ELEVEN DIMENSIONAL ORIGIN OF STRING/STRING DUALITY: A ONE LOOP TEST[1]

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ABSTRACT

Membrane/fivebrane duality in $D = 11$ implies Type IIA string/Type IIA fivebrane duality in $D = 10$, which in turn implies Type IIA string/heterotic string duality in $D = 6$. To test the conjecture, we reproduce the corrections to the 3-form field equations of the $D = 10$ Type IIA string (a mixture of tree-level and one-loop effects) starting from the Chern-Simons corrections to the 7-form Bianchi identities of the $D = 11$ fivebrane (a purely tree-level effect). $K3$ compactification of the latter then yields the familiar gauge and Lorentz Chern-Simons corrections to 3-form Bianchi identities of the heterotic string. We note that the absence of a dilaton in the $D = 11$ theory allows us to fix both the gravitational constant and the fivebrane tension in terms of the membrane tension. We also comment on an apparent conflict between fundamental and solitonic heterotic strings and on the puzzle of a fivebrane origin of S-duality.

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1 Introduction

With the arrival of the 1984 superstring revolution [1], eleven-dimensional Kaluza Klein supergravity [2] fell out of favor, where it more or less remained until the recent observation by Witten [3] that $D = 11$ supergravity corresponds to the strong coupling limit of the $D = 10$ Type IIA superstring, coupled with the realization that there is a web of interconnections between Type IIA and all the other known superstrings: Type IIB, heterotic $E_8 \times E_8$, heterotic $SO(32)$ and open $SO(32)$. In particular, string/string duality [4–10] implies that the $D = 10$ heterotic string compactified to $D = 6$ on T^4 is dual to the $D = 10$ Type IIA string compactified to $D = 6$ on $K3$ [11]. Moreover, this automatically accounts for the conjectured strong/weak coupling S-duality in $D = 4$, $N = 4$ supersymmetric theories, since S-duality for one string is just target-space T-duality for the other [8]. In this paper we find further evidence for an eleven-dimensional origin of string/string duality and hence for S-duality.

$D = 10$ string/fivebrane duality and $D = 6$ string/string duality can interchange the roles of spacetime and worldsheet loop expansions [4]. For example, tree-level Chern-Simons corrections to the Bianchi identities in one theory may become one-loop Green-Schwarz corrections to the field equations in the other. In a series of papers [4, 7, 12–17], it has been argued that this provides a useful way of putting various duality conjectures to the test. In particular, we can compare quantum spacetime effects in string theory with the σ-model anomalies for the dual p-branes [18–22] even though we do not yet know how to quantize the p-branes! This is the method we shall employ in the present paper. We reproduce the corrections to the 3-form field equations of the $D = 10$ Type IIA string (a mixture of tree-level and one-loop effects) starting from the Chern-Simons corrections to the 7-form $\tilde{K}_7 = *K_4$ Bianchi identities of the $D = 11$ fivebrane (a purely tree-level effect):

$$d\tilde{K}_7 = -\frac{1}{2}K_4{}^2 + (2\pi)^4\tilde{\beta}'\tilde{X}_8 , \qquad (1.1)$$

where the fivebrane tension is given by $\tilde{T}_6 = 1/(2\pi)^3\tilde{\beta}'$ and where the 8-form polynomial $\tilde{X}_8$ describes the $d = 6$ σ-model Lorentz anomaly of the $D = 11$ fivebrane:

$$\tilde{X}_8 = \frac{1}{(2\pi)^4}\Big[-\frac{1}{768}(\mathrm{tr}R^2)^2 + \frac{1}{192}\mathrm{tr}R^4\Big] . \qquad (1.2)$$

$K3$ compactification of (1.1) then yields the familiar gauge and Lorentz Chern-Simons cor-

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rections to 3-form Bianchi identities of the heterotic string:

$$d\tilde{H}_3 = \frac{1}{4}\tilde{\alpha}'(\mathrm{tr}F^2 - \mathrm{tr}R^2) \ . \tag{1.3}$$

The present paper thus provides evidence not only for the importance of eleven dimensions in string theory but also (in contrast to Witten's paper) for the importance of supersymmetric extended objects with $d = p + 1 > 2$ worldvolume dimensions: the super p-branes[2].

2 Ten to eleven: it is not too late

In fact it should have come as no surprise that string theory makes use of eleven dimensions, as there were already tantalizing hints in this direction:

i) In 1986, it was pointed out [25] that $D = 11$ supergravity compactified on $K3 \times T^{n-3}$ [26] and the $D = 10$ heterotic string compactified on T^n [27, 28] have the same moduli spaces of vacua, namely

$$\mathcal{M} = \frac{SO(16 + n, n)}{SO(16 + n) \times SO(n)} \ . \tag{2.1}$$

It was subsequently confirmed [29, 30], in the context of the $D = 10$ Type IIA theory compactified on $K3 \times T^{n-4}$, that this equivalence holds globally as well as locally.

ii) In 1987 the $D = 11$ supermembrane was discovered [31, 32]. It was then pointed out [33] that the $(d = 2, D = 10)$ Green-Schwarz action of the Type IIA superstring follows by simultaneous worldvolume/spacetime dimensional reduction of the $(d = 3, D = 11)$ Green-Schwarz action of the supermembrane.

iii) In 1990, based on considerations of this $D = 11$ supermembrane which treats the dilaton and moduli fields on the same footing, it was conjectured [34, 35] that discrete subgroups of all the old non-compact global symmetries of compactified supergravity [36–39] (*e.g.* $SL(2, R)$, $O(22, 6)$, $O(24, 8)$, E_7, E_8, E_9, E_{10}) should be promoted to duality symmetries of either heterotic or Type II superstrings. The case for a target space $O(22, 6; Z)$ (*T-duality*) had already been made, of course [40]. Stronger evidence for a strong/weak coupling $SL(2, Z)$ (*S-duality*) in string theory was subsequently provided in [5, 9, 41–51]. Stronger evidence for their combination into an $O(24, 8; Z)$ duality in heterotic strings was provided in [10, 50, 52, 53] and stronger evidence for their combination into a discrete E_7 in Type II strings was provided in [11], where it was dubbed *U-duality*.

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[2]Super p-branes are reviewed in [9, 23, 24]

iv) In 1991, the supermembrane was recovered as an elementary solution of $D = 11$ supergravity which preserves half of the spacetime supersymmetry [54]. (*Elementary* solutions are singular and carry a Noether "electric" charge, in contrast to *solitons* which are non-singular solutions of the source-free equations and carry a topological "magnetic" charge.) The preservation of half the supersymmetries is intimately linked with the worldvolume kappa symmetry. It followed by the same simultaneous dimensional reduction in (*ii*) above that the elementary Type IIA string could be recovered as a solution of Type IIA supergravity. By truncation, one then obtains the $N = 1, D = 10$ elementary string [55].

v) In 1991, the elementary superfivebrane was recovered as a solution of the dual formulation of $N = 1, D = 10$ supergravity which preserves half of the spacetime supersymmetry [56]. It was then reinterpreted [57, 58] as a non-singular soliton solution of the usual formulation. Moreover, it was pointed out that it also provides a solution of both the Type IIA and Type IIB field equations preserving half of the spacetime supersymmetry and therefore that there exist both Type IIA and Type IIB superfivebranes. This naturally suggested a Type II string/fivebrane duality in analogy with the earlier heterotic string/fivebrane duality conjecture [23, 59]. Although no Green-Schwarz action for the $d = 6$ worldvolumes is known, consideration of the soliton zero modes means that the gauged fixed actions must be described by a chiral antisymmetric tensor multiplet $(B_{\mu\nu}^-, \lambda^I, \phi^{[IJ]})$ in the case of IIA and a non-chiral vector multiplet $(B_\mu, \chi^I, A^I{}_J, \xi)$ in the case of IIB [57, 58].

vi) Also in 1991, black p-brane solutions of $D = 10$ superstrings were found [60] for $d = 1$ (IIA only), $d = 2$ (Heterotic, IIA and IIB), $d = 3$ (IIA only), $d = 4$ (IIB only) $d = 5$ (IIA only), $d = 6$ (Heterotic, IIA and IIB) and $d = 7$ (IIA only). Moreover, in the extreme mass=charge limit, they each preserve half of the spacetime supersymmetry [61]. Hence there exist all the corresponding super p-branes, giving rise to $D = 10$ particle/sixbrane, membrane/fourbrane and self-dual threebrane duality conjectures in addition to the existing string/fivebrane conjectures. The soliton zero modes are described by the supermultiplets listed in Table (1). Note that in contrast to the fivebranes, *both* Type IIA and Type IIB string worldsheet supermultiplets are non-chiral[3]. As such, they follow from T^4 compactification of the Type IIA fivebrane worldvolume supermultiplets.

vii) In 1992, a fivebrane was discovered as a soliton of $D = 11$ supergravity preserving half the spacetime supersymmetry [62]. Hence there exists a $D = 11$ superfivebrane and

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[3]This corrects an error in [9, 61]

it forms the subject of the present paper. Once again, its covariant action is unknown but consideration of the soliton zero modes means that the gauged fixed action must be described by the same chiral antisymmetric tensor multiplet in (v) above [9, 63, 64]. This naturally suggests a $D = 11$ membrane/fivebrane duality.

$viii$) In 1993, it was recognized [61] that by dualizing a vector into a scalar on the gauge-fixed $d = 3$ worldvolume of the Type IIA supermembrane, one increases the number of worldvolume scalars (*i.e.* transverse dimensions) from 7 to 8 and hence obtains the corresponding worldvolume action of the $D = 11$ supermembrane. Thus the $D = 10$ Type IIA theory contains a hidden $D = 11$ Lorentz invariance!

ix) In 1994 [65] and 1995 [66], all the $D = 10$ Type IIA p-branes of (vi) above were related to either the $D = 11$ supermembrane or the $D = 11$ superfivebrane.

x) Also in 1994, the (extreme electric and magnetic black hole [50, 67]) Bogomol'nyi spectrum necessary for the E_7 U-duality of the $D = 10$ Type IIA string compactified to $D = 4$ on T^6 was given an explanation in terms of the wrapping of either the $D = 11$ membrane or $D = 11$ fivebrane around the extra dimensions [11].

xi) In 1995, it was conjectured [64] that the $D = 10$ Type IIA superstring should be identified with the $D = 11$ supermembrane compactified on S^1, with the charged extreme black holes of the former interpreted as the Kaluza-Klein modes of the latter.

xii) Also in 1995, the conjectured duality of the $D = 10$ heterotic string compactified on T^4 and the $D = 10$ Type IIA string compactified on $K3$ [3, 11], combined with the above conjecture implies that the $d = 2$ worldsheet action of the $D = 6$ ($D = 7$) heterotic string may be obtained by $K3$ compactification[4] of the $d = 6$ worldvolume action of the $D = 10$ Type IIA fivebrane ($D = 11$ fivebrane) [68, 69]. We shall shortly make use of this result.

Following Witten's paper [3] it was furthermore proposed [70] that the combination of perturbative and non-perturbative states of the $D = 10$ Type IIA string could be assembled into $D = 11$ supermultiplets. It has even been claimed [71] that both the $E_8 \times E_8$ and $SO(32)$ heterotic strings in $D = 10$ may be obtained by compactifying the $D = 11$ theory on Ξ_1 and Ξ_2 respectively, where Ξ_1 and Ξ_2 are one-dimensional structures obtained by squashing $K3$!

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[4]The wrapping of the $D = 10$ *heterotic* fivebrane worldvolume around $K3$ to obtain a $D = 6$ *heterotic* string was considered in [7].

$d = 7$	Type IIA	$(A_\mu, \lambda, 3\phi)$		$n = 1$
$d = 6$	Type IIA	$(B^-_{\mu\nu}, \lambda_R^I, \phi^{[IJ]})$	$I = 1, \ldots, 4$	$(n_+, n_-) = (2, 0)$
	Type IIB	$(B_\mu, \chi^I, A^I{}_J, \xi)$	$I = 1, 2$	$(n_+, n_-) = (1, 1)$
	Heterotic	(ψ^a, ϕ^α)	$a = 1, \ldots, 60$	
			$\alpha = 1, \ldots, 120$	$(n_+, n_-) = (1, 0)$
$d = 5$	Type IIA	$(A_\mu, \lambda^I, \phi^{[IJ]|})$	$I = 1, \ldots, 4$	$n = 2$
$d = 4$	Type IIB	$(B_\mu, \chi^I, \phi^{[IJ]})$	$I = 1, \ldots, 4$	$n = 4$
$d = 3$	Type IIA	(χ^I, ϕ^I)	$I = 1, \ldots, 8$	$n = 8$
$d = 2$	Type IIA	$(\chi_L{}^I, \phi_L{}^I), (\chi_R{}^I, \phi_R{}^I)$	$I = 1, \ldots, 8$	$(n_+, n_-) = (8, 8)$
	Type IIB	$(\chi_L{}^I, \phi_L{}^I), (\chi_R{}^I, \phi_R{}^I)$	$I = 1, \ldots, 8$	$(n_+, n_-) = (8, 8)$
	Heterotic	$(0, \phi_L{}^M), (\chi_R{}^I, \phi_R{}^I)$	$M = 1, \ldots, 24$	
			$I = 1, \ldots, 8$	$(n_+, n_-) = (8, 0)$

Table 1: Gauge-fixed $D = 10$ theories on the worldvolume, corresponding to the zero modes of the soliton, are described by the above supermultiplets and worldvolume supersymmetries. The $D = 11$ membrane and fivebrane supermultiplets are the same as Type IIA in $D = 10$.

3 $D = 11$ membrane/fivebrane duality

We begin with the bosonic sector of the $d = 3$ worldvolume of the $D = 11$ supermembrane:

$$
\begin{aligned}
S_3 &= T_3 \int d^3\xi \Big[-\frac{1}{2}\sqrt{-\gamma}\gamma^{ij}\partial_i X^M \partial_j X^N G_{MN}(X) + \frac{1}{2}\sqrt{-\gamma} \\
&\quad - \frac{1}{3!}\epsilon^{ijk}\partial_i X^M \partial_j X^N \partial_k X^P C_{MNP}(X) \Big] ,
\end{aligned}
\tag{3.1}
$$

where T_3 is the membrane tension, ξ^i ($i = 1, 2, 3$) are the worldvolume coordinates, γ^{ij} is the worldvolume metric and $X^M(\xi)$ are the spacetime coordinates ($M = 0, 1, \ldots, 10$). Kappa symmetry [31, 32] then demands that the background metric G_{MN} and background 3-form potential C_{MNP} obey the classical field equations of $D = 11$ supergravity, whose bosonic action is

$$
I_{11} = \frac{1}{2\kappa_{11}{}^2} \int d^{11}x \sqrt{-G}\left[R_G - \frac{1}{2 \cdot 4!}K^2_{MNPQ} \right] - \frac{1}{12\kappa_{11}{}^2} \int C_3 \wedge K_4 \wedge K_4 ,
\tag{3.2}
$$

where $K_4 = dC_3$ is the 4-form field strength. In particular, K_4 obeys the field equation

$$
d * K_4 = -\frac{1}{2}K_4{}^2
\tag{3.3}
$$

and the Bianchi identity

$$
dK_4 = 0 .
\tag{3.4}
$$

While there are two dimensionful parameters, the membrane tension T_3 and the eleven-dimensional gravitational constant κ_{11}, they are in fact not independent. To see this, we note from (3.1) that C_3 has period $2\pi/T_3$ so that K_4 is quantized according to

$$\int K_4 = \frac{2\pi n}{T_3} \qquad n = \text{integer} . \tag{3.5}$$

Consistency of such C_3 periods with the spacetime action, (3.2), gives the relation

$$\frac{(2\pi)^2}{\kappa_{11}{}^2 T_3{}^3} \in 4\mathbb{Z} . \tag{3.6}$$

The $D = 11$ classical field equations admit as a soliton a dual superfivebrane [6,62] whose worldvolume action is unknown, but which couples to the dual field strength $\tilde{K}_7 = *K_4$. The fivebrane tension $\tilde{T}_6$ is given by the Dirac quantization rule [6]

$$2\kappa_{11}{}^2 T_3 \tilde{T}_6 = 2\pi n \qquad n = \text{integer} . \tag{3.7}$$

Using (3.6), this may also be written as

$$\pi \frac{\tilde{T}_6}{T_3{}^2} \in \mathbb{Z} , \tag{3.8}$$

which we will find useful below. Although Dirac quantization rules of the type (3.7) appear for other p-branes and their duals in lower dimensions [6], it is the absence of a dilaton in the $D = 11$ theory that allows us to fix both the gravitational constant and the dual tension in terms of the fundamental tension.

From (3.3), the fivebrane Bianchi identity reads

$$d\tilde{K}_7 = -\frac{1}{2}K_4{}^2 . \tag{3.9}$$

However, such a Bianchi identity will in general require gravitational Chern-Simons corrections arising from a sigma-model anomaly on the fivebrane worldvolume [7, 14, 18–22]:

$$d\tilde{K}_7 = -\frac{1}{2}K_4{}^2 + (2\pi)^4 \tilde{\beta}' \tilde{X}_8 , \tag{3.10}$$

where $\tilde{\beta}'$ is related to the fivebrane tension by $T_6 = 1/(2\pi)^3 \tilde{\beta}'$ and where the 8-form polynomial $\tilde{X}_8$, quartic in the gravitational curvature R, describes the $d = 6$ σ-model Lorentz anomaly of the $D = 11$ fivebrane. Although the covariant fivebrane action is unknown, we know from section 2 that the gauge fixed theory is described by the chiral antisymmetric

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tensor multiplet $(B_{\mu\nu}^-, \lambda^I, \phi^{[IJ]})$, and it is a straightforward matter to read off the anomaly polynomial from the literature. See, for example [72, 73]. The contribution from the anti self-dual tensor is

$$\tilde{X}_B = \frac{1}{(2\pi)^4}\frac{1}{5760}\left[-10(\mathrm{tr}R^2)^2 + 28\,\mathrm{tr}R^4\right] \tag{3.11}$$

and the contribution from the four left-handed (symplectic) Majorana-Weyl fermions is

$$\tilde{X}_\lambda = \frac{1}{(2\pi)^4}\frac{1}{5760}\left[\frac{10}{4}(\mathrm{tr}R^2)^2 + 2\,\mathrm{tr}R^4\right] . \tag{3.12}$$

Hence $\tilde{X}_8$ takes the form quoted in the introduction:

$$\tilde{X}_8 = \frac{1}{(2\pi)^4}\left[-\frac{1}{768}(\mathrm{tr}R^2)^2 + \frac{1}{192}\mathrm{tr}R^4\right] . \tag{3.13}$$

Thus membrane/fivebrane duality predicts a spacetime correction to the $D = 11$ supermembrane action

$$I_{11}(\mathrm{Lorentz}) = T_3 \int C_3 \wedge \frac{1}{(2\pi)^4}\left[-\frac{1}{768}(\mathrm{tr}R^2)^2 + \frac{1}{192}\mathrm{tr}R^4\right] . \tag{3.14}$$

Unfortunately, since the correct quantization of the supermembrane is unknown, this prediction is difficult to check. However, by simultaneous dimensional reduction [33] of ($d = 3, D = 11$) to ($d = 2, D = 10$) on S^1, this prediction translates into a corresponding prediction for the Type IIA string:

$$I_{10}(\mathrm{Lorentz}) = T_2 \int B_2 \wedge \frac{1}{(2\pi)^4}\left[-\frac{1}{768}(\mathrm{tr}R^2)^2 + \frac{1}{192}\mathrm{tr}R^4\right] , \tag{3.15}$$

where B_2 is the string 2-form, T_2 is the string tension, $T_2 = 1/2\pi\alpha'$, related to the membrane tension by

$$T_2 = 2\pi R T_3 , \tag{3.16}$$

where R is the S^1 radius.

As a consistency check we can compare this prediction with previous results found by explicit string one-loop calculations. These have been done in two ways: either by computing directly in $D = 10$ the one-loop amplitude involving four gravitons and one B_2 [74–77], or by compactifying to $D = 2$ on an 8-manifold M and computing the B_2 one-point function [17]. We indeed find agreement. In particular, we note that

$$\tilde{X}_8 = \frac{1}{6}[2Y_8^{NS,R} - Y_8^{R,R}] , \tag{3.17}$$

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where

$$Y_8^{NS,R} = \frac{1}{(2\pi)^4} \frac{1}{2880} \left[-\frac{25}{4} (\mathrm{tr}R^2)^2 + 31 \, \mathrm{tr}R^4 \right]$$

$$Y_8^{R,R} = \frac{1}{(2\pi)^4} \frac{1}{2880} \left[10(\mathrm{tr}R^2)^2 - 28 \, \mathrm{tr}R^4 \right] . \qquad (3.18)$$

Upon compactification to $D = 2$, we arrive at

$$n_{NS,R} = \int_M Y_8^{NS,R}$$

$$n_{R,R} = \int_M Y_8^{R,R} , \qquad (3.19)$$

where in the (NS,R) sector $n_{NS,R}$ computes the index of the Dirac operator coupled to the tangent bundle on M and in the (R,R) sector $n_{R,R}$ computes the index of the Dirac operator coupled to the spin bundle on M. We also find agreement with the well-known tree-level terms

$$\frac{1}{2\kappa_{10}^2} \int \frac{1}{2} B_2 \wedge K_4 \wedge K_4 , \qquad (3.20)$$

where

$$\kappa_{11}^2 = 2\pi R \kappa_{10}^2 . \qquad (3.21)$$

Thus using $D = 11$ membrane/fivebrane duality we have correctly reproduced the corrections to the B_2 field equations of the $D = 10$ Type IIA string (a mixture of tree-level and string one-loop effects) starting from the Chern-Simons corrections to the Bianchi identities of the $D = 11$ superfivebrane (a purely tree-level effect). It is now instructive to derive this same result from $D = 10$ string/fivebrane duality.

4 $D = 10$ Type IIA string/fivebrane duality

To see how a double worldvolume/spacetime compactification of the $D = 11$ supermembrane theory on S^1 leads to the Type IIA string in $D = 10$ [33], let us denote all $(d = 3, D = 11)$ quantities by a hat and all $(d = 2, D = 10)$ quantities without. We then make a ten-one split of the spacetime coordinates

$$\hat{X}^{\hat{M}} = (X^M, Y) \qquad M = 0, 1, \ldots, 9 \qquad (4.1)$$

and a two-one split of the worldvolume coordinates

$$\hat{\xi}^{\hat{i}} = (\xi^i, \rho) \qquad i = 1, 2 \qquad (4.2)$$

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in order to make the partial gauge choice

$$\rho = Y \ , \tag{4.3}$$

which identifies the eleventh dimension of spacetime with the third dimension of the world-volume. The dimensional reduction is then effected by taking Y to be the coordinate on a circle of radius R and discarding all but the zero modes. In practice, this means taking the background fields $\hat{G}_{\hat{M}\hat{N}}$ and $\hat{C}_{\hat{M}\hat{N}\hat{P}}$ to be independent of Y. The string backgrounds of dilaton Φ, string σ-model metric G_{MN}, 1-form A_M, 2-form B_{MN} and 3-form C_{MNP} are given by[5]

$$
\begin{aligned}
\hat{G}_{MN} &= e^{-\Phi/3}\begin{pmatrix} G_{MN} + e^{\Phi}A_M A_N & e^{\Phi}A_M \\ e^{\Phi}A_N & e^{\Phi} \end{pmatrix} \\
\hat{C}_{MNP} &= C_{MNP} \\
\hat{C}_{MNY} &= B_{MN} \ .
\end{aligned}
\tag{4.4}
$$

The actions (3.1) and (3.2) now reduce to

$$
\begin{aligned}
S_2 &= T_2 \int d^2\xi \Big[-\frac{1}{2}\sqrt{-\gamma}\gamma^{ij}\partial_i X^M \partial_j X^N G_{MN}(X) \\
&\quad -\frac{1}{2!}\epsilon^{ij}\partial_i X^M \partial_j X^N B_{MN}(X) + \cdots \Big]
\end{aligned}
\tag{4.5}
$$

and

$$
\begin{aligned}
I_{10} &= \frac{1}{2\kappa_{10}{}^2} \int d^{10}x \sqrt{-G}\, e^{-\Phi} \Big[R_G + (\partial_M \Phi)^2 - \frac{1}{2\cdot 3!}H^2_{MNP} - \frac{1}{2\cdot 2!}e^{\Phi}F^2_{MN} - \frac{1}{2\cdot 4!}e^{\Phi}J^2_{MNPQ} \Big] \\
&\quad -\frac{1}{2\kappa_{10}{}^2}\int \frac{1}{2}K_4 \wedge K_4 \wedge B_2 \ ,
\end{aligned}
\tag{4.6}
$$

where the field strengths are given by $J_4 = K_4 + A_1 H_3$, $H_3 = dB_2$ and $F_2 = dA_1$. Let us now furthermore consider a simple spacetime compactification of the fivebrane theory on the same S^1 to obtain the Type IIA fivebrane in $D = 10$. From (3.4) and (3.10), the field equations and Bianchi identities for the field strengths J_4, H_3, F_2 and their duals $\tilde{J}_6 = *J_4$,

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[5]The choice of dilaton prefactor, $e^{-\Phi/3}$, is dictated by the requirement that G_{MN} be the $D = 10$ string σ-model metric. To obtain the $D = 10$ fivebrane σ-model metric, the prefactor is unity because the reduction is then spacetime only and not simultaneous worldvolume/spacetime. This explains the remarkable "coincidence" [6] between $\hat{G}_{MN}$ and the fivebrane σ-model metric.

$\tilde{H}_7 = e^{-\Phi} * H_3$, $\tilde{F}_8 = *F_2$ now read

$$dJ_4 \quad = \quad F_2 H_3 \qquad d\tilde{J}_6 = H_3 J_4 \tag{4.7}$$

$$dH_3 \quad = \quad 0 \qquad d\tilde{H}_7 = -\frac{1}{2}J_4{}^2 + F_2\tilde{J}_6 + (2\pi)^4\tilde{\beta}'\tilde{X}_8 \tag{4.8}$$

$$dF_2 \quad = \quad 0 \qquad d\tilde{F}_8 = -H_3\tilde{J}_6 \ . \tag{4.9}$$

Of course, the Lorentz corrections to the Bianchi identity for $\tilde{H}_7$ could have been derived directly from the Type IIA fivebrane in $D = 10$ since its worldvolume is described by the same antisymmetric tensor supermultiplet. Note that of all the Type IIA p-branes in Table (1), only the fivebrane supermultiplet is chiral, so only the $\tilde{H}_7$ Bianchi identity acquires corrections.

From (3.7), (3.16) and (3.21), or from first principles of string/fivebrane duality [78], the Dirac quantization rule for $n = 1$ is now

$$2\kappa_{10}{}^2 = (2\pi)^5\alpha'\tilde{\beta}' \ . \tag{4.10}$$

So from either $D = 10$ string/fivebrane duality or from compactification of $D = 11$ membrane/fivebrane duality, the B_2 field equation with its string one-loop correction is

$$d(e^{-\Phi} * H_3) = -\frac{1}{2}J_4{}^2 + F_2 * J_4 + \frac{2\kappa_{10}{}^2}{2\pi\alpha'}\tilde{X}_8 \ , \tag{4.11}$$

which once again agrees with explicit string one-loop calculations [17, 74].

5 $D = 7$ string/membrane duality

Simultaneous worldvolume/spacetime compactification of the $D = 11$ fivebrane on $K3$ gives a heterotic string in $D = 7$ [68,69]. The five worldvolume scalars produce $(5_L, 5_R)$ worldsheet scalars, the four worldvolume fermions produce $(0_L, 8_R)$ worldsheet fermions and the worldvolume self-dual 3-form produces $(19_L, 3_R)$ worldsheet scalars, which together constitute the field content of the heterotic string. We may thus derive the Bianchi identity for this string starting from the fivebrane Bianchi identity, (1.1):

$$d\tilde{K}_7 = -\frac{1}{2}K_4{}^2 + (2\pi)^4\tilde{\beta}'\tilde{X}_8 \ . \tag{5.1}$$

We begin by performing a seven-four split of the eleven-dimensional coordinates

$$X^M = (x^\mu, y^i) \qquad \mu = 0, 1, \ldots, 6 \ ; \ i = 7, 8, 9, 10 \tag{5.2}$$

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so that the original set of ten-dimensional fields $\{\mathcal{A}_n\}$ may be decomposed in a basis of harmonic p-forms on $K3$:

$$\mathcal{A}_n(X) = \sum \mathcal{A}_{n-p}(x)\omega_p(y) \ . \tag{5.3}$$

In particular, we expand C_3 as

$$C_3(X) = C_3(x) + \frac{1}{2T_3}\sum C_1^I(x)\omega_2^I(y) \ , \tag{5.4}$$

where ω_2^I, $I = 1,\ldots,22$ are an integral basis of b_2 harmonic two-forms on $K3$. We have chosen a normalization where the seven-dimensional $U(1)$ field strengths $K_2^I = dC_1^I$ are coupled to even charges

$$\int K_2^I \in 4\pi\,\mathbb{Z} \ , \tag{5.5}$$

which follows from the eleven-dimensional quantization condition, (3.5).

Following [7], let us define the dual (heterotic) string tension $\tilde{T}_2 = 1/2\pi\tilde{\alpha}'$ by

$$\frac{1}{2\pi\tilde{\alpha}'} = \frac{1}{(2\pi)^3\tilde{\beta}'}V \ , \tag{5.6}$$

where V is the volume of $K3$, and the dual string 3-form $\tilde{H}_3$ by

$$\frac{1}{2\pi\tilde{\alpha}'}\tilde{H}_3 = \frac{1}{(2\pi)^3\tilde{\beta}'}\int_{K3}\tilde{K}_7 \ , \tag{5.7}$$

so that $\tilde{H}_3$ satisfies the conventional quantization condition

$$\int \tilde{H}_3 = 4\pi^2 n\tilde{\alpha}' \ , \tag{5.8}$$

which follows from the underlying $\tilde{K}_7$ quantization. The dual string Lorentz anomaly polynomial, $\tilde{X}_4$, is given by

$$\begin{aligned}
\tilde{X}_4 = \int_{K3}\tilde{X}_8 &= \frac{1}{(2\pi)^4}\int_{K3}\left[-\frac{1}{768}(\mathrm{tr}R^2 + \mathrm{tr}R_0^2)^2 + \frac{1}{192}(\mathrm{tr}R^4 + \mathrm{tr}R_0^4)\right] \\
&= \frac{1}{(2\pi)^2}\frac{1}{192}\mathrm{tr}R^2 p_1(K3) \\
&= -\frac{1}{(2\pi)^2}\frac{1}{4}\mathrm{tr}R^2 \ , \tag{5.9}
\end{aligned}$$

where $p_1(K3)$ is the Pontryagin number of $K3$

$$p_1(K3) = -\frac{1}{8\pi^2}\int_{K3}\mathrm{tr}R_0^2 = -48 \ . \tag{5.10}$$

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We may now integrate (5.1) over $K3$, using the Dirac quantization rule, (3.8), to find

$$d\tilde{H}_3 = -\frac{\tilde{\alpha}'}{4}\left[K_2^I K_2^J d_{IJ} + \text{tr} R^2\right] , \tag{5.11}$$

where d_{IJ} is the intersection matrix on $K3$, given by

$$d_{IJ} = \int_{K3} \omega_2^I \wedge \omega_2^J \tag{5.12}$$

and has $b_2^+ = 3$ positive and $b_2^- = 19$ negative eigenvalues. Therefore we see that this form of the Bianchi identity corresponds to a $D = 7$ toroidal compactification of a heterotic string at a generic point on the Narain lattice [27,28]. Thus we have reproduced *exactly* the $D = 7$ Bianchi identity of the heterotic string, starting from a $D = 11$ fivebrane!

6 $D = 6$ string/string duality

Further compactification of (5.11) on S^1 clearly yields the six-dimensional Bianchi identity with two additional $U(1)$ fields coming from S^1, giving $\text{tr} F^2$ with signature $(4, 20)$. Alternatively, this may be obtained from $K3$ compactification of the $D = 10$ fivebrane, with Bianchi identity

$$d\tilde{H}_7 = -\frac{1}{2}J_4{}^2 + F_2\tilde{J}_6 + (2\pi)^4\tilde{\beta}'\tilde{X}_8 . \tag{6.1}$$

Although in this section we focus just on this identity, we present the compactification of the complete bosonic $D = 10$ Type IIA action, (4.6), in the Appendix.

The reduction from ten dimensions is similar to that from eleven. There is one subtlety, however, which is that J_4 is the $D = 10$ gauge invariant combination, $J_4 = K_4 + A_1 H_3$. Compactifying (6.1) to six dimensions on $K3$, we may identify 22 $U(1)$ fields coming from the reduction of J_4 and one each coming from F_2 and $\tilde{J}_6$. Normalizing these 24 six-dimensional $U(1)$ fields according to (5.5), we obtain

$$d\tilde{H}_3 = -\frac{\tilde{\alpha}'}{4}\left[J_2^I J_2^J d_{IJ} - 2F_2\tilde{J}_2 - 16\pi^2\tilde{X}_4\right] , \tag{6.2}$$

where $J_2^I = dC_2^I + A_1 db^I$ and $J_4 = dC_3 + A_1 H_3$. The 22 scalars b^I are torsion moduli of $K3$. While we may be tempted to identify these two-forms with $U(1)$ field strengths, this would not be correct since $dJ_2^I = F_2 db^I \neq 0$ and $d\tilde{J}_2 = J_2^I db^J d_{IJ} \neq 0$. Thus the actual field strengths must be shifted according to

$$\hat{K}_2^I = J_2^I - F_2 b^I$$
$$\hat{J}_2 = \tilde{J}_2 - J_2^I b^J d_{IJ} + \tfrac{1}{2}F_2 b^I b^J d_{IJ} , \tag{6.3}$$

so that $d\hat{K}_2^I = d\hat{J}_2 = 0$. Inverting these definitions and inserting them into (6.2) gives finally

$$d\tilde{H}_3 = -\frac{\tilde{\alpha}'}{4}\left[\hat{K}_2^I\hat{K}_2^J d_{IJ} - 2F_2\hat{J}_2 + \mathrm{tr}R^2\right] \ . \tag{6.4}$$

In order to compare this result with the toroidally compactified heterotic string, it is useful to group the $U(1)$ field-strengths into a 24-dimensional vector

$$\mathcal{F}_2 = [F_2, \ \hat{J}_2, \ \hat{K}_2^I]^T \ , \tag{6.5}$$

in which case the $D = 6$ Bianchi identity now reads

$$d\tilde{H}_3 = -\frac{\tilde{\alpha}'}{4}\left[\mathcal{F}^T L \mathcal{F} + \mathrm{tr}R^2\right] \ , \tag{6.6}$$

where the matrix $L = [(-\sigma^1) \oplus d_{IJ}]$ has 4 positive and 20 negative eigenvalues. This is in perfect agreement with the reduction of the $D = 7$ result, (5.11), and corresponds to a Narain compactification on $\Gamma_{4,20}$.

Note that the heterotic string tension $1/2\pi\tilde{\alpha}'$ and the Type IIA string tension $1/2\pi\alpha'$ are related by the Dirac quantization rule [6, 7]

$$2\kappa_6{}^2 = (2\pi)^3 n\alpha'\tilde{\alpha}' \ , \tag{6.7}$$

where $\kappa_6{}^2 = \kappa_{10}{}^2/V$ is the $D = 6$ gravitational constant. Some string theorists, while happy to endorse string/string duality, eschew the *soliton* interpretation. It is perhaps worth emphasizing, therefore, that without such an interpretation with its Dirac quantization rule, there is no way to relate the two string tensions.

7   Elementary versus solitonic heterotic strings

Our success in correctly reproducing the *fundamental* heterotic string σ-model anomaly polynomial

$$X_4 = \frac{1}{4}\frac{1}{(2\pi)^2}(\mathrm{tr}R^2 - \mathrm{tr}F^2) \ , \tag{7.1}$$

by treating the string as a ($K3$ compactified fivebrane) *soliton*, now permits a re-evaluation of a previous controversy concerning fundamental [79] versus solitonic [9, 12, 78] heterotic strings. In an earlier one loop test of $D = 10$ heterotic string/heterotic fivebrane duality [14],

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X_4 was obtained by the following logic: the $d = 2$ gravitational anomaly for complex fermions in a representation $\mathcal{R}$ of the gauge group is [72,73]

$$I_4 = \frac{1}{2}\frac{1}{(2\pi)^2}(\frac{r}{24}\text{tr}R^2 - \text{tr}_{\mathcal{R}}F^2) \,, \tag{7.2}$$

where r is the dimensionality of the representation and R is the *two-dimensional* curvature. Since the $SO(32)$ heterotic string has 32 left-moving gauge Majorana fermions (or, if we bosonize, 16 chiral scalars) and 8 physical right-moving spacetime Majorana fermions, Dixon, Duff and Plefka [14] set $\mathcal{R}$ to be the fundamental representation and put $r = 32 - 8 = 24$ to obtain $X_4 = I_4/2$, on the understanding that R is now to be interpreted as the pull-back of the *spacetime* curvature. Exactly the same logic was used in [14] in obtaining the heterotic[6] fivebrane $\tilde{X}_8$

$$\tilde{X}_8 = \frac{1}{(2\pi)^4}\left[\frac{1}{24}\,\text{tr}F^4 - \frac{1}{192}\,\text{tr}F^2\,\text{tr}R^2 + \frac{1}{768}\,(\text{tr}R^2)^2 + \frac{1}{192}\,\text{tr}R^4\right] \tag{7.3}$$

and in sections 3 and 4 above in obtaining the Type IIA fivebrane $\tilde{X}_8$ of (3.13). This logic was however criticized by Izquierdo and Townsend [15] and also by Blum and Harvey [16]. They emphasize the difference between the *gravitational anomaly* (which vanishes for the *fundamental* heterotic string [79]) involving the *two-dimensional* curvature and the σ-*model* anomaly (which is given by X_4 [80]) involving the pull-back of the *spacetime* curvature. Moreover, they go on to point out that the 32 left-moving gauge Majorana fermions (or 16 chiral scalars) of the fundamental heterotic string do not couple at all to the spin connections of this latter curvature. They conclude that the equivalence between X_4 and $I_4/2$ is a "curious fact" with no physical significance. They would thus be forced to conclude that the derivation of the Type IIA string field equations presented in the present paper is also a gigantic coincidence!

An attempt to make sense of all this was made by Blum and Harvey. They observed that the zero modes of *solitonic* strings (and fivebranes) necessarily couple to the spacetime spin connections because they inherit this coupling from the spacetime fields from which they are constructed. For these objects, therefore, they would agree that the logic of Dixon, Duff and Plefka (and, by inference, the logic of the present paper) is correct. But they went on

---

[6]Note that the heterotic string X_4, the heterotic fivebrane $\tilde{X}_8$ and the Type IIA fivebrane $\tilde{X}_8$ are the only non-vanishing anomaly polynomials, since from Table (1), these are the only theories with chiral supermultiplets.

to speculate that although *fundamental* and *solitonic* heterotic strings may both exist, they are *not* to be identified! Recent developments in string/string duality [3, 8, 11, 68, 69, 81], however, have convinced many physicists that the fundamental heterotic string is a soliton after all and so it seems we must look for an alternative explanation.

The correct way to resolve the apparent conflict is, we believe, rather mundane. The solitonic string and p-brane solitons are invariably presented in a *physical gauge* where one identifies d of the D spacetime dimensions with the $d = p + 1$ dimensions of the p-brane worldvolume. As discussed in [14], this is best seen in the Green-Schwarz formalism, which is in fact the only formalism available for $d > 2$. In such a physical gauge (which is only well-defined for vanishing *worldvolume* gravitational anomaly) the worldvolume curvatures and pulled-back spacetime curvatures are mixed up. So, in this sense, the gauge fermions do couple to the spacetime curvature after all.

8 Fivebrane origin of S-duality?
Discard worldvolume Kaluza-Klein modes?

In a recent paper [8], it was explained how S-duality in $D = 4$ follows as a consequence of $D = 6$ string/string duality: S-duality for one theory is just T-duality for the other. Since we have presented evidence in this paper that Type IIA string/heterotic string duality in $D = 6$ follows as a consequence of Type IIA string/Type IIA fivebrane duality in $D = 10$, which in turn follows from membrane/fivebrane duality in $D = 11$, it seems natural to expect a fivebrane origin of S-duality. (Indeed, a fivebrane explanation for S-duality has already been proposed by Schwarz and Sen [46] and by Binetruy [48], although they considered a T^6 compactification of the *heterotic* fivebrane rather than a $K3 \times T^2$ compactification of the *Type IIA fivebrane*.)

The explanation of [8] relied on the observation that the roles of the axion/dilaton fields S and the modulus fields T trade places in going from the fundamental string to the dual string. It was proved that, for a dual string compactified from $D = 6$ to $D = 4$ on T^2, $SL(2, Z)_S$ is a symmetry that interchanges the roles of the dual string worldsheet Bianchi identities and the field equations for the internal coordinates y^m ($m = 4, 5$). However, in unpublished work along the lines of [34, 35], Duff, Schwarz and Sen tried and failed to prove

that, for a fivebrane compactified from $D = 10$ to $D = 6$, $SL(2, Z)_S$ is a symmetry that interchanges the roles of the fivebrane worldvolume Bianchi identities and the field equations for the internal coordinates y^m ($m = 4, 5, 6, 7, 8, 9$). A similar negative result was reported by Percacci and Sezgin [82].

Another way to state the problem is in terms of massive worldvolume Kaluza-Klein modes. In the double dimensional reduction of the $D = 10$ fivebrane to $D = 6$ heterotic string considered in section 6, we obtained the heterotic string worldsheet multiplet of 24 left-moving scalars, 8 right moving scalars and 8 chiral fermions as the *massless modes* of a Kaluza-Klein compactification on $K3$. Taken in isolation, these massless modes on the dual string worldsheet will display the usual T-duality when the string is compactified from $D = 6$ to $D = 4$ and hence the fundamental string will display the desired S-duality. However, no-one has yet succeeded in showing that this T-duality survives when the *massive* Kaluza-Klein modes on the fivebrane worldvolume are included. Since these modes are just what distinguishes a string $X^M(\tau, \sigma)$ from a fivebrane $X^M(\tau, \sigma, \rho^i)$ ($i = 1, 2, 3, 4$), this was precisely the reason in [8] for preferring a $D = 6$ string/string duality explanation for $SL(2, Z)$ over a $D = 10$ string/fivebrane duality explanation. (Another reason, of course, is that the quantization of strings is understood, but that of fivebranes is not!) The same question about whether or not to discard massive worldvolume Kaluza-Klein modes also arises in going from the membrane in $D = 11$ to the Type IIA string in $D = 10$. For the moment therefore, this inability to provide a fivebrane origin for $SL(2, Z)$ remains the Achilles heel of the super p-brane programme[7].

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[7]Another unexplained phenomenon, even in pure string theory, is the conjectured $SL(2, Z)$ duality of the $D = 10$ Type IIB string [11], which gives rise to U-duality in $D = 4$. In this connection, it is perhaps worth noting from Table (1) that the gauged-fixed worldvolume of the self-dual Type IIB superthreebrane is described by the $d = 4, n = 4$ Maxwell supermultiplet [83]. Now $d = 4, n = 4$ abelian gauge theories are expected to display an $SL(2, Z)$ duality. See [84, 85] for a recent discussion. Could this be the origin of the $SL(2, Z)$ of the Type IIB string which follows from a T^2 compactification of the threebrane? Note moreover, that the threebrane supermultiplet itself follows from T^2 compactification of either the Type IIA or Type IIB fivebrane supermultiplet. Compactifications of such $d = 6$ self-dual antisymmetric tensors have, in fact, recently been invoked precisely in the context of S-duality in abelian gauge theories [85]. Of course, the gauged-fixed action for the superthreebrane is presumably not simply the Maxwell action but some non-linear (possibly Born-Infeld [83]) version. Nevertheless, S-duality might still hold [86].

9   Web of interconnections

We have discussed membrane and fivebranes in $D = 11$, heterotic strings and Type II fivebranes in $D = 10$, heterotic strings and membranes in $D = 7$, heterotic and Type II strings in $D = 6$ and how they are related by various compactifications. This somewhat bewildering mesh of interconnections is summarized in Fig. (1a). There are two types of dimensional reduction to consider: lines sloping down left to right represent spacetime reduction $(d, D) \rightarrow (d, D - k)$ and lines sloping down right to left represent simultaneous spacetime/worldsheet reduction $(d, D) \rightarrow (d - k, D - k)$. The worldsheet reductions may be checked against Table (1). Note that the simultaneous reduction on Ξ of the $D = 11$ membrane to yield the $D = 10$ heterotic string is still somewhat speculative [71], but we have included it since it nicely completes the diagram.

According to Townsend [68], a similar picture may be drawn relating the Type IIA string and heterotic fivebrane, which we show in Fig. (1b), where we have once again speculated on a spacetime reduction on Ξ of the $D = 11$ fivebrane to yield the $D = 10$ heterotic fivebrane. However, one must now explain how T^3 (or T^4) compactification of the $(120, 120)$ degrees of freedom of the gauge-fixed $D = 10$ heterotic fivebrane [59] can yield only the $(8, 8)$ of the $D = 7$ membrane (or the $(8_L, 8_L), (8_R, 8_R)$ of the $D = 6$ Type IIA string). Townsend has given arguments to support this claim. There are more interrelationships one can illustrate by including horizontal lines representing worldsheet reduction only[8], $(d, D) \rightarrow (d - k, D)$, some of which are shown in Figs. (2a,b).

Note that these diagrams describe theories related by compactification and so relate weak coupling to weak coupling and strong to strong. In Fig. (3), we have superimposed Figs. (1a) and (1b) to indicate how the various theories are also related by *duality* (denoted by the dotted horizontal lines) which relates weak coupling to strong. We believe that these interrelationships, which have in particular enabled us to deduce supermembrane effects in agreement with explicit string one-loop calculations, strengthen the claim that eleven dimensions and supermembranes have a part to play in string theory: a triumph of *diversification* over *unification* [87].

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[8]Townsend claims [68] that the worldvolume action of the $D = 10$ Type IIA string (or $D = 11$ supermembrane) may be obtained by T^4 (or T^3) compactification of the worldvolume action of the $D = 10$ heterotic fivebrane. However, we do not subscribe to this interpretation. Toroidal compactification can never produce $(n_+, n_-) = (8, 8)$ supersymmetry in $d = 2$ (or $n = 8$ in $d = 3$) from $(n_+, n_-) = (1, 0)$ in $d = 6$.

10 Acknowledgements

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A Reduction of the $D = 10$ Type IIA model on $K3$

In section 6 we presented the reduction of the fivebrane Bianchi identity on $K3$. For completeness, we present the reduction of the bosonic part of the $D = 10$ Type IIA supergravity action, (4.6), which we write here in a form notation:

$$
\begin{aligned}
I_{10} &= \frac{1}{2\kappa_{10}{}^2} \int d^{10}x \sqrt{-G} e^{-\Phi} \left[R_G + (\partial_M \Phi)^2 \right] \\
&\quad + \frac{1}{4\kappa_{10}{}^2} \int \left[F_2 \wedge *F_2 + e^{-\Phi} H_3 \wedge *H_3 + J_4 \wedge *J_4 - K_4 \wedge K_4 \wedge B_2 \right] ,
\end{aligned} \tag{A.1}
$$

where the ten-dimensional bosonic fields are the metric G, dilaton Φ and the 1-, 2- and 3-form fields A_1, B_2 and C_3. Eleven-dimensional K_4 quantization, (3.5), as well as the usual Kaluza-Klein condition for F_2, give rise to the ten-dimensional conditions

$$
\begin{aligned}
\int K_4 &= \frac{4\pi^2 n R}{T_2} \\
\int H_3 &= \frac{2\pi n}{T_2} \\
\int F_2 &= 2\pi n R .
\end{aligned} \tag{A.2}
$$

Following the decomposition of the fields in section 5, we write

$$
\begin{aligned}
A_1(X) &= \frac{R}{2} A_1(x) \\
B_2(X) &= B_2(x) + \frac{2\pi}{T_2} \sum b^I(x) \omega_2^I(y) \\
C_3(X) &= \frac{R}{2} C_3(x) + \frac{\pi R}{T_2} \sum C_1^I(x) \omega_2^I(y) ,
\end{aligned} \tag{A.3}
$$

in which case the four-form J_4 is given by

$$
J_4(X) = \frac{R}{2}[K_4(x) + A_1(x)H_3(x)] + \frac{\pi R}{T_2} \sum [K_2^I(x) + A_1(x)db^I(x)]\omega_2^I(y) . \tag{A.4}
$$

The constants are chosen so the six-dimensional $U(1)$ fields will be coupled to even charges

$$
\int \mathcal{F}_2 \in 4\pi \, \mathbb{Z} . \tag{A.5}
$$

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For $K3$, with Betti numbers $b_0 = 1$, $b_1 = 0$, $b_2^+ = 3$ and $b_2^- = 19$, we may choose an integral basis of harmonic two-forms, ω_2^I with intersection matrix

$$d_{IJ} = \int_{K3} \omega_2^I \wedge \omega_2^J \ . \tag{A.6}$$

Since taking a Hodge dual of ω_2^I on $K3$ gives another harmonic two-form, we may expand the dual in terms of the original basis

$$\hat{*}\omega_2^I = \omega_2^J H^J{}_I \ , \tag{A.7}$$

where we use $\hat{*}$ to denote Hodge duals on $K3$. In this case, we find

$$\int_{K3} \omega_2^I \wedge \hat{*}\omega_2^J = d_{IK} H^K{}_J \ . \tag{A.8}$$

The matrix $H^I{}_J$ depends on the metric on $K3$, $i.e.$ the $b_2^+ \cdot b_2^- = 57$ $K3$ moduli. Because of the fact that $\hat{*}\hat{*} = 1$, $H^I{}_J$ satisfies the properties [69]

$$
\begin{aligned}
H^I{}_J H^J{}_K &= \delta^I{}_K \\
d_{IJ} H^J{}_K &= d_{KJ} H^J{}_I \ ,
\end{aligned}
\tag{A.9}
$$

so that

$$H^J{}_I d_{JK} H^K{}_L = d_{IL} \tag{A.10}$$

and hence is an element of $SO(3, 19)/SO(3) \times SO(19)$.

Using these properties of $K3$, we may compactify the second line of (A.1) to obtain

$$
\begin{aligned}
I_6 &= \frac{1}{2\kappa_6{}^2} \int \Big[\tfrac{1}{2} e^{-\phi} H_3 \wedge *H_3 + \tfrac{1}{2} e^{-\phi} e^\rho db^I \wedge *db^J d_{IK} H^K{}_J \\
&\quad + \frac{\tilde{\alpha}'}{4} \Big(e^{-\rho} F_2 \wedge *F_2 + e^{-\rho} J_4 \wedge *J_4 + J_2^I \wedge *J_2^J d_{IK} H^K{}_J \\
&\qquad - K_2^I \wedge K_2^J \wedge B_2\, d_{IJ} - 2 K_4 \wedge K_2^I b^J d_{IJ} \Big) \Big] \ .
\end{aligned}
\tag{A.11}
$$

The six-dimensional dilaton is given by $\phi = \Phi + \rho$ where Φ is the ten-dimensional dilaton and ρ is the breathing mode of $K3$:

$$e^{-\rho} = \frac{1}{V} \int_{K3} \hat{*}1 \ . \tag{A.12}$$

In order to make contact with the compactified heterotic string, we wish to dualize the four-form J_4. Note, however, that since $d(e^{-\rho}\hat{*}J_4) = J_2^I db^J d_{IJ}$, the proper expression for dualizing

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J_4 is given by (6.3). Performing such a step and rewriting J_2^I as well, we finally arrive at

$$
\begin{aligned}
I_6 \;=\; & \frac{1}{2\kappa_6{}^2} \int \Big[\tfrac{1}{2} e^{-\phi} H_3 \wedge *H_3 + \tfrac{1}{2} e^{-\phi} e^{\rho} db^I \wedge *db^J d_{IK} H^K{}_J \\
& + \frac{\tilde{\alpha}'}{4} \Big(e^{-\rho} F_2 \wedge *F_2 + (\hat{K}_2^I + F_2 b^I) \wedge *(\hat{K}_2^J + F_2 b^J) d_{IK} H^K{}_J \\
& \qquad + e^{\rho} (\hat{J}_2 + \hat{K}_2^I b^J d_{IJ} + \tfrac{1}{2} F_2 b^I b^J d_{IJ}) \wedge *(\hat{J}_2 + \hat{K}_2^K b^L d_{KL} + \tfrac{1}{2} F_2 b^K b^L d_{KL}) \\
& \qquad - (\hat{K}_2^I \wedge \hat{K}_2^J d_{IJ} - 2 F_2 \wedge \hat{J}_2) \wedge B_2 \Big) \Big] \, .
\end{aligned}
\tag{A.13}
$$

This expression can be brought into a $SO(4,20)/SO(4) \times SO(20)$ invariant form. As in section 6, we group the $U(1)$ field strengths into the 24 component vector

$$
\mathcal{F}_2 = [F_2, \ \hat{J}_2, \ \hat{K}_2^I]^T \, ,
\tag{A.14}
$$

which allows us to rewrite the bosonic lagrangian as

$$
\begin{aligned}
I_6 \;=\; & \frac{1}{2\kappa_6{}^2} \int d^6 x \sqrt{-G} \, e^{-\phi} \left(R + (\partial_\mu \phi)^2 - \frac{1}{2 \cdot 3!} H^2_{\mu\nu\lambda} + \frac{1}{8} \mathrm{Tr}[\partial_\mu M L \partial_\mu M L] \right) \\
& + \frac{1}{2\kappa_6{}^2} \int \frac{\tilde{\alpha}'}{4} \left(\mathcal{F}_2{}^T (LML) \wedge *\mathcal{F}_2 - \mathcal{F}_2{}^T \wedge L \mathcal{F}_2 \wedge B_2 \right) \, .
\end{aligned}
\tag{A.15}
$$

The matrix L is given by

$$
L = \begin{bmatrix} -\sigma^1 & 0 \\ 0 & d_{IJ} \end{bmatrix} \, ,
\tag{A.16}
$$

where $\sigma^1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right)$. The matrix M contains the $1 + 57 + 22 = 80$ moduli of $K3$ with torsion, broken up in terms of e^{ρ}, $H^I{}_J$ and b^I respectively:

$$
M = \begin{bmatrix} e^{\rho} & -\tfrac{1}{2} e^{\rho} (b^I b^J d_{IJ}) & e^{\rho} b^I \\ -\tfrac{1}{2} e^{\rho} (b^I b^J d_{IJ}) & e^{-\rho} + b^I b^J d_{IK} H^K{}_J + \tfrac{1}{4} e^{\rho} (b^I b^J d_{IJ})^2 & -b^K H^I{}_K - \tfrac{1}{2} e^{\rho} b^I (b^K b^L d_{KL}) \\ e^{\rho} b^J & -b^K H^J{}_K - \tfrac{1}{2} e^{\rho} b^J (b^K b^L d_{KL}) & H^I{}_K d^{JK} + e^{\rho} b^I b^J \end{bmatrix} \, .
\tag{A.17}
$$

In the last entry of M, d^{IJ} is the inverse of d_{IJ}. We verify that

$$
M^T = M, \qquad MLM^T = L^{-1} \, .
\tag{A.18}
$$

This agrees with the bosonic action given in [81].

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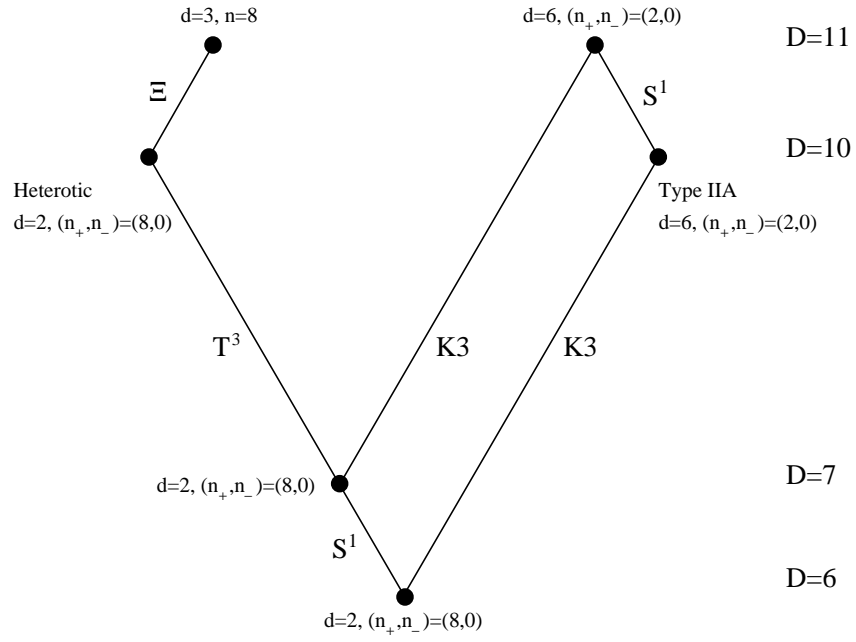
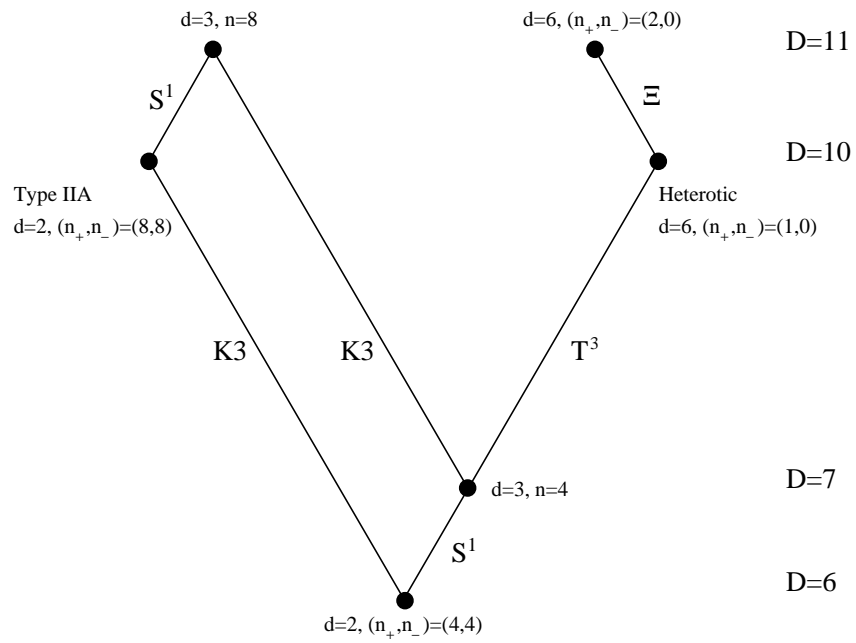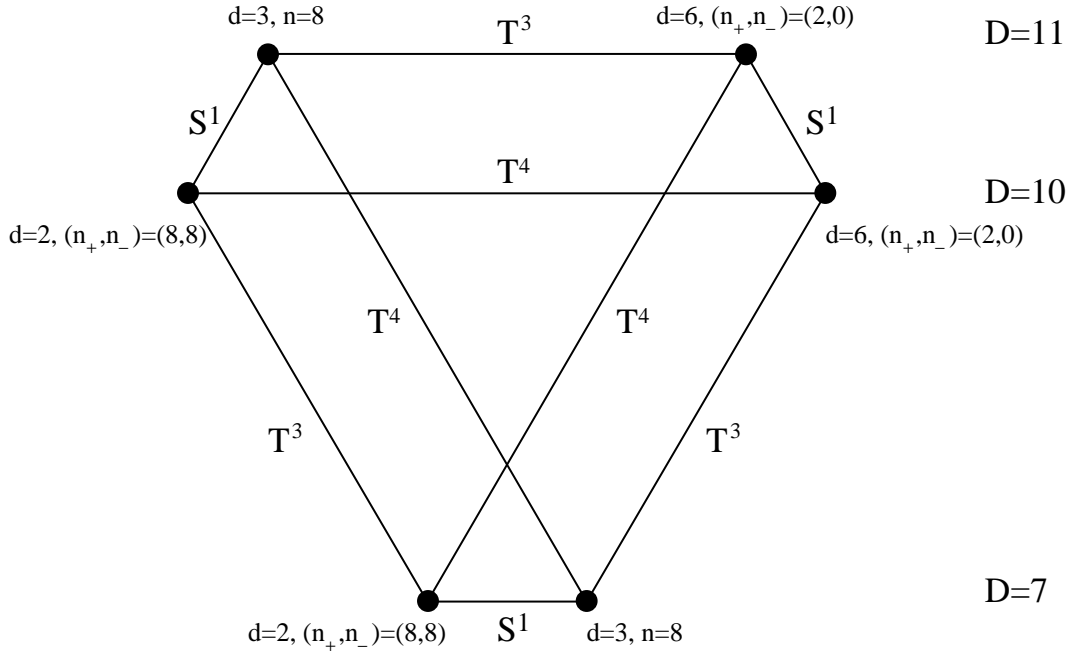Figure 1: Compactifications relating (a) the Type IIA fivebrane to the heterotic string and (b) the heterotic fivebrane to the Type IIA string. Worldvolume supersymmetries are indicated.
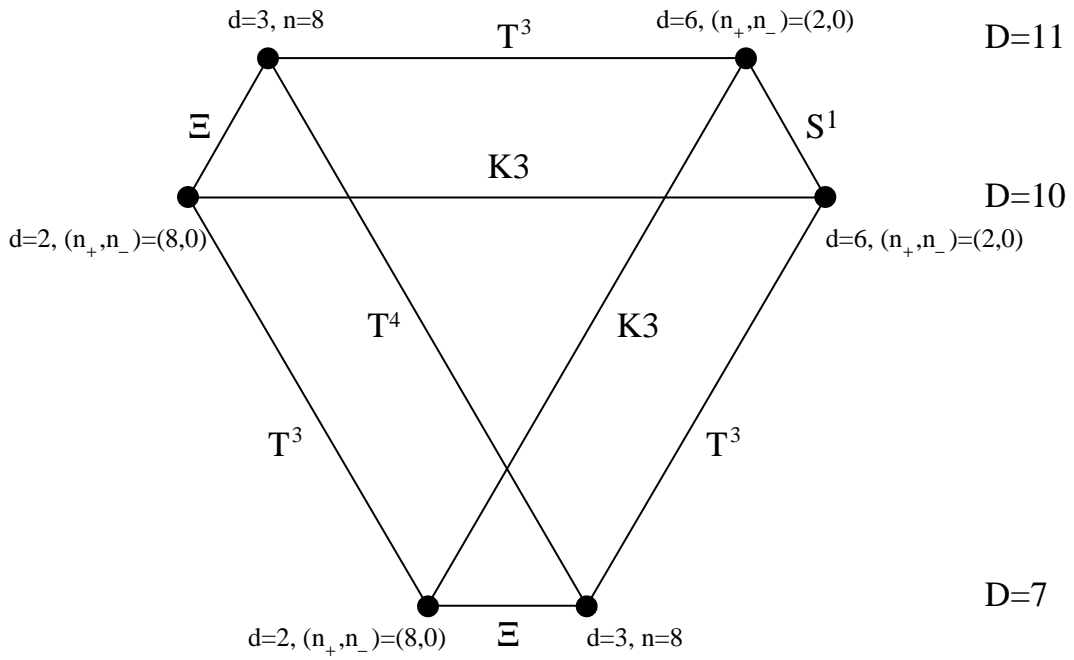
a)

d=3, n=8   T³   d=6, (n₊,n₋)=(2,0)   D=11

S¹   S¹

T⁴

d=2, (n₊,n₋)=(8,8)   d=6, (n₊,n₋)=(2,0)   D=10

T⁴   T⁴

T³   T³

D=7

d=2, (n₊,n₋)=(8,8)   S¹   d=3, n=8

b)

d=3, n=8   T³   d=6, (n₊,n₋)=(2,0)   D=11

Ξ   S¹

K3

d=2, (n₊,n₋)=(8,0)   d=6, (n₊,n₋)=(2,0)   D=10

T⁴   K3

T³   T³

D=7

d=2, (n₊,n₋)=(8,0)   Ξ   d=3, n=8

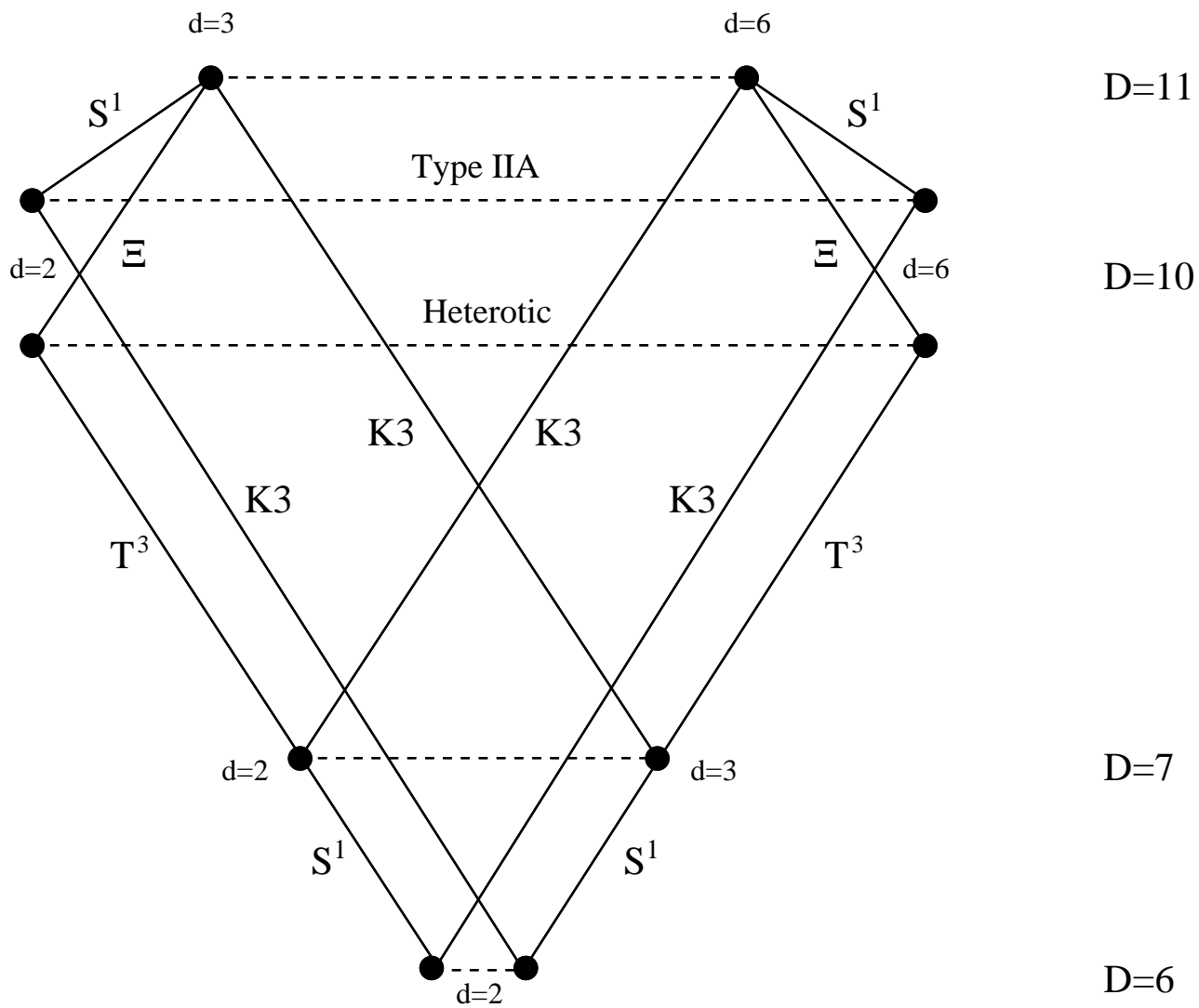Figure 2: Compactifications incorporating worldvolume reductions.

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Figure 3: A superposition of Figs. 1 (*a*) and (*b*), illustrating strong/weak coupling dualities (denoted by the dotted lines).