), and hence of the field equations of the five dimensional action (13). It follows that the latter field equations admit the solution

$$\begin{aligned}
ds_5^2 &= -dt^2 + U[(dx^5)^2 + d\mathbf{x}^2] \\
V &= \frac{1}{2U}dt \\
e^{2\phi} &= U \\
\mathbf{Y} &= \mathbf{\Phi}dx^5 + \mathbf{B}_i dx^i \ .
\end{aligned} \tag{21}$$

where $\mathbf{\Phi}$ and $\mathbf{B}_i$ solve the flat space Bogomolnyi equations. This is the solution used in [1]. We conclude that, for the special case of dilaton coupling $a = \sqrt{3}$, the multi-monopole solution of [2] is equivalent to that of [1].

The five-dimensional interpretation of the monopole solutions enables a class of dyon solutions to be found by the method of boosting in the fifth dimension [7]. This changes the asymptotic value, η, of the length of the Higgs field $\mathbf{\Phi}$ but, since η was arbitrary, this problem can be simply overcome by choosing the initial asymptotic value of $|\mathbf{\Phi}|$ to have some other value, η', and then adjusting eta' such that $|\mathbf{\Phi}| \to \eta$. Thus, we first make the replacement

$$dx^5 \to \gamma(dx^5 + \beta dt) \qquad , \qquad dt \to \gamma(dt + \beta dx^5) \ , \tag{22}$$

in (21), where $\gamma = (1 - \beta^2)^{-\frac{1}{2}}$. If we denote by $\mathbf{\Phi}^0(\mathbf{x}, \eta'), \mathbf{B}^0_i(\mathbf{x}, \eta')$, the solution of the flat space Bogomolnyi equations (η' being the expectation value of the Higgs field that we start with) and by $U^0(\mathbf{x}, \eta')$ the associated solution of Poisson's equation, then the new fields are given by

$$\begin{aligned} ds_5^2 &= -\frac{U^0}{\gamma^2(U^0 - \beta^2)} dt^2 + \gamma^2(U^0 - \beta^2)\big[dx^5 + \beta\frac{U^0 - 1}{U^0 - \beta^2}dt\big]^2 + U^0 d\mathbf{x}^2 \\ V &= \frac{\gamma}{2U^0}\, dt + \frac{\beta\gamma}{2U^0}\, dx^5 \\ e^{2\phi} &= U^0 \\ \mathbf{Y} &= \gamma\mathbf{\Phi}^0(\mathbf{x}, \eta')dx^5 + \beta\gamma\mathbf{\Phi}^0(\mathbf{x}, \eta')dt + \mathbf{B}^0_i(\mathbf{x}, \eta')dx^i \ . \end{aligned} \tag{23}$$

By comparison with (15) we can now read off all the four-dimensional fields of the dyon solution of the field equations of (17), except that we learn only the combination $e^{2\psi}ds_4^2$ of the scalar ψ and the 4-metric. However, since $\psi = \phi - \frac{1}{2}\rho$ and ϕ is boost invariant we can deduce the new value of ψ from that of ρ, and hence the new 4-metric. The result is

$$
\begin{aligned}
ds_4^2 &= -\frac{1}{\gamma\sqrt{U^0-\beta^2}}dt^2 + \gamma\sqrt{U^0-\beta^2}\,d\mathbf{x}^2 \\
K &= -\frac{1}{2}\beta\frac{(U^0-1)}{(U^0-\beta^2)}\,dt \\
e^{2\rho} &= \gamma^2(U^0-\beta^2) \\
e^{2\psi} &= \frac{U^0}{\gamma\sqrt{(U^0-\beta^2)}} \\
v &= \frac{\beta\gamma}{2U^0} \\
A &= \frac{1}{2\gamma(U^0-\beta^2)}dt
\end{aligned}
\tag{24}
$$

and

$$
\begin{aligned}
\mathbf{\Phi} &= \gamma\mathbf{\Phi}^0(\mathbf{x},\eta') \\
\mathbf{B} &= \beta\gamma\mathbf{\Phi}^0(\mathbf{x},\eta')dt + \mathbf{B}_i^{(0)}(\mathbf{x},\eta')dx^i dx^i \ .
\end{aligned}
\tag{25}
$$

If we now set

$$\eta' = \eta/\gamma \ , \tag{26}$$

then the Higgs field is

$$\mathbf{\Phi} = \gamma\mathbf{\Phi}^0(\mathbf{x},\gamma^{-1}\eta) \ , \tag{27}$$

which has the property that $|\mathbf{\Phi}| \to \eta$ as $|\mathbf{x}| \to \infty$. We have thus arranged for the dyon to have the same asymptotic value for the Higgs field as it originally had for the monopole solution. Observe also that (for all values of η') both ρ and ψ still vanish as $|\mathbf{x}| \to \infty$, as they did in the monopole solution, and that the 4-metric remains asymptotically flat. The v field however is now non-zero at spatial infinity, so the dyon is nevertheless not a solution in the same vacuum as that of the monopole. This feature seems to be the principal difference between the flat space case and its gravitational generalization.

We conclude with some remarks about the moduli space of the antigravitating multi-monopole solutions discussed above. First, because the solutions are constructed entirely in terms of a solution of the *flat space* Bogomol'nyi equations the moduli space of these solutions is *topologically* the same as as for the flat space solutions, i.e. it is diffeomorphic to the space of rational functions of a complex variable of degree k where k is the monopole number. Let us now turn to the *metric* on this moduli space. Because the monopole is a solution of an $N=4$ supergravity theory that breaks half the supersymmetry, the metric should be hyper-Kahler. Moreover, it should be invariant under the action of the Euclidean group. Hyper-Kahler metrics are rather rigid and given the boundary conditions and topology it is difficult to see how the metric can differ from the hyper-Kahler metric of the flat space theory. Consider, for example, the case of two monopoles. The metric on the 'relative' moduli space is four-dimensional and admits an $SO(3)$ action which rotates the complex structures. This fixes it to be the Atiyah-Hitchin metric.

If indeed the metric on the moduli space of k BPS monopoles is the same as in the flat space theory, it is presumably because the gravitational, gravivector and graviscalar interactions cancel against one another. In particular, this cancellation must occur for large monopole separations, where it may easily be checked. The lowest order two-body velocity dependent forces at large separation are given by the Darwin Lagrangian, which contains terms of the form [8]

$$\begin{aligned}
&\frac{\mathbf{v}_1^2+\mathbf{v}_2^2}{r_{12}}\left(3M_1M_2-\Sigma_1\Sigma_2\right)\\
&\frac{\mathbf{v}_1\cdot\mathbf{v}_2}{r_{12}}\left(Q_1Q_2+\Sigma_1\Sigma_2-7M_1M_2\right)\\
&\frac{(\mathbf{v}_1\cdot\hat{\mathbf{r}})(\mathbf{v}_2\cdot\hat{\mathbf{r}})}{r_{12}}\left(Q_1Q_2-\Sigma_1\Sigma_2-M_1M_2\right)
\end{aligned}\tag{28}$$

where Σ_i are the scalar charges, Q_i are the vector charges and M_i are the masses of the monopoles. All three terms vanish if and only $a^2=3$ [9] which, as we have shown earlier, is the value required to interpret the four-dimensional BPS monopoles as solutions of string theory. It is interesting to note that the moduli space of extreme electrically charged dilaton black holes is not only asymptotically flat (i.e. to order $\frac{1}{r^2}$) when $a=\sqrt{3}$ but everywhere flat [10]. The same is true of extreme magnetically charged $a=\sqrt{3}$ dilaton black holes [9] which can be viewed four-dimensional projections of Kaluza-Klein monopoles [11,12]. In other words, the phenomenon of *enhanced* anti-gravity, i.e. the cancellation of gravitationally induced forces to first non-trivial order in velocities, seems to be a general feature $a = \sqrt{3}$

If the metric on the moduli space is, as we suggest, unchanged by graviton, gravi-photon and gravi-scalar exchange forces then the result of Sen [13] concerning a unique L^2 harmonic form on the relative moduli space remains true in our case. Tensoring with the sixteen-plet of forms on the $S^1 \times \mathbb{R}^3$ factor (due to the centre of mass motion and the total electric charge) will give a short Bogolmol'nyi 16-fold supermultiplet of bound monopole-dyon pairs.

Eigenfunctions of the Hodge-De-Rham Laplacian on the relative moduli space with *non-vanishing* eigenvalues yield long, non-Bogolmolnyi, 256-fold supermultiplets of bound monople-dyon pairs. This follows from the fact that the non-vanishing eigenvalues come in multiples of sixteen, and hence give 256-plets on tensoring with the centre of mass 16-plet. To see that the multiplicity of non-zero eigenvalues is a multiple of sixteen it suffices to note, following Hawking and Pope [14], that the moduli space admits two covariantly constant chiral spinors as a consequence of the fact that its holonomy lies in $Sp(1) \equiv SU(2)$. Using these spinors, Hawking and Pope show that the non-zero spectrum of the the Hodge-De-Rham Laplacian on p-forms, $p = 1, 2, 3, 4$ is given entirely in terms of the spectrum of the ordinary Laplacian on zero-forms, i.e. scalar functions. For each scalar eigenfunction they showed that there are four eigen one-forms, six eigen two-forms, four eigen three-forms and one eigen four-form, all with the same eigenvalue. This implies a multiplicity of sixteen for all but the zero-mode spectrum of the Hodge-De-Rham operator on the relative moduli space. Of course, this argument does not establish the existence of L^2 eigenforms with non-zero eigenvalues, but merely that if they do exist then they must do so in multiples of sixteen. However, the results of Gibbons and Manton [15] indicated strongly that L^2 scalar eigenfunctions exist on the relative moduli space and this suggestion has been confirmed by detailed calculations of Shroers and Manton [16].

Appendix: Five-dimensional Einstein-Maxwell Theory

In this appendix we shall justify our claim that the coefficient of the 'topological' interaction term in (14) arises from the requirements of supersymmetry.

The coupling of five-dimensional supergravity to n vector multiplets has been described in detail in [6]. The bosonic field content comprises the metric, g_{AB}, $(n+1)$ vector fields A^I_A, $I = 1, \ldots, n+1$ and n scalar fields ϕ^i, $i = 1, \ldots, n$. If the gauge group is abelian then the bosonic Lagrangian is

$$R - g_{ij}(\phi)\partial\phi^i\partial\phi^j - \frac{1}{2}m_{IJ}(\phi)F^I F^J + \frac{1}{3\sqrt{6}}\epsilon^{ABCDE}A^I_A\ F^J_{BC}\ F^J_{DE}\ C_{IJK}\ , \quad (A.1)$$

where $g_{ij}(\phi)$ is the metric on the scalar field target space, $m_{IJ}(\phi)$ is a positive definite matrix-valued function of ϕ^i, and C_{IJK} are *constants*. These constants determine a symmetric homogeneous polynomial of degree three:

$$\mathcal{N}(\xi) = \beta^3 C_{IJK}\xi^I\xi^J\xi^K \quad (A.2)$$

with $\beta = \sqrt{\frac{2}{3}}$, where ξ^I are the components of a vector in an $n+1$ dimensional vector space J. The scalar field target space is the $\mathcal{N} = 1$ hypersurface in J. All couplings of the theory are determined by $\mathcal{N}$. For example

$$m_{IJ} = -\frac{1}{2}\partial_I\partial_J \ln\mathcal{N}\big|_{\mathcal{N}=1}\ . \quad (A.3)$$

The target space metric g_{ij} is given by

$$g_{ij} = \frac{1}{\beta^2}m_{IJ}h^I{}_{,i}\,h^J{}_{,j}\,\big|_{\mathcal{N}=1} \quad (A.4)$$

where

$$h^I = \frac{1}{3\beta}m^{IJ}\partial_J\mathcal{N}\big|_{\mathcal{N}=1}\ . \quad (A.5)$$

The pure five-dimensional supergravity corresponds to the case $n = 0$. Let $C_{111} = \gamma^3$. Then $\mathcal{N} = (\gamma\beta\xi)^3$ and thus the hypersurface $\mathcal{N} = 1$ is given by

$\xi = (\beta\gamma)^{-1}$. There are no scalars and just one component of m_{IJ}, $m_{11} = \gamma^2$. The resulting Lagrangian is

$$R - \frac{1}{2}\gamma^2 F^2 + \frac{\gamma^3}{3\sqrt{6}}\epsilon^{ABCDE} A_A F_{BC} F_{DE} \ . \qquad (A.6)$$

We recover the Lagrangian used in [2] by setting $\gamma = \sqrt{2}$.

In the case $n = 1$ we set $C_{122} = \frac{\gamma^3}{3}$. Thus

$$\mathcal{N} = \beta\gamma^3 \xi^1 (\xi^2)^2 \ . \qquad (A.7)$$

The hypersurface $\mathcal{N} = 1$ can be parametrised by the scalar ϕ by setting

$$\xi^1 = (\beta\gamma)^{-1}\alpha^2 e^{-\frac{4}{3}\phi} \qquad (A.8)$$

and

$$\xi^2 = (\beta\gamma)^{-1}\alpha^{-1} e^{\frac{2}{3}\phi} \ , \qquad (A.9)$$

where α is a constant. We find that the matrix m_{IJ} is given by

$$m = \frac{(\beta\gamma)^2}{2\alpha^4}\begin{pmatrix} e^{\frac{8}{3}\phi} & 0 \\ 0 & 2\alpha^6 e^{-\frac{4}{3}\phi} \end{pmatrix} \ , \qquad (A.10)$$

and therefore

$$h^I = \gamma^{-1}\alpha^2 \left(e^{-2\phi}, \alpha^{-3} e^{\frac{2}{3}\phi}\right) \ , \qquad (A.11)$$

whence $g_{ij}\partial\phi^i\partial\phi^j = \frac{4}{3}(\partial\phi)^2$. We may choose $\alpha^6 = 1$ in order to arrange that $m_{IJ} = \delta_{IJ}$ at the origin $\phi = 0$, and we then choose γ such that $(\beta\gamma)^3 = -4$. The resulting lagrangian is

$$\begin{aligned} R - &\frac{4}{3}(\partial\phi)^2 - e^{\frac{8}{3}\phi}\ F^{(1)}_{AE}\ F^{(1)AE} - e^{-\frac{4}{3}\phi}\ F^{(2)}_{AE}\ F^{(2)AE} \\ &- \epsilon^{ABCDE} A^{(1)}_E\ F^{(2)}_{AB}\ F^{(2)}_{CD} \ . \end{aligned} \qquad (A.12)$$

which is the five-dimensional string related action (14) for the special case of a $U(1)$ Yang Mills gauge group. As a further check we note that (A.6) is recovered by the consistent truncation $\phi = 0$, $A^{(2)} = \sqrt{2}A^{(1)} = \sqrt{\frac{2}{3}}A$.

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