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# Parton Gas Model for the Nucleon Structure Functions

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## Abstract

A phenomenological model for the nucleon structure functions is presented. Visualising the nucleon as a cavity filled with parton gas in thermal equilibrium and parametrizing the effects due to the finiteness of the nucleon volume, we obtain a good fit to the data on the unpolarized nucleon structure functions.

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Recent experiments have revealed some remarkable features of the nucleon structure functions $F_2^{p,n}$. Data on deep inelastic scattering of muons off proton and deuteron targets [1] show that the quark sea in the nucleon is not flavor-symmetric, $\bar{u}(x) \neq \bar{d}(x)$; the Gottfried sum [2] $S_G \equiv \int (F_2^p - F_2^n)\,(dx/x)$, at $Q^2 = 4$ GeV$^2$, has the value $0.235 \pm 0.026$ compared to the usual quark model prediction of 1/3. This result has been confirmed by the observed asymmetry in Drell-Yan production of dileptons in $pp$ and $pn$ collisions [3]. Most notably, the HERA electron-proton scattering data [4] reveal a rapid rise of the proton structure function $F_2^p(x)$ as $x$ decreases. Indeed over a wide range of small $x$, data from the various groups [4,5], for fixed $Q^2$, are all well described by a single inverse power of $x$. Figure 1 is a log-log plot of the data on $F_2^p(x)/x$ (the combination that enters $S_G$) versus $x$. We see that, for fixed $Q^2$, the data fall on straight lines defined by

$$\frac{F_2^p(x)}{x} = \frac{c}{x^m}\ , \ (0.0004 \lesssim x \lesssim 0.2). \qquad (1)$$

For instance, at $Q^2 = 15$ GeV$^2$, the best-fit parameters are $c = 0.229 \pm 0.005$ and $m = 1.22 \pm 0.01$ [6].

Global fits to the nucleon structure data involve parametrizing the various parton densities at some low $Q^2$ and evolving them to higher values of $Q^2$ relevant to observations. The fits so obtained [7] have very high precision but contain several (typically ~15-20) arbitrary parameters and provide little physical insight into the structure of the nucleon. On the other hand, phenomenological models could give us some valuable clues into the physics of parton distributions in the nucleon. From this point of view the statistical models of the nucleon structure functions [8] have been quite interesting due to their intuitive appeal and simplicity.

We present here a phenomenological model for the unpolarized nucleon structure functions by regarding the nucleonic contents as constituting a gas of noninteracting partons in thermal equilibrium. An attractive feature of this general framework is the natural explanation of the violation of the Gottfried sum rule: the excess of $u$-quarks over $d$-quarks in the proton implies unequal chemical potentials, hence unequal $u\bar{u}$ and $d\bar{d}$ seas, which leads to $S_G \neq 1/3$. However, the ensuing structure function $F_2(x)$ vanishes like $x^2$ as $x \to 0$, in violent conflict with the data. To remedy this we invoke corrections arising from the finiteness of the nucleon volume, by multiplying the parton density of states by a factor having inverse powers of the radial dimension, $[1 + \mathcal{O}(1/R^\delta)]$. We find that the small-$x$ rise and other features of the nucleon structure functions are reproduced quite well.

*The Model*

We picture the nucleon (mass $M$) to consist of a gas of massless partons (quarks, antiquarks and gluons) in thermal equilibrium at temperature $T$ in a spherical volume $V$ with radius $R$. We consider two frames, the proton rest frame and the infinite-momentum frame (IMF) moving with velocity $-v(\simeq -1)$ along the common $z$ axis. Our interest lies in the limit when the Lorentz factor $\gamma \equiv (1-v^2)^{-1/2} \to \infty$. The invariant parton number density in phase space [9] is given by (quantities in the IMF are denoted by the index $i$)

$$\frac{dn^i}{d^3p^i\ d^3r^i} = \frac{dn}{d^3p\ d^3r} = \frac{g}{(2\pi)^3}\left[\frac{1}{\exp[\beta(E-\mu)] \pm 1}\right] \equiv f(E), \tag{2}$$

where $\beta \equiv T^{-1}$, $g$ is the degeneracy ($g = 16$ for gluons and $g = 6$ for $q$ or $\bar{q}$ of a given flavor), $(E, p)$ is the parton four-momentum in the proton rest frame and $f(E)$ is the usual distribution for noninteracting fermions or bosons. In terms of the Bjorken scaling variable $x = p_z^i/(Mv\gamma)$, the phase space element can be expressed as

$$\begin{aligned} d^3p^i\ d^3r^i &= 2\pi p_T^i dp_T^i (Mv\gamma dx)\frac{d^3r}{\gamma} \\ &= 2\pi[Mxv^3 + \frac{Ev}{\gamma^2}]dE\ Mdx\ d^3r, \end{aligned}$$

where for $\gamma \to \infty$ the expression in square brackets becomes $Mx$. For fixed $x$ the parton energy $E$ varies between the kinematic limits $Mx/2 < E < M/2$, where the lower limit is attained when $p_T^i = 0$. Consequently the parton number distribution $dn^i/dx$ in the IMF is simply proportional to an integral of the *rest-frame* distribution $f(E)$:

$$dn^i/dx = 2\pi V M^2 x \int_{xM/2}^{M/2} dE\ \ f(E), \tag{3}$$

where the factor $V$ results from $d^3r$ integration. The structure function $F_2(x)$ is given by

$$F_2(x) = x \sum_q e_q^2 \left[ \frac{dn_q^i}{dx} + \frac{dn_{\bar{q}}^i}{dx} \right].$$

The number distribution vanishes linearly as $x \to 0$ (and also as $x \to 1$) and leads to the behavior of the structure function $F_2(x) \sim x^2$ at small $x$, which disagrees with the observations noted in Eq. (1).

In order to obtain the rise of $F_2^p(x)$ at small $x$, we shall modify the model to reflect effects arising from the finiteness of the nucleon volume $V$. Various studies of finite-size corrections (FSC) show that they are sensitive to the precise shape and size of the enclosure, the type of boundary conditions imposed on the wave function, and to the details such as whether the particles are strictly massless, whether chemical potentials are nonzero, etc. [10]. Moreover, these studies invariably involve some simplifying assumptions and thus their use is difficult to justify in the present context.

In keeping with the phenomenological nature of the model, we have chosen to parametrize the correction due to the finiteness of the nucleon volume. This is implemented through the use of the dimensionless combination $1/(ER)$. We have chosen two alternative forms of parametrization, a form prompted by the empirical observation in Eq. (1):

$$\Phi_1 = 1 + \frac{B}{(ER)^\delta} \ , \tag{4}$$

and a general power series expansion in the variable $1/(ER)$:

$$\Phi_2 = 1 + \frac{a}{ER} + \frac{b}{(ER)^2} + ... \ , \tag{5}$$

where $B$, $\delta(> 0)$, $a$, $b$, ... are arbitrary constants. We multiply the integrand in Eq. (3) by the function $\Phi$ $(= \Phi_1 \ or \ \Phi_2)$ in order to incorporate the finite-volume effects in our model.

The model described above is assumed to hold at a certain input momentum scale $Q_0^2$, and if necessary can be evolved to higher $Q^2$ by means of the standard techniques in quantum chromodynamics (QCD). To complete the statement of the model, we demand the thermal parton distributions to obey the following *three* constraints at the input scale. The constraints on the net quark numbers in the proton are $n_u - n_{\bar{u}} = 2$ and $n_d - n_{\bar{d}} = 1$, i.e.,

$$\frac{VM^2}{(2\pi)^2}\int_0^1 dx\ x\ \int_{xM/2}^{M/2} dE \quad \left\{\frac{6}{\exp\left[\beta(E-\mu_\alpha)\right]+1}-\frac{6}{\exp\left[\beta(E+\mu_\alpha)\right]+1}\right\}\Phi(ER)$$
$$= n_\alpha - n_{\bar{\alpha}}. \qquad (\alpha = u, d) \tag{6}$$

Obviously, chemical potentials for heavy flavors are necessarily zero. As regards the third constraint, we assume that the longitudinal momentum fractions in the $u$, $d$ flavors and the gluons add up to unity:

$$\frac{VM^2}{(2\pi)^2}\int_0^1 dx\ x^2\int_{xM/2}^{M/2} dE \ \left\{\frac{6}{\exp\left[\beta(E-\mu_u)\right]+1}+\frac{6}{\exp\left[\beta(E+\mu_u)\right]+1}+\right.$$
$$\left.\frac{6}{\exp\left[\beta(E-\mu_d)\right]+1}+\frac{6}{\exp\left[\beta(E+\mu_d)\right]+1}+\frac{16}{\exp(\beta E)-1}\right\}\Phi(ER) \ = 1\ . \tag{7}$$

The quark flavors $s$, $c$, ... which are not introduced in Eq. (7) show up at higher $Q^2$ as a result of QCD evolution.

By interchanging the order of $x$ and $E$ integrations in Eqs. (6-7) and performing the $x$-integration analytically, we see that in order to keep the integrals finite, large powers of $1/E$ are not allowed in the integrand. This requires that while using $\Phi_1$ the exponent should be bounded, $\delta < 3$, and while using $\Phi_2$ only the first three terms can be present. Thus the model effectively has only two free parameters.

To determine $\mu_u$, $\mu_d$ and $T$, we solved the three coupled nonlinear equations (6-7) by the Davidenko-Broyden method [11]. The resulting values of $\mu_u$, $\mu_d$ and $T$ are such that the left and right hand sides of these equations agree with each other to typically one part in $10^6$. The parton densities were evolved by means of the Gribov-Lipatov-Altarelli-Parisi equations [12] in leading order (LO), taking the input scale $Q_0^2 = M^2$ and $\Lambda_{QCD} = 0.3$ GeV. Finally, the root-mean-square (rms) radius of the parton distribution was taken to be the same as the charge rms radius ($\rho$) of the proton; since $\rho \simeq 0.862$ fm [13], this yields $R = \sqrt{5/3}\ \rho = 1.11$ fm.

*Results and Discussion*

Since the two arbitrary constants $B$ and $\delta$, or $a$ and $b$ in Eq. (4) or (5) are not known, we have determined them by fitting the deep inelastic scattering data on $F_2^p(x)$ at $Q^2 = 15$ GeV$^2$ [4,5]. The results of our fit incorporating the finite-size corrections and QCD evolution are shown by the solid curve in Fig. 2. (Results presented here are based on $\Phi_2$; the alternative form $\Phi_1$ gives an equally good fit.) Also shown for comparison in Fig. 2 are: (a) the (dot-dashed) curve labeled 'GAS' giving the prediction of the parton gas model which has no free parameters by virtue of the constraints, (b) the (dashed) curve labeled 'QCD' showing the effect of QCD evolution on the gas model, and (c) the (dotted) curve labeled 'FSC' showing a *fit* to the data when only the finite-size corrections are introduced in the gas model. If $\Phi_1$ is used in order to incorporate FSC, the fitted values of the two parameters are

$$B = 0.269 \quad and \quad \delta = 2.14,$$

and the corresponding temperature and chemical potentials are $T = 63$ MeV, $\mu_u = 124$ MeV and $\mu_d = 64$ MeV. If, on the other hand, $\Phi_2$ is used, the fitted values of the two parameters are [14]

$$a = -1.88 \quad and \quad b = 2.24,$$

and the corresponding temperature and chemical potentials are $T = 72$ MeV, $\mu_u = 162$ MeV and $\mu_d = 81$ MeV.

To comment on the relative importance of the inputs, we focus on the curves in Fig. 2 at, say, $x = 10^{-3}$: a fit with FSC gives a very small value of $F_2^p \sim 0.02$, reflecting the restrictive nature of the constraints. Leading-order QCD evolution does result in a value of $F_2$ which is significantly large but not large enough, $F_2^p \sim 0.23$. However, when the effects due to both FSC and QCD are included in the model, we obtain $F_2^p \sim 1.1$, which is consistent with the data. The presence of inverse powers of $(ER)$ in $\Phi$ is thus partially responsible for increase in $F_2^p$ at small $x$.

As a test of the model, we show in Fig. 3 the prediction (solid curve) for the difference $[F_2^p(x) - F_2^n(x)]$. Also shown for comparison is the result (dashed curve) based on the parametrization of Glück *et al.* [7]. The agreement with the NMC data is reasonable.

As for the Gottfried sum $S_G$, we have

$$S_G = \frac{1}{3} - \frac{2}{3}\int_0^1 (\bar{d} - \bar{u})dx$$
$$= \frac{1}{3} - \frac{2}{3}\frac{VM^2}{(2\pi)^2}\int_0^1 dx\ x \int_{xM/2}^{M/2} dE \left\{\frac{6}{\exp\left[\beta(E+\mu_d)\right]+1} - \frac{6}{\exp\left[\beta(E+\mu_u)\right]+1}\right\}\Phi(ER). \tag{8}$$

The inequality $S_G < \frac{1}{3}$ is thus a result of having in the proton, more valence $u$ quarks than valence $d$ quarks, $(n_u - n_{\bar{u}}) > (n_d - n_{\bar{d}})$, implying that $\mu_u > \mu_d$ and hence the integral in Eq. (8) is positive. Our model predicts at $Q^2 = 4$ GeV$^2$, the value $S_G = 0.22$ which is consistent with the experimental value $S_G = 0.235 \pm 0.026$.

The rapidity dependence of the $W$ charge asymmetry in the reactions $\bar{p}p \rightarrow W^{\pm}\ +\ ...$ is now known to a very high precision [15]. It is a sensitive function of the quark flavor ratio $d(x)/u(x)$ in the proton, in the range $0.007 < x < 0.24$ at $Q^2 = M_W^2$. The ratio $\bar{u}(x)/\bar{d}(x)$ at $< x >= 0.18$ has been deduced to be about 0.51 by the NA51 collaboration [3]. These and other predictions of the model, on the ratio $(F_2^n(x)/F_2^p(x))$, the quark and antiquark distributions $q(x)$, $\bar{q}(x)$, $q_v(x) = q(x) - \bar{q}(x)$ for various flavors, the gluon distribution $g(x)$, the longitudinal momentum fraction carried by the charged partons, etc. will be given elsewhere [16].

Now we briefly describe the salient features of some of the recent calculations of the nucleon structure functions, which use ideas from statistical mechanics. Mac and Ugaz [8a] calculated first-order QCD corrections to the statistical distributions and obtained a crude but reasonable agreement with $F_2^p(x)$ data for $x \gtrsim 0.2$. The momentum constraint was not imposed and the fitted value of the proton radius ($R$) was 2.6 fm. Cleymans *et al.* [8b] used the framework of the finite temperature quantum field theory. They considered $\mathcal{O}(\alpha_s)$ corrections to the statistical distributions and obtained a good fit to the $F_2^p(x)$ data for $x \geq 0.25$. They also calculated the ratio $\sigma_L/\sigma_T$ in this region; it was a factor of 6 above the experimental value. Bourrely *et al.* [8c] considered polarized as well as unpolarized structure functions and presented a statistical parametrization (with eight parameters) of parton distributions in the IMF. Their framework allowed chemical potential for quarks as well as for gluons. The number constraints were not satisfied very accurately. QCD effects were not considered. $x\bar{q}(x)$ vanished as $x \to 0$ and so it was not possible to reproduce the fast increase of the antiquark distributions for $x < 0.1$. Bourrely and Soffer's [8d] approach was similar to that in [8c]. By incorporating QCD evolution of parton distributions and allowing the antiquark chemical potential to depend on $x$, they were able to reproduce the HERA data on $F_2^p$.

In conclusion, it is noteworthy that the application of ideas of statistical mechanics to the point constituents of the nucleon can provide a simple description of all the observed features of the (unpolarized) nucleon structure functions down to the lowest $x$ values so far explored. The model presented here has two free parameters which arise from our treatment of the finite-size corrections.


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## FIGURES

FIG. 1. Log-log plot of the proton structure function data. Experimental data are from Refs. [4,5]; the error bars show statistical and systematic errors combined in quadrature. The straight lines are our fits described in Eq. (1), and are labeled by $Q^2 = 15$, 35, and 120 GeV$^2$. Numbers have been scaled by the factors shown in parentheses for convenience in plotting.

FIG. 2. Proton structure function $F_2^p(x)$ at $Q^2 = 15$ GeV$^2$. Data points are as in Fig. 1. Solid curve is our best fit to the data. Also shown for comparison are: the (dot-dashed) curve labeled 'GAS' giving the gas model prediction, the (dashed) curve labeled 'QCD' showing the QCD-evolved gas model, and the (dotted) curve labeled 'FSC' which is a fit to the data when finite-size corrections are included in the gas model (without QCD).

FIG. 3. Difference $(F_2^p - F_2^n)$ versus $x$, at $Q^2 = 4$ GeV$^2$. Experimental data are from Ref. [1]; errors are statistical only. Solid curve is the prediction of our model. Dashed curve is based on the parametrization of Glück *et al.* [7].