SISSA-ISAS 68/95/EP
hep-th/yymmxxx

Multi–matrix models: integrability properties and topological content

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HEP-TH-9506124

Abstract

We analyze multi–matrix chain models. They can be considered as multi–component Toda lattice hierarchies subject to suitable coupling conditions. The extension of such models to include extra discrete states requires a weak form of integrability. The discrete states of the q–matrix model are organized in representations of sl_q. We solve exactly the Gaussian–type models, of which we compute several all-genus correlators. Among the latter models one can classify also the discretized $c = 1$ string theory, which we revisit using Toda lattice hierarchy methods. Finally we analyze the topological field theory content of the $2q$–matrix models: we define primary fields (which are ∞^q), metrics and structure constants and prove that they satisfy the axioms of topological field theories. We outline a possible method to extract interesting topological field theories with a finite number of primaries.

1 Introduction

In this paper we intend to analyze matrix models made of q Hermitean $N \times N$ matrices with bilinear couplings between different matrices. Unless otherwise specified, by this we mean an open chain of q matrices, each linearly interacting with the nearest neighbours. These models have been already introduced and partially analyzed in [1] (for other approaches to multi–matrix models, see [2],[3],[4],[5], [6],[7]). The reasons to go beyond two–matrix models are diverse. The extended two–matrix model provides a useful representation of $c = 1$ string theory at the self–dual point, [8]; in particular it naturally incorporates the so–called discrete states, which appear to be organized in sl_2 multiplets. We find it natural to ask ourselves whether such a construction can be generalized. The answer is affirmative: in the extended q–matrix model we do find discrete states organized according to representations of sl_q. More recently it has been shown, [9], that $c = 1$ string theory at the self–dual point, i.e. the two–matrix model, is a huge topological field theory in which we can distinguish primaries, puncture operators and descendants. As we shall see, this holds for $2q$ matrix model too, although with new features (for example, the number of primaries is ∞^n).

On a more speculative ground one may remark that two–matrix models lead via hamiltonian reduction to reduced models characterized by classical hierarchies [9],[10] which can be interpreted in terms of topological field theories coupled to topological gravity; in turn the latter can be put in correspondence with string vacua. The two–matrix model analysis suggests that, if we want to reach more interesting string or W–string vacua, we have to shift to matrix models with several matrices. Although we do not go as far as proving this, nevertheless many elements we find seem to support such a conjecture.

Finally, to end the list of the reasons of interest on a more formal ground, we recall that the integrable hierarchy characterizing two–matrix models is the discrete Toda hierarchy, while the discrete integrable hierarchy characterizing multi–matrix models is a generalization of the latter. As we have already pointed out, we can extend these models by introducing additional (extra) states and couplings. While the extended two–matrix model does not present any essentially new features, the q–matrix models with $q > 2$ do. In fact we can have in general only a *weak* form of integrability of the extra flows (as opposed to the *strong* integrability of the ordinary cases). This form of integrability is nevertheless sufficient for all our purposes.

The paper is organized as follows. In section 2 we review, mostly from [1], the main results concerning multi–matrix models and derive the flows of the extended q–matrix models. In section 3 we solve the coupling condition of Gaussian q–matrix models. In section 4 we introduce the discrete states and discuss their group properties. We then compute several examples of correlators in $2q$–matrix models. Section 5 is devoted to the topological field theory properties alluded to above. In section 6 we introduce a few simple examples of non–Gaussian matrix models. Finally section 7 is devoted to an analysis of the discretized 1D string. The latter can in fact be envisaged as a chain matrix model with bilinear couplings. It is interesting to rederive known properties of $c = 1$ string in our formalism. Finally two Appendices are devoted to the W–constraints in q–matrix models and to an explicit computation, respectively.

2 Multi–matrix models: general introduction

We review here some general results concerning q–matrix models, [1]. The partition function of the q–matrix model is given by

$$Z_N(t,c) = \int dM_1 dM_2 \dots dM_q e^{TrU} \tag{2.1}$$

where $M_1, \dots, M_q$ are Hermitian $N \times N$ matrices and

$$U = \sum_{\alpha=1}^{q} V_\alpha + \sum_{\alpha=1}^{q-1} c_{\alpha,\alpha+1} M_\alpha M_{\alpha+1}$$

with potentials

$$V_\alpha = \sum_{r=1}^{p_\alpha} \bar{t}_{\alpha,r} M_\alpha^r \qquad \alpha = 1, 2 \dots, q \tag{2.2}$$

The p_α's are finite positive integers.

We denote by $\mathcal{M}_{p_1,p_2,\dots,p_q}$ the corresponding q–matrix model. It has become moreover customary to associate to the generic q–matrix model (2.1) the Dynkin diagram A_q. Occasionally we will stick to this convention and speak abou nodes and links.

We are interested in computing correlation functions (CF's) of the operators

$$\tau_{\alpha,k} = tr M_\alpha^k$$

and possibly of other composite operators (see below). For this reason we complete the above model by replacing (2.2) with the more general potentials

$$V_\alpha = \sum_{r=1}^{\infty} t_{\alpha,r} M_\alpha^r, \qquad \alpha = 1, \dots q \tag{2.3}$$

where $t_{\alpha,r} \equiv \bar{t}_{\alpha,r}$ for $r \leq p_\alpha$.

In other words we have embedded the original couplings $\bar{t}_{\alpha,r}$ into infinite sets of couplings. Therefore we have two types of couplings. The first type consists of those couplings (the barred ones) that define the model: they represent the true *dynamical* parameters of the theory; they are kept non-vanishing throughout the calculations. The second type encompasses the remaining couplings, which are introduced only for computational purposes. In terms of ordinary field theory the former are analogous to the interaction couplings, while the latter correspond to external sources (coupled to composite operators). Any CF is obtained by differentiating lnZ_N with respect to the couplings associated to the operators that appear in the correlator and then setting to zero (only) the external couplings.

From now on we will not make any formal distinction between interacting and external couplings. Case by case we will specify which are the interaction couplings and which are the external ones. Finally, it is sometime convenient to consider N on the same footing as the couplings and to set $t_{\alpha,0} \equiv N$.

The most popular procedure to calculate the partition function consists of three steps [13],[14],[15]:

(i). One integrates out the angular parts such that only the integrations over the eigenvalues are left,

$$Z_N(t,c) = const \int \prod_{\alpha=1}^{q} \prod_{i=1}^{N} d\lambda_{\alpha,i} \Delta(\lambda_1) e^U \Delta(\lambda_q), \tag{2.4}$$

where

$$U = \sum_{\alpha=1}^{q} \sum_{i=1}^{N} V_\alpha(\lambda_{\alpha,i}) + \sum_{\alpha=1}^{q-1} \sum_{i=1}^{N} c_{\alpha,\alpha+1} \lambda_{\alpha,i} \lambda_{\alpha+1,i}, \tag{2.5}$$

and $\Delta(\lambda_1)$ and $\Delta(\lambda_q)$ are Vandermonde determinants.
(ii). One introduces the orthogonal polynomials

$$\xi_n(\lambda_1) = \lambda_1^n + \text{lower powers}, \qquad \eta_n(\lambda_q) = \lambda_q^n + \text{lower powers}$$

which satisfy the orthogonality relations

$$\int d\lambda_1 \ldots d\lambda_q \xi_n(\lambda_1) e^\mu \eta_m(\lambda_q) = h_n(t,c) \delta_{nm} \tag{2.6}$$

where

$$\mu \equiv \sum_{\alpha=1}^{q} \sum_{r=1}^{\infty} t_{\alpha,r} \lambda_\alpha^r + \sum_{\alpha=1}^{q-1} c_{\alpha,\alpha+1} \lambda_\alpha \lambda_{\alpha+1}. \tag{2.7}$$

(iii). If one expands the Vandermonde determinants in terms of these orthogonal polynomials and using the orthogonality relation (2.6), one can easily calculate the partition function

$$Z_N(t,c) = const\ N! \prod_{i=0}^{N-1} h_i \tag{2.8}$$

Knowing the $h(c,t)$'s amounts to knowing the partition function, up to an N-dependent constant. In turn the information concerning the $h(c,t)$'s can be encoded in suitable *flow equations*, subject to specific conditions, *the coupling conditions*. Before we come to that, however, we recall some necessary notations.

For any matrix M, we define the conjugate $\mathcal{M}$

$$\mathcal{M} = H^{-1} M H, \qquad H_{ij} = h_i \delta_{ij}, \qquad \bar{M}_{ij} = M_{ji}, \qquad M_l(j) \equiv M_{j,j-l}.$$

As usual we introduce the natural gradation

$$deg[E_{ij}] = j - i, \qquad where \qquad (E_{i,j})_{k,l} = \delta_{i,k} \delta_{j,l}$$

and, for any given matrix M, if all its non–zero elements have degrees in the interval $[a,b]$, then we will simply write: $M \in [a,b]$. Moreover M_+ will denote the upper triangular part of M (including the main diagonal), while $M_- = M - M_+$. We will write

$$Tr(M) = \sum_{i=0}^{N-1} M_{ii}$$

The latter operation will be referred to as taking the finite trace.

Coupling conditions.
First we introduce the Q–type matrices

$$\int \prod_{\alpha=1}^{q} d\lambda_\alpha \xi_n(\lambda_1) e^{\mu} \lambda_\alpha \eta_m(\lambda_q) \equiv Q_{nm}(\alpha) h_m = \bar{\mathcal{Q}}_{mn}(\alpha) h_n, \quad \alpha = 1, \dots, q. \tag{2.9}$$

Among them, $Q(1), \bar{\mathcal{Q}}(q)$ are Jacobi matrices: their pure upper triangular part is $I_+ = \sum_i E_{i,i+1}$. We will need two P–type matrices, defined by

$$\int \prod_{\alpha=1}^{q} d\lambda_\alpha \Big(\frac{\partial}{\partial \lambda_1} \xi_n(\lambda_1)\Big) e^{\mu} \eta_m(\lambda_q) \equiv P_{nm}(1) h_m \tag{2.10}$$

$$\int d\lambda_1 \dots d\lambda_q \xi_n(\lambda_1) e^{\mu} \Big(\frac{\partial}{\partial \lambda_q} \eta_m(\lambda_q)\Big) \equiv P_{mn}(q) h_n \tag{2.11}$$

The matrices (3.8) we introduced above are not completely independent. More precisely all the $Q(\alpha)$'s can be expressed in terms of only one of them and one matrix P. Expressing the trivial fact that the integral of the total derivative of the integrand in eq.(2.6) with respect to $\lambda_\alpha, 1 \le \alpha \le q$ vanishes, we can easily derive the constraints or *coupling conditions*

$$P(1) + V_1' + c_{12} Q(2) = 0, \tag{2.12a}$$

$$c_{\alpha-1,\alpha} Q(\alpha-1) + V_\alpha' + c_{\alpha,\alpha+1} Q(\alpha+1) = 0, \qquad 2 \le \alpha \le q-1, \tag{2.12b}$$

$$c_{q-1,q} Q(q-1) + V_q' + \bar{\mathcal{P}}(q) = 0. \tag{2.12c}$$

where we use the notation

$$V_\alpha' = \sum_{r=1}^{p_\alpha} r\, t_{\alpha,r} Q^{r-1}(\alpha), \qquad \alpha = 1, 2, \dots, q$$

These conditions explicitly show that the Jacobi matrices depend on the choice of the potentials. In fact they completely determine the degrees of the matrices $Q(\alpha)$. A simple calculation shows that

$$Q(\alpha) \in [-m_\alpha, n_\alpha], \qquad \alpha = 1, 2, \dots, q$$

where

$$\begin{aligned} m_1 &= (p_q - 1) \dots (p_3 - 1)(p_2 - 1) \\ m_\alpha &= (p_q - 1)(p_{q-1} - 1) \dots (p_{\alpha+1} - 1), \qquad 2 \le \alpha \le q-1 \\ m_q &= 1 \end{aligned}$$

and

$$\begin{aligned} n_1 &= 1 \\ n_\alpha &= (p_{\alpha-1} - 1) \dots (p_2 - 1)(p_1 - 1), \qquad 2 \le \alpha \le q-1 \\ n_q &= (p_{q-1} - 1) \dots (p_2 - 1)(p_1 - 1) \end{aligned}$$

Throughout the paper we will refer to the following coordinatization of the Jacobi matrices

$$Q(1) = I_+ + \sum_i \sum_{l=0}^{m_1} a_l(i) E_{i,i-l}, \qquad \bar{\mathcal{Q}}(q) = I_+ + \sum_i \sum_{l=0}^{m_q} b_l(i) E_{i,i-l} \tag{2.13}$$

and for the supplementary matrices

$$Q(\alpha) = \sum_i \sum_{l=-n_\alpha}^{m_\alpha} T_l^{(\alpha)}(i) E_{i,i-l}, \quad 2 \leq \alpha \leq q-1 \tag{2.14}$$

Flow equations

The flow equations of the q–matrix model can be expressed by means of the following hierarchies of equations for the matrices $Q(\alpha)$.

$$\frac{\partial}{\partial t_{\beta,k}} Q(\alpha) = [Q_+^k(\beta), \quad Q(\alpha)], \qquad 1 \leq \beta \leq \alpha \tag{2.15a}$$

$$\frac{\partial}{\partial t_{\beta,k}} Q(\alpha) = [Q(\alpha), \quad Q_-^k(\beta)], \qquad \alpha \leq \beta \leq q \tag{2.15b}$$

These flows commute and define a multi–component Toda lattice hierarchy, [12],[6].

Reconstruction formulae.

The coupling conditions and the flow equations allow us to calculate the matrix elements of $Q(\alpha)$. From the latter we can reconstruct the partition function as follows. We start from the following main formula

$$\frac{\partial}{\partial t_{\alpha,r}} lnZ_N(t,c) = Tr\Big(Q^r(\alpha)\Big), \qquad 1 \leq \alpha \leq q \tag{2.16}$$

It is evident that, by means of the flow equations for $Q(\alpha)$, we can express all the derivatives of lnZ_N with respect to the couplings $t_{\alpha,k}$ (i.e. all the correlators) as finite traces of commutators of the $Q(\alpha)$'s themselves. In other words, knowing the $Q(\alpha)$'s, we can reconstruct the partition function (up to a constant depending only on N). In particular we can get

$$\frac{\partial^2}{\partial t_{1,1} \partial t_{\alpha,r}} lnZ_N(t,c) = \Big(Q^r(\alpha)\Big)_{N,N-1}, \qquad 1 \leq \alpha \leq q \tag{2.17}$$

It was already noticed in [1] that this equation leads to the two-dimensional Toda lattice equation.

2.1 Extended q–matrix models.

It is important to be able to compute the correlators not only of the states considered above, but also of new states, the *extra states*. To this end we enlarge the q–matrix model by introducing in the potential U new interaction terms, as follows. We change

$$U \to \hat{U} = \sum_{i=1}^{N} \sum_{b_1,\ldots,b_q} g_{b_1,\ldots,b_q} \lambda_{1,i}^{b_1} \ldots \lambda_{q,i}^{b_q} \tag{2.18}$$

in (2.1,2.5), and, accordingly,

$$\mu \to \hat{\mu} = \sum_{b_1,\ldots,b_q} g_{b_1,\ldots,b_q} \lambda_1^{b_1} \ldots \lambda_q^{b_q} \tag{2.19}$$

in (2.7). Henceforth $a_i, b_i, c_i, \ldots$ will denote non–negative indices.

We denote by $\chi_{b_1,\ldots,b_q}$ the state specified (classically) by $\sum_{i=1}^N \lambda_{1,i}^{b_1}\ldots\lambda_{q,i}^{b_q}$. It is clear that when $b_i = 0$ for all $i \neq \alpha$, this state reduces to τ_{α,b_α}, while the corresponding coupling g boils down to t_{α,b_α}. Moreover the previously introduced bilinear coupling $c_{\alpha,\alpha+1}$ is nothing but the above g when all the $b_i = 0$ except $b_\alpha = b_{\alpha+1} = 1$.

All the couplings and states that do not appear in the original model (2.5) are called *extra*. Exactly as in the original q–matrix model, we can introduce orthogonal monic polynomials $\xi_n(\lambda_1)$ and $\eta_m(\lambda_q)$ and define the $Q(\alpha)$ matrices. This is parallel to what happens in the the extended two-matrix model, [8].

However, unlike the extended two–matrix model, in the extended q–matrix model, we cannot in general define flow equations in matrix form like eqs.(2.15a,2.15b). This is a remarkable difference between extended two– and q–matrix models (with $q > 2$), and, at first sight, seems to spoil integrability and any possibility of exact calculation of the CF's. Fortunately this is not the case. What one has to do is not to calculate the flows of the matrices $Q(\alpha)$, but the multiple derivatives w.r.t. the couplings of lnZ_N, i.e. the multiple derivatives of h_n, and express them in terms of matrices $Q(\alpha)$. One can verify that such 'weak flows' commute, and thus integrability is preserved, although in a weak sense.

The procedure is as follows. We first introduce two series of functions, [1],

$$\xi_n^{(\alpha)}(t,\lambda_\alpha) \equiv \int \prod_{\beta=1}^{\alpha-1} d\lambda_\beta \xi_n(\lambda_1) e^{\mu_\alpha^L}. \tag{2.20}$$

and

$$\eta_n^{(\alpha)}(t,\lambda_\alpha) \equiv \int \prod_{\beta=\alpha+1}^{q} d\lambda_\beta e^{\mu_\alpha^R} \eta_m(\lambda_q). \tag{2.21}$$

where

$$\mu_\alpha^L \equiv \sum_{\beta=1}^{\alpha-1}\sum_{k=1}^{\infty} t_{\beta,k}\lambda_\beta^k + \sum_{\beta=1}^{\alpha-1} c_{\beta,\beta+1}\lambda_\beta\lambda_{\beta+1}.$$

$$\mu_\alpha^R \equiv \sum_{\beta=\alpha+1}^{q}\sum_{k=1}^{\infty} t_{\beta,k}\lambda_\beta^k + \sum_{\beta=\alpha}^{q-1} c_{\beta,\beta+1}\lambda_\beta\lambda_{\beta+1}.$$

Obviously we have

$$\xi_n^{(1)}(t,\lambda_1) = \xi_n(\lambda_1), \qquad \eta_n^{(q)}(t,\lambda_q) = \eta_m(\lambda_q).$$

but for other values of α one sees immediately that $\xi^{(\alpha)}$ and $\eta^{(\alpha)}$ are not polynomials. But they satisfy the orthogonality relations

$$\int d\lambda_\alpha \xi_n^{(\alpha)}(t,\lambda_\alpha) e^{V_\alpha(\lambda_\alpha)} \eta_m^{(\alpha)}(t,\lambda_\alpha) = \delta_{nm} h_n(t,c), \qquad 1 \leq \alpha \leq q. \tag{2.22}$$

Eq.(2.9) provides a definition of the $Q(\alpha)$ matrix in this basis

$$\int d\lambda_\alpha \xi_n^{(\alpha)}(t,\lambda_\alpha)\lambda_\alpha e^{V_\alpha(\lambda_\alpha)} \eta_m^{(\alpha)}(t,\lambda_\alpha) = Q_{nm}(\alpha) h_m(t,c), \qquad \forall 1 \leq \alpha \leq q. \tag{2.23}$$

Therefore the spectral equations follow

$$\lambda_\alpha \xi^{(\alpha)} = Q(\alpha)\xi^{(\alpha)}, \qquad 1 \leq \alpha \leq q. \tag{2.24}$$

$$\lambda_\alpha \eta^{(\alpha)} = \bar{\mathcal{Q}}(\alpha)\eta^{(\alpha)}, \qquad 1 \leq \alpha \leq q. \tag{2.25}$$

where ξ^α and η^α represent the infinite vectors with components $\xi^\alpha_0, \xi^\alpha_1, \ldots$ and $\eta^\alpha_0, \eta^\alpha_1, \ldots$, respectively.

With these bases at hand one differentiates h_n, i.e. (2.6) for $n = m$, w.r.t the appropriate couplings and evaluate the results when the extra couplings vanish. The result contains derivatives of ξ_n and η_n w.r.t to the couplings, which in turn can be evaluated differentiating (2.6) with $n > m$ or $n < m$. Finally one can express the result in terms of elements of the matrices $Q(\alpha)$, by making use of the above defined bases ξ^α_n and η^α_m. Inserting this into the expressions of the correlators, i.e. into the derivatives of lnZ_N w.r.t. the appropriate couplings, one can express the latter in terms of finite traces of polynomials in the $Q(\alpha)$'s.

From now on, whenever it is not confusing, we use the simplified notation $Q(\alpha) \equiv Q_\alpha$.

The 1–point CF is easily found to be given by

$$< \chi_{a_1,\ldots,a_q} >= Tr\left(Q_1^{a_1} \cdots Q_q^{a_q}\right) \tag{2.26}$$

The derivation of the two point functions, by the above procedure, is as follows

$$< \chi_{a_1\ldots a_q}\chi_{b_1\ldots b_q} >= \sum_{n=0}^{N-1} \frac{\partial^2 lnh_n}{\partial g_{a_1\ldots a_q}\partial g_{b_1\ldots b_q}}$$

$$\begin{aligned}\frac{\partial^2 h_n}{\partial g_{a_1\ldots a_q}\partial g_{b_1\ldots b_q}} &= \int d\lambda \frac{\partial}{\partial g_{b_1\ldots b_q}}\xi_n\lambda_1^{a_1}\ldots\lambda_q^{a_q}\eta_n + \int d\lambda\xi_n\lambda_1^{a_1+b_1}\ldots\lambda_q^{a_q+b_q}\eta_n + \\ &+ \int d\lambda\xi_n\lambda_1^{b_1}\ldots\lambda_q^{b_q}\frac{\partial}{\partial g_{b_1\ldots b_q}}\eta_n\end{aligned}$$

Then, using

$$\frac{\partial}{\partial g_{b_1\ldots b_q}}\xi_n = -\sum_{m=0}^{n-1}(Q_1^{b_1}\ldots Q_q^{b_q})_{nm}\xi_m, \qquad \frac{\partial}{\partial g_{b_1\ldots b_q}}\eta_n = -\sum_{m=0}^{n-1}\eta_m(Q_1^{b_1}\ldots Q_q^{b_q})_{mn}\frac{h_n}{h_m},$$

we obtain

$$\begin{aligned}< \chi_{a_1,\ldots,a_q}\chi_{b_1,\ldots,b_q} > &= Tr\Big[Q_1^{a_1+b_1}\cdots Q_q^{a_q+b_q} - \left(Q_1^{b_1}\cdots Q_q^{b_q}\right)_-\left(Q_1^{a_1}\cdots Q_q^{a_q}\right) \\ &\quad -\left(Q_1^{a_1}\cdots Q_q^{a_q}\right)\left(Q_1^{b_1}\cdots Q_q^{b_q}\right)_+\Big]\end{aligned} \tag{2.27}$$

Along the same lines, we get

$$\begin{aligned}&< \chi_{a_1,\ldots,a_q}\chi_{b_1,\ldots,b_q}\chi_{c_1,\ldots,c_q} >= Tr\Big\{Q_1^{a_1+b_1+c_1}\ldots Q_q^{a_q+b_q+c_q} \\ &-\left[\left(Q_1^{a_1}\ldots Q_q^{a_q}\right)_-\left(Q_1^{b_1+c_1}\ldots Q_q^{b_q+c_q}\right) + \left(Q_1^{b_1+c_1}\ldots Q_q^{b_q+c_q}\right)\left(Q_1^{a_1}\ldots Q_q^{a_q}\right)_+ + \ c.p.\right] \\ &+\left[\left(Q_1^{a_1}\ldots Q_q^{a_q}\right)_-\left(Q_1^{b_1}\ldots Q_q^{b_q}\right)\left(Q_1^{c_1}\ldots Q_q^{c_q}\right)_+ + \ p.\right] \\ &+2\left(Q_1^{a_1}\ldots Q_q^{a_q}\right)_+\left(Q_1^{b_1}\ldots Q_q^{b_q}\right)_+\left(Q_1^{c_1}\ldots Q_q^{c_q}\right)_+\Big\}.\end{aligned} \tag{2.28}$$

where p. (c.p.) means permutations (cyclic permutations) of the sets $\{a_1, ..., a_q\}$, $\{b_1, ..., b_q\}$ and $\{c_1, ..., c_q\}$. The RHS's of both (2.27) and (2.28) are symmetric under the exchange of the χ operators. This property together with the fact that the RHS's can be written down in terms of the Q_α's, which are calculable, expresses what we call *weak integrability*.

3 Coupling conditions in Gaussian models.

There are several methods to solve matrix models. One is based on W–constraints (see Appendix A) and will be occasionally used also in this paper. The most powerful however consists of solving the coupling conditions to obtain explicit expressions of the $Q(\alpha)$ matrices and, then, inserting these into the expressions of the correlators. In this paper we will mostly consider q–matrix models in which the interacting terms are at most quadratic. These models can be solved in general and the solutions can be expressed by means of very general formulas (see below), the reason being that the coupling conditions reduce to a system of linear equations in the $Q(\alpha)$'s. We would like however to point out that much more general (than Gaussian) q–matrix models can in principle be exactly solved. The only trouble is that, when the potential terms are more than quadratic, the coupling conditions are non-linear equations in the $Q(\alpha)$'s and we cannot find such compact formulas as in the Gaussian models but we have to proceed case by case. We will see later on examples of non-Gaussian models. For the time being let us concentrate on the Gaussian ones. They are sufficient to reveal the topological properties of the corresponding matrix models.

Let us first introduce a more convenient notation for Gaussian models. The q-matrix models with quadratic potential have the partition function of the form:

$$Z = \int \prod_{\alpha=1}^{q} dM_\alpha \exp(\sum_{\alpha=1}^{q}(t_\alpha M_\alpha^2 + u_\alpha M_\alpha) + \sum_{\alpha=1}^{q-1} c_\alpha M_\alpha M_{\alpha+1}) \tag{3.1}$$

We notice that the linear terms can be eliminated by suitable redefinitions of the matrices M_α. However it is often useful to keep them distinct (for example to study the topological field theory properties). Therefore, whenever this does not complicate the formulas too much, we will keep the linear terms.

We will solve first the coupling conditions of 3– and 4–matrix models, both for pedagogical reasons and in order to have explicit formulas of the simplest cases, and then the general case.

3.1 The 3-matrix model

The coupling conditions are :

$$\begin{array}{ll} P_1 & +2t_1Q_1 + u_1 + c_1Q_2 = 0 \\ & 2t_2Q_2 + u_2 + c_1Q_1 + c_2Q_3 = 0 \\ \overline{\mathcal{P}_3} & +2t_3Q_3 + u_3 + c_2Q_2 = 0 \end{array} \tag{3.2}$$

Eliminating the matrix Q we get the following two-matrix model type coupling conditions:

$$\begin{array}{ll} P_1 & +2(t_1 - \frac{c_1^2}{4t_2})Q_1 + (u_1 - \frac{u_2c_1}{2t_2}) - \frac{c_1c_2}{2t_2}Q_3 = 0 \\ \overline{\mathcal{P}_3} & +2(t_3 - \frac{c_2^2}{4t_2})Q_3 + (u_3 - \frac{u_2c_2}{2t_2}) - \frac{c_1c_2}{2t_2}Q_1 = 0 \end{array}$$

Solving the system we obtain the following form of the Q matrices (with reference to the coordinatization (2.13,2.14)

$$b_0 \;=\; -\frac{2t_2}{B}\,(c_1c_2u_1 - 2c_2t_1u_2 + (4t_1t_2 - c_1^2)u_3)$$

$$
\begin{aligned}
T_0^{(2)} &= \frac{2t_2}{B}\left(c_1 t_3 u_1 - 2t_1 t_3 u_2 + c_2 t_1 u_3\right) \\
a_0 &= -\frac{2t_2}{B}\left((4t_2 t_3 - c_1^2)u_1 - 2c_1 t_3 u_2 + c_1 c_2 u_3\right)
\end{aligned}
$$

and

$$
\begin{aligned}
b_1 &= -\frac{n}{B}\left(2t_2(4t_1 t_2 - c_2^2)\right), \qquad a_1 = -\frac{n}{B}\left(2t_2(4t_2 t_3 - c_1^2)\right), \\
T_1^{(2)} &= \frac{4c_1 t_2 t_3 n}{B}, \qquad T_{-1}^{(2)} = -\frac{2t_1 n}{c_1}, \qquad R_3 = -\frac{2c_1 c_2 t_2 n}{B}
\end{aligned}
$$

where

$$
B = (4t_2 t_3 - c_2^2)(4t_2 t_1 - c_1^2) - (c_1 c_2)^2
$$

while all the other coordinates vanish.

So far we have used a basis ξ_n corresponding to the first matrix or first node and to a basis η_n corresponding to the third matrix or node. One may wonder what happens if one switches from the node 1,3 to the nodes 1,2. The coupling constraints are of course modified: $\overline{\mathcal{P}_3}$ disappears from the third eq. (3.3) and in the second eq.(3.3) there appears $\overline{\mathcal{P}_2}$ .Eliminating now the matrix Q_3 we obtain the 2–matrix model coupling conditions:

$$
\begin{aligned}
P_1 \qquad & +2t_1 Q_1 + u_1 - c_1 Q_2 = 0 \\
\overline{\mathcal{P}_2} \qquad & +2(t_2 - \frac{c_2^2}{4t_3})Q_2 + (u_2 - \frac{u_3 c_2}{2t_3}) - c_1 Q_1 = 0
\end{aligned}
$$

Calculating the form of the Q matrices we get the same result as above. Hence,changing the basis does not modify the model. This is the simplest example of a base independence property which must of course hold for all multi–matrix models with generic potentials.

3.1.1 The 4-matrix model

For the 4-matrix model the coupling conditions are:

$$
\begin{aligned}
P_1 + 2t_1 Q_1 + u_1 + c_1 Q_2 &= 0 \\
2t_2 Q_2 + u_2 + c_1 Q_1 + c_2 Q_3 &= 0 \\
2t_3 Q_3 + u_3 + c_4 Q_4 + c_2 Q_2 &= 0 \\
\overline{\mathcal{P}_4} + 2t_4 Q_4 + u_4 + c_3 Q_2 &= 0
\end{aligned}
\tag{3.3}
$$

Eliminating the Q_2 and Q_3 matrices we get the following constraints:

$$
\begin{aligned}
P_1 \qquad & +2(t_1 - \frac{c_1^2 t_3}{4t_2 t_3 - c_2^2})Q_1 + (u_1 + c_1 \frac{2t_3 u_2 - u_3 c_2}{4t_2 t_3 - c_2^2}) - \frac{c_1 c_2 c_3}{4t_2 t_3 - c_2^2} Q_4 = 0 \\
\overline{\mathcal{P}_4} \qquad & +2(t_4 - \frac{c_3^2 t_2}{4t_2 t_3 - c_2^2})Q_4 + (u_4 - c_3 \frac{2t_2 u_3 - u_2 c_2}{4t_2 t_3 - c_2^2}) - \frac{c_1 c_2 c_3}{4t_2 t_3 - c_2^2} Q_1 = 0
\end{aligned}
$$

For Q matrices we obtain the following form:

$$
\begin{aligned}
a_0 &= \frac{s_1}{A}, \qquad & T_0^{(2)} &= \frac{s_2}{A(t_2 t_3 - c_2^2)}, \\
b_0 &= \frac{s_4}{A}, \qquad & T_0^{(3)} &= \frac{s_3}{A(t_2 t_3 - c_2^2)},
\end{aligned}
\tag{3.4}
$$

where

$$A = 4t_1t_2t_3t_4 - 4c_1^2t_3t_4 - 4c_2^2t_1t_4 - 4c_3^2t_1t_2 + (c_1c_3)^2 \tag{3.5}$$

while we do not write down the explicit expressions for the s_α's; they are linear functions of u_α, therefore when $u_\alpha = 0$ the $Q(\alpha)$ matrices are traceless. The other coordinates are:

$$T_{-1}^{(2)} = -\frac{2t_1 n}{c_1}, \qquad T_{-1}^{(3)} = n\frac{4t_1t_2 - c_1^2}{c_1c_2}, \qquad R_4 = \frac{c_1c_2c_3n}{A}$$

and

$$\begin{aligned} a_1 &= \frac{2n(c_3^2t_2 + c_2^2t_4 - 4t_2t_3t_4)}{A}, \\ T_1^{(2)} &= \frac{c_1n(4t_3t_4 - c_3^2)}{A}, \qquad T_1^{(3)} = \frac{2c_1c_2t_4n}{A} \\ b_1 &= \frac{2n(c_1^2t_3 + c_2^2t_1 - 4t_2t_3t_1)}{A}. \end{aligned}$$

Let us consider the two previous models at the *cosmological point*, i.e. when all the couplings are set to zero except the bilinear ones (the c_α's). The reason of the name is that in such a case the CF's essentially depend only on N, which is interpreted as the renormalized cosmological constant (see section 5).

We see immediately that, while such a point is well-defined for the 4–matrix model, it is singular for the 3–matrix model (in fact $A \neq 0$ but $B = 0$). These two models reveal the difference between odd and even q–matrix models. The cosmological point is well-defined only for even q–matrix models.

3.2 Gaussian q–matrix models

Let us now concentrate on the most general case (3.1). In particular μ takes the form

$$\mu = \mu(\lambda_1, \ldots, \lambda_q) = \sum_{\alpha=1}^{q} u_\alpha\lambda_\alpha + \sum_{\alpha=1}^{q} t_\alpha\lambda_\alpha^2 + \sum_{\alpha=1}^{q-1} c_\alpha\lambda_\alpha\lambda_{\alpha+1} \tag{3.6}$$

The coupling conditions are

$$P(1) + u_1 + 2t_1Q(1) + c_1Q(2) = 0 \tag{3.7a}$$

$$u_\alpha + 2t_\alpha Q(\alpha) + c_\alpha Q(l+1) + c_{\alpha-1}Q(\alpha-1) = 0, \quad \alpha = 2, \ldots, q-1 \tag{3.7b}$$

$$\bar{\mathcal{P}}(q) + u_q + 2t_qQ(q) + c_{q-1}Q(q-1) = 0 \tag{3.7c}$$

These coupling conditions imply that $Q(\alpha)$ has only three non–vanishing diagonal lines, the main diagonal and the two adjacent lines. Now let us simplify the coordinatization of such matrix as follows

$$Q(\alpha) = \epsilon_+(\alpha) + \epsilon_0(\alpha) + \epsilon_-(\alpha) \tag{3.8}$$

where

$$\epsilon_-(\alpha) = \sum_n g_\alpha(n)E_{n,n-1}, \qquad \epsilon_0(\alpha) = \sum_n s_\alpha(n)E_{n,n}, \qquad \epsilon_+(\alpha) = \sum_n h_\alpha(n)E_{n,n+1}$$

with the understanding that $h_1(n) = 1$ and $g_q(n) = R(n)$. In terms of these coordinates the above coupling equations take the form of the following linear system

$$
\begin{aligned}
&2t_1 + c_1 h_2(n) = 0 \\
&2t_1 s_1(n) + c_1 s_2(n) + u_1 = 0 \qquad (3.9a)\\
&n + 2t_1 g_1(n) + c_1 g_2(n) = 0 \\
&2t_\alpha h_\alpha(n) + c_\alpha h_{\alpha+1}(n) + c_{\alpha-1} h_{\alpha-1}(n) = 0, \qquad \alpha = 2, \ldots, q-1 \\
&2t_\alpha s_\alpha(n) + c_\alpha s_{\alpha+1}(n) + c_{\alpha-1} s_{\alpha-1}(n) + u_\alpha = 0, \quad \alpha = 2, \ldots, q-1 \qquad (3.9b)\\
&2t_\alpha g_\alpha(n) + c_\alpha g_{\alpha+1}(n) + c_{\alpha-1} g_{\alpha-1}(n) = 0, \qquad \alpha = 2, \ldots, q-1 \\
&\frac{n+1}{R(n+1)} + 2t_q h_q(n) + c_{q-1} h_{q-1}(n) = 0 \\
&2t_q s_q(n) + c_{q-1} s_{q-1}(n) = 0 \qquad (3.9c)\\
&2t_q R(n) + c_{q-1} g_{q-1}(n) = 0
\end{aligned}
$$

The solution of this system is expressed in terms of the matrices X_α and Y_α, defined as follows

$$
X_\alpha = \begin{pmatrix}
2t_1 & c_1 & 0 & \ldots & 0 & 0 \\
c_1 & 2t_2 & c_2 & \ldots & 0 & 0 \\
0 & c_2 & 2t_3 & \ldots & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & \ldots & 2t_{\alpha-1} & c_{\alpha-1} \\
0 & 0 & 0 & \ldots & c_{\alpha-1} & 2t_\alpha
\end{pmatrix} \qquad (3.10)
$$

and

$$
Y_\alpha = \begin{pmatrix}
2t_\alpha & c_\alpha & 0 & \ldots & 0 & 0 \\
c_\alpha & 2t_{\alpha+1} & c_{\alpha+1} & \ldots & 0 & 0 \\
0 & c_{\alpha+1} & 2t_{\alpha+2} & \ldots & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & \ldots & 2t_{q-1} & c_{q-1} \\
0 & 0 & 0 & \ldots & c_{q-1} & 2t_q
\end{pmatrix} \qquad (3.11)
$$

Of course $Y_1 \equiv X_q$. One finds

$$
\begin{aligned}
h_\alpha(n) &= (-1)^\alpha (c_1 c_2 \ldots c_{\alpha-1})^{-1} Det X_{\alpha-1} \\
R(n) &= (-1)^q n c_1 c_2 \ldots c_{q-1} \Big(Det X_q \Big)^{-1} \qquad (3.12)\\
g_\alpha(n) &= (-1)^\alpha n c_1 c_2 \ldots c_{\alpha-1} \frac{Det Y_{\alpha+1}}{Det X_q}
\end{aligned}
$$

Moreover, if we denote by S and U the vectors $(s_1, s_2, \ldots, s_q)^t$ and $(u_1, \ldots, u_q)^t$, respectively, we have

$$
S = -X_q^{-1} U \qquad (3.13)
$$

As we have already remarked we can always without loss of generality suppress the linear terms in u_α by constant shifts of M_α. In such a case $S = 0$.

It is now easy to see that, at the cosmological point ($t_\alpha = u_\alpha = 0$), the solution (3.12) is well defined when q is even, while it is singular when q is odd – in the latter case, for example, $DetX_q = 0$.

In the last part of this section we would like to dispel a seemingly obvious objection to the very content of this paper. Take the generic quadratic model of q matrices with nearest neighbour interactions

$$U = \sum_{\alpha=1}^{q} t_\alpha M_\alpha^2 + \sum_{\alpha=1}^{q-1} c_{\alpha,\alpha+1} M_\alpha M_{\alpha+1} \equiv \sum_{\alpha,\beta}^{q} M_\alpha A_{\alpha\beta} M_\beta . \tag{3.14}$$

The $q \times q$ matrix A is symmetric, and, for the theory of central quadrics, it can be brought to a canonical diagonal form with all ones or minus ones on the diagonal. The signature of A is of course a characteristic of the potential.

Let us see the consequences of this simple remark as far as the matrix model is concerned. The diagonalization of A can be achieved by integrating in the path integral over suitable linear combinations of the matrices M_α, instead of integrating simply over the M_α's. Of course this gives rise to a Jacobian factor, which is however one if one uses only shifts of the M_α's. In this way one brings A to the diagonal form

$$A = Diag(f_1, \ldots, f_q) \tag{3.15}$$

but does not rescale its elements to unity. However this form is sufficient for our subsequent discussion. The initial matrix model appears at this point to be equivalent to the decoupled model with potential

$$U' = \sum_{\alpha} f_\alpha M_\alpha^2 .$$

with partition function $Z = Const(N)(f_1 f_2 ... f_q)^{-N^2/2}$. We remark however that this procedure is of no help if one has to compute correlation functions of composite operators, in that it screws up the definition of the states and renders the calculation of the correlators practically impossible. The procedure followed in this paper, i.e. the use of the generalized Toda lattice hierarchy, has precisely the virtue that it allows the calculation of the exact correlators of significant composite operators.

Finally let us remark that we can easily generalize the results of this subsection to the cases when in the potential are present, beside the terms of (3.14), also interactions of the type $c_{\alpha,\beta} D_\alpha D_\beta$ where $D_\alpha = Diag\ M_\alpha$ and $\beta \neq \alpha - 1, \alpha, \alpha + 1$. In such cases the method is the same as in the chain models, with the only difference that the matrices X_α and Y_α will have, at the position (α, β), additional non–vanishing entries $c_{\alpha\beta}$ if the latter are present in the potential.

4 Correlation function of discrete sl_q states

In the previous section we have shown how to solve the coupling conditions of a given model. In this section we show how to calculate various correlation functions of composite operators (or discrete states). To start with let us illustrate a basic property of the latter: in the q matrix model they are organized in finite dimensional representations of sl_q.

4.1 sl_q symmetry of the discrete states.

We have shown in section 2.1 that we can enlarge the q–matrix model by introducing in the potential terms of the form:

$$g_{a_1,\dots a_q}\prod_{\alpha=1}^{q} D_\alpha^{a_\alpha}, \qquad with\ D_\alpha = Diag(M_\alpha)$$

We call *discrete states* the operators $\chi_{a_1,\dots a_q}$ coupled to $g_{a_1,\dots a_q}$. We introduce also $\chi_{0,\dots 0}\equiv Q$ as the operator coupled to $g_{0,\dots 0}\equiv N$. Classically, $\chi_{a_1,\dots a_q}$ is represented by $\sum_{k=1}^{N}\lambda_{1,k}^{a_1}\dots\lambda_{q,k}^{a_q}$. These states carry a built–in sl_q structure. To see this one has to consider the following generators

$$H_i = \frac{1}{2}\sum_{k=0}^{N}\left(\lambda_{i,k}\frac{\partial}{\partial\lambda_{i,k}} - \lambda_{i+1,k}\frac{\partial}{\partial\lambda_{i+1,k}}\right), \qquad 1\le i\le q-1$$

$$E_{i,j}^{+} = \sum_{i=1}^{N}\lambda_{i,k}\frac{\partial}{\partial\lambda_{j,k}}, \qquad E_{i,j}^{-} = \sum_{i=1}^{N}\lambda_{j,k}\frac{\partial}{\partial\lambda_{i,k}}, \qquad 1\le i<j\le q$$

H_i form the Cartan subalgebra of sl_q, while $E_{i,j}^{+}$ and $E_{i,j}^{-}$ are, respectively, the raising and lowering operators of the Lie algebra sl_q, corresponding to the roots:

$$\alpha_{ij} = \varepsilon_i - \varepsilon_j, \quad i<j$$

in the standard notation. The action on the states is as follows:

$$H_i\chi_{a_1\dots a_q} = \frac{1}{2}(a_i - a_{i+1})\chi_{a_1\dots a_q}, \qquad E_{i,j}^{\pm}\chi_{a_1,\dots,a_i,\dots,a_j,\dots,a_q} = \chi_{a_1,\dots,a_i\pm 1,\dots,a_j\mp 1,\dots,a_q}$$

Therefore the set $\{\chi_{a_1,\dots a_q} = \sum_{i=1}^{N}\lambda_{1,k}^{a_1}\dots\lambda_{q,k}^{a_q},\ \ \sum_{i=1}^{q}a_i = n\}$ form an (unnormalized) representation of this algebra of dimension $\underline{\left(\begin{array}{c} n+q-1 \\ n \end{array}\right)}$.

Although everything we do here can be repeated for q–matrix model with q odd, we concentrate from now on on the far more interesting case of even q. The main reason for this is the well-definedness of the cosmological point when q is even. This will allow us to give an unambiguous topological field theory interpretation of the corresponding matrix models, while such an interpretation does not seem to be possible for odd q. Therefore, from now on, unless otherwise specified we consider $2q$–matrix models.

4.2 General properties of correlators

The correlation functions of the extended multi-matrix model are in general defined by

$$< \chi_{a_1^{(1)},\dots a_{2q}^{(1)}}\dots\chi_{a_1^{(n)},\dots a_{2q}^{(n)}} >= \frac{\partial}{\partial g_{a_1^{(1)},\dots a_{2q}^{(1)}}}\dots\frac{\partial}{\partial g_{a_1^{(n)},\dots a_{2q}^{(n)}}} lnZ_N$$

Our purpose in this section is to calculate the correlation functions in two simple special cases: *the pure chain models* where we set $g_{a_1,\dots a_{2q}} = 0$ except for $g_{0\dots a_\alpha a_{\alpha+1}\dots 0}\equiv c_\alpha$ and *the*

quadratic models where we have also the following nonzero coupling constants $g_{0\dots a_\alpha=2\dots 0}\equiv t_\alpha, g_{0\dots a_\alpha=1\dots 0}\equiv u_\alpha$. As a consequence the CF's will be functions of $c_\alpha, t_\alpha, u_\alpha$ and N. The chain models were referred to above as the *cosmological point* of the relevant $2q$–mm, while the quadratic models can be considered as quadratic perturbations of the latter. This second terminology is related to the topological field theory interpretation of section 5.

To see some general properties of the CF's, it is convenient to use the W–constraints (see Appendix A). We write down the W constraints in terms of them and obtain a set of (overdetermined) algebraic equations which in general one can solve recursively.

The CF's, in the chain models, have the following symmetry property:

$$\begin{aligned}&< \chi_{a_1^{(1)},\dots a_i^{(1)}\dots a_j^{(1)}\dots a_{2q}^{(1)}}\dots\chi_{a_1^{(n)},\dots a_i^{(n)}\dots a_j^{(n)}\dots a_{2q}^{(n)}} >\\ &\qquad =< \chi_{a_1^{(1)},\dots a_j^{(1)}\dots a_i^{(1)}\dots a_{2q}^{(1)}}\dots\chi_{a_1^{(n)},\dots a_j^{(n)}\dots a_i^{(n)}\dots a_{2q}^{(n)}} >\end{aligned}$$

This is due to the symmetry of the W constraints and to the invariance of the chain models under the exchange $i\leftrightarrow j$.

In the chain models the CF's satisfy (charge conservation):

$$\sum_{\alpha=1}^{q}[(a_{2\alpha-1}^{(1)}+\dots a_{2\alpha-1}^{(n)})-(a_{2\alpha}^{(1)}+\dots a_{2\alpha}^{(n)})]<\chi_{a_1^{(1)}\dots a_{2q}^{(1)}}\dots\chi_{a_1^{(n)}\dots a_{2q}^{(n)}}>=0 \tag{4.1}$$

To prove the last statement we rewrite the $W_0^{[1]}$ constraint as follows:

$$\begin{aligned}&\sum_{a_1\geq 1, a_2\geq 0\dots a_{2q}\geq 0} a_1 g_{a_1\dots a_{2q}}<\chi_{a_1\dots a_{2q}}>+\frac{1}{2}N(N+1)=0\\ &\sum_{a_1\geq,\dots a_\alpha\geq 1\dots a_{2q}\geq 0} a_\alpha g_{a_1\dots a_{2q}}<\chi_{a_1\dots a_{2q}}>=0,\qquad 2\leq\alpha\leq 2q-1\\ &\sum_{a_1\geq 0, a_{2q-1}\geq 0 a_{2q}\geq 1} a_q g_{a_1\dots a_{2q}}<\chi_{a_1\dots a_{2q}}>+\frac{1}{2}N(N+1)=0\end{aligned}$$

We differentiate these equations w.r.t. $g_{a_1^{(1)}\dots a_{2q}^{(1)}}$, …, $g_{a_1^{(n)}\dots a_{2q}^{(n)}}$ and set $g_{a_1\dots a_{2q}}=0$ except $g_{0\dots a_\alpha a_{\alpha+1}0}=c_\alpha$. One gets

$$\begin{aligned}&\sum_{k=1}^{n}a_1^{(k)}<\chi_{a_1^{(1)}\dots a_{2q}^{(1)}}\dots\chi_{a_1^{(n)}\dots a_{2q}^{(n)}}>+c_1<\chi_{110\dots 0}\chi_{a_1^{(1)}\dots a_{2q}^{(1)}}\dots\chi_{a_1^{(n)}\dots a_{2q}^{(n)}}>=0\\ &\sum_{k=1}^{n}a_\alpha^{(k)}<\chi_{a_1^{(1)}\dots a_{2q}^{(1)}}\dots\chi_{a_1^{(n)}\dots a_{2q}^{(n)}}>+<(c_{\alpha-1}\chi_{0\dots 1,a_\alpha=1\dots 0}+\\ &\qquad c_\alpha\chi_{0\dots a_\alpha=1,1\dots 0})\chi_{a_1^{(1)}\dots a_{2q}^{(1)}}\dots\chi_{a_1^{(n)}\dots a_{2q}^{(n)}}>=0,\qquad 2\leq\alpha\leq 2q-1\\ &\sum_{k=1}^{n}a_{2q}^{(k)}<\chi_{a_1^{(1)}\dots a_{2q}^{(1)}}\dots\chi_{a_1^{(n)}\dots a_{2q}^{(n)}}>+c_{2q}<\chi_{0\dots 011}\chi_{a_1^{(1)}\dots a_{2q}^{(1)}}\dots\chi_{a_1^{(n)}\dots a_{2q}^{(n)}}>=0\end{aligned}$$

Subtracting the even equations from the odd ones we obtain the result.

The last property partially reflects the sl_{2q} structure of the discrete states as it means, at the cosmological point, the conservation of the eigenvalue of $H=H_1+H_3+\dots+H_{2q-1}$.

In the remaining part of this section we are going to compute exact correlators, i.e. all–genus expressions, from which we can extract the genus by genus expansion. To obtain such

an expansion it is of fundamental importance that we can assign to each coupling a degree, denoted $[\cdot]$, as follows

$$[g_{a_1,\ldots,a_q}] = y + \sum_{i=1}^{q} y_i a_i, \qquad [N] = y \tag{4.2}$$

and for each quantity such as free energies, correlators, etc., we can define a genus expansion, each genus contribution having a definite degree,

$$F = lnZ_N, \qquad F = \sum_{h=0}^{\infty} F_h, \qquad [F_h] = 2(1-h)y \tag{4.3}$$

In eqs.(4.2,4.3), y, y_i are arbitrary nonvanishing real numbers.

4.3 The 1-point correlation functions

To find explicit expressions of the correlators it is more convenient to switch from the $W-$ constraints to the method based on the solution of the coupling conditions.

4.3.1 Pure chain models

We specialize here to the case in which all the couplings, except the bilinear ones, are turned off:

$$t_{r,\alpha} = 0. \tag{4.4}$$

In this case the coupling conditions take the form:

$$\begin{aligned} P_1 + c_1 Q_2 &= 0, \\ c_{\alpha-1} Q_{\alpha-1} + c_\alpha Q_{\alpha+1} &= 0 \quad (\alpha = 2, \ldots, 2q-1) \\ \bar{\mathcal{P}}_{2q} + c_{2q} Q_{2q-1} &= 0 \end{aligned} \tag{4.5}$$

This linear system is so simple that we do not need to rely on the formulas of the previous section. We note that Q_1 has only the first upper diagonal, and P_1, which represents a derivative, has only the first lower one

$$Q_1 = I_+ = \begin{pmatrix} 0 & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & \ddots \end{pmatrix} \qquad P_1 = \epsilon_- = \begin{pmatrix} 0 & & & & \\ 1 & & & & \\ & 2 & & & \\ & & 3 & & \\ & & & \ddots & \end{pmatrix}$$

The first coupling condition (4.5) gives now $Q_2 = -1/c_1 \epsilon_-$. Using the second for $\alpha = 3$ one finds then Q_4, and so on for all the even Q's up to Q_{2q}. For the odd ones the procedure is the same starting from Q_1.

$$\begin{aligned} Q_{2k+1} &= (-1)^k \frac{c_1}{c_2} \ldots \frac{c_{2k-1}}{c_{2k}} I_+ \\ Q_{2k} &= (-1)^k \frac{c_2}{c_1} \ldots \frac{1}{c_{2k-1}} \epsilon_- \end{aligned} \tag{4.6}$$

Now we come to the correlation functions which are expressed in terms of the Q-matrices by means of the formula:

$$< \chi_{a_1,\ldots,a_{2q}} >= Tr({Q_1}^{a_1}{Q_2}^{a_2}\ldots{Q_{2q}}^{a_{2q}}). \tag{4.7}$$

Due to the particular form of the Q's it is immediate to verify the conservation law (4.1). In order to have nonvanishing trace the number of I_+ and I_- must be the same, i.e.

$$a_1 + a_3 + \ldots + a_{2q-1} - a_2 - a_4 - \ldots - a_{2q} = 0$$

The result for the one point functions can be found by means of the identities $[I_+, \epsilon_-] = I_0$ and $I_+\epsilon_- = \underline{n}$ (I_0 being the identity matrix and $\underline{n} = diag(1, 2, 3, ...)$):

$$\begin{aligned} < \chi_{a_1,\ldots,a_{2q}} > \quad = \sum_{n=x}^{N-1} \quad & (n + a_1 - a_2)(n + a_1 - a_2 + 1)\ldots(n + a_1 - 1) \\ & (n + a_1 - a_2 + a_3 - a_4)\ldots(n + a_1 - a_2 + a_3 - 1)\ldots \\ & (n + a_1 - a_2 + \ldots + a_{2q-1} - a_{2q})\ldots(n + a_{2q} - 1) \end{aligned} \tag{4.8}$$

where $x = -min[a_1 - a_2, a_1 - a_2 + a_3 - a_4, \ldots, a_1 - a_2 + ... - a_{2q} = 0]$.

4.3.2 Quadratic models

We write down some special 1-point correlation functions in the quadratic models. The derivation can be found in Appendix B.

$$< \tau_{\alpha,r} >= \mathrm{Tr}(Q_\alpha^r) = \sum_{2l=0}^{r}\sum_{k=0}^{l} \frac{(-1)^k 2^{-k} r!}{(r-2l)!k!(l-k)!} \begin{pmatrix} N+l-k \\ l-k+1 \end{pmatrix} (h_\alpha g_\alpha)^l s_\alpha^{r-2l} \tag{4.9}$$

where h_α, g_α are defined by eqs.(3.12).

This is an all–genus expression. In order to extract the genus h contribution follow the above described recipe. In particular for the 1-point functions we have the following expansion

$$< \tau_{\alpha,k} >= \sum_{h=0}^{\infty} < \tau_{\alpha,k} >_h \; N^{1+k-2h}$$

The only dependence on N comes from $\begin{pmatrix} N+r \\ r=1 \end{pmatrix}$ and we can expand it as follows

$$\begin{pmatrix} N+r \\ r+1 \end{pmatrix} = \sum_{h=0}^{\infty} N^{1+k-2h} b_{2h}(k) \tag{4.10}$$

Using the last relation we can extract the genus h contribution:

$$\begin{aligned} < \tau_{\alpha,r} >_h = \mathrm{Tr}(Q_\alpha^r) \quad = \quad & \sum_{2l=0}^{r}\sum_{k=0}^{l} \frac{(-1)^k 2^{-k} r! b_{2h}(l-k+1)}{(r-2l)!k!(l-k)!} N^{l-k+1-2h} \cdot \\ & \cdot (h_\alpha g_\alpha)^l s_\alpha^{r-2l} \end{aligned} \tag{4.11}$$

with:

$$b_{2h}(n) = \sum_{1 \le r_1 \ldots r_{2h} \le n} r_1 r_2 \ldots r_{2h}, \qquad b_0(n) = 1$$

4.4 Two-point functions

4.4.1 Pure chain models

For the 2–point functions we have to use eq.(2.27). As an example we calculate the correlation functions of the form:

$$< \chi_{0\dots a_\alpha=r\dots 0}\chi_{0\dots a_\beta=s\dots 0} >=< \tau_{\alpha,r}\tau_{\beta,s} >$$

The formula (2.27) becomes, in this case,

$$< \chi_{0\dots a_\alpha=r\dots 0}\chi_{0\dots a_\beta=s\dots 0} >= \begin{cases} Tr([(Q_\alpha^r)_+, Q_\beta^s]) & (\alpha < \beta) \\ Tr(-[(Q_\alpha^r)_-, Q_\beta^s]) & (\alpha \geq \beta) \end{cases}$$

We take first the case α=even, ($Q_\alpha \simeq \epsilon_-$) and β=odd ($Q_\beta \simeq I_+$) and the two point function is written as:

$$< \tau_{\alpha,r}\tau_{\beta,s} >= Tr([Q_\alpha^r, Q_\beta^s]) \quad (\alpha > \beta \text{ and zero otherwise}) \tag{4.12}$$

Here the number of Q_α and Q_β in each trace must be the same, to balance the I_+ and ϵ_-'s (remember that α and β have different parity) so that r and s are forced to be equal. The traces can be evaluated as above and one gets:

$$\begin{aligned} < \tau_{\alpha,r}\tau_{\beta,s} > &= (q_\alpha q_\beta)^r Tr[(\epsilon_-)^r, I_+^r]\delta_{r=s} = \\ &= (q_\alpha q_\beta)^r \delta_{r=s} \left(\sum_{n=1}^{N-r} - \sum_{n=1}^{N}\right) n(n+1)...(n+r-1) \end{aligned} \tag{4.13}$$

where $q_\alpha = \frac{c_2}{c_1}\frac{c_4}{c_3}...\frac{1}{c_{\alpha-1}}$, $q_\beta = \frac{c_1}{c_2}\frac{c_3}{c_4}...\frac{c_{\beta-2}}{c_{\beta-1}}$.

When α is odd and β even the result is the same with r exchanged with s. When α and β are either both even or both odd the 2-point function identically vanishes.

As an example:

$< \tau_{1,r}\tau_{2,s} >= \delta_{rs}\left(\sum_{n=1}^{N-r} - \sum_{n=1}^{N}\right) n(n+1)...(n+r-1)$

$< \tau_{5,r}\tau_{2,s} >= 0$ because $\alpha = 2 < \beta = 5$.

4.4.2 Quadratic models

The 2-point correlation functions is a very important quantity in matrix models because its singularity indicate the existence of critical points and its scaling near them evaluate the anomalous dimensions of corresponding operators. In our case 2-point correlation functions permits also direct the calculation of the metric for the associated topological model (when the puncture operator is $Q = \frac{\partial}{\partial N}$).

Using again the equation (2.27) we write down the two-point correlation functions:

$$< \tau_{1,r}\tau_{\alpha,s} >= \mathrm{Tr}[(Q_1^r)_+, (Q_\alpha^s)_-] \tag{4.14}$$

Using the form of the Q matrices (3.8,3.12) we get the result:

$$< \tau_{1,r}\tau_{\alpha,s} >= \sum_{\substack{0\leq 2l\leq r \\ 0\leq 2l'\leq s}} \sum_{0\leq k\leq l \atop 0\leq k'\leq l'} \sum_{\substack{0\leq i\leq l-k \\ l'-k'\leq j\leq 2(l'-k') \\ i+j=l+l'-k-k'}}$$

$$\left(\begin{array}{c} r \\ 2l \end{array}\right)\left(\begin{array}{c} s \\ 2l' \end{array}\right)\left(\begin{array}{c} 2l-2k \\ i \end{array}\right)\left(\begin{array}{c} 2l'-2k' \\ j \end{array}\right)(-1)^{k+k'}A_k^{(2l)}A_{k'}^{(2l')}f(S_1,T_1,S_\alpha,T_\alpha,R_\alpha)$$
$$i!j!\sum_{n=0}^{N-1}\left[\left(\begin{array}{c} n+2l-2k-i \\ i \end{array}\right)\left(\begin{array}{c} n+j \\ j \end{array}\right)-\left(\begin{array}{c} n+2l'-2k'-j \\ j \end{array}\right)\left(\begin{array}{c} n+i \\ i \end{array}\right)\right]$$

where $f(\alpha)$ is:

$$f(\alpha)=(g_1h_\alpha/g_al)^{i+k}g_\alpha^{l'+l}s_0^{r-2l}h_\alpha^{l'-l}s_\alpha^{s-2l}, \qquad \alpha=1\ldots q$$

To calculate the 2-point correlation we needed the quantity $Tr(I_+^n\epsilon_-^mI_+^p\epsilon_-^q)$. We use the fact that for $(n>m)$:

$$I_+^n\epsilon_-^m=\sum_{k=0}^{m}\epsilon_-^kI_+^{n-m-k}A_k^{(n,m)}, \quad A_k^{(n,m)}=\frac{n!m!}{k!(n-m+k)!(m-k)!} \tag{4.15}$$

Using this sum we get :

$$Tr(I_+^n\epsilon_-^mI_+^p\epsilon_-^q)=\sum_{k=0}^{m}\frac{n!m!(q+k)!}{k!(n-m+k)!(m-k)!}\left(\begin{array}{c} N \\ q+k+1 \end{array}\right)\delta_{n+p,m+q} \tag{4.16}$$

The evaluation of higher genus contribution follows the same method we have followed at the calculation of 1-point correlation functions. The only dependence of N comes from $\sum_{p=0}^{N-1}\left(\begin{array}{c} p+s \\ s \end{array}\right)\left(\begin{array}{c} p+n \\ m \end{array}\right)$ and we are looking for the contribution of order N^{1+k-2h}.

We define the function $B(r,s|n,m)$ as the coefficient of the genus h expansion:

$$\sum_{p=0}^{N-1}\left(\begin{array}{c} p+s \\ s \end{array}\right)\left(\begin{array}{c} p+n \\ m \end{array}\right)=N^{m+s-2h}B_h(r,s|n,m), \quad r+s=n+m \tag{4.17}$$

The explicit expression is:

$$B_h(r,s|r',s')=\textstyle\sum_{l=0}^{s+s'}\gamma_l(s,r,s')\left[\frac{1}{2}\left(\begin{array}{c} s+s'-l \\ 2h-l \end{array}\right)(r-s)^{2h-l}+\right.$$
$$+\tfrac{(r-s)^{1+2h-l}}{s+s'-l+1}\left(\begin{array}{c} s+s'-l+1 \\ 1-2h-l \end{array}\right)+$$
$$\left.+\textstyle\sum_{2\le 2t\le s=s'-l}\frac{B_{2t}}{2t}\left(\begin{array}{c} s+s'-l \\ 2t-1 \end{array}\right)\left(\begin{array}{c} s+s'-l-2t+1 \\ 2h-l-2t+1 \end{array}\right)(r-s)^{2h-l-2t+1}\right]$$

$(r+r'=s+s')$; where B_{2t} are Bernoulli numbers and γ are:

$$\gamma_l(s,r,s')=\sum_{k=0}^{l}\sum_{0\le i_1\ldots i_k\le s-1}\ \sum_{n-m\le j_{k+1}\ldots j_l\le n-1} i_1\ldots i_kj_{k+1}\ldots j_l$$

The genus h contribution is:

$$<\tau_{1,r}\tau_{\alpha,s}>_h=\sum_{\substack{0\le 2l\le r\\ 0\le 2l'\le s}}\ \sum_{\substack{0\le k\le l\\ 0\le k'\le l'}}\ \sum_{\substack{0\le i\le l-k\\ l'-k'\le j\le 2(l'-k')\\ i+j=l+l'-k-k'}}$$

$$\frac{r!s!(-1)^{k+k'}2^{-(k+k')}}{(r-2l)!(s-2l')!(2l-2k-i)!(2l'-2k'-j)!k!k'!i!j!}f(\alpha)N^{l+l'-k-k'-2h}$$
$$(B_h(2(i+k-l)+j,j|2l-2k-i,i)-B_h(2(j+k'-l')+i,i|2l'-2k'-j,j))$$

5 Topological field theory properties of $2q$–matrix models

.

We study in this section the content of $2q$–matrix models in terms topological field theories. The motivation is offered by the example of 2–matrix model, which can be interpreted as a topological field theory with an infinite number of primary fields, [9]. We want to see whether a similar conclusion can be drawn also for multi–matrix models. The easiest way to identify a possible topological field theory (TFT) content is to go to the cosmological point. We have seen previously that such a point is not well defined for odd q multi–matrix models. Consequently, in this section, we concentrate on even q multi–matrix models. To be definite we start with the 4–matrix model. We recall that the cosmological point is identified by setting all the couplings to zero except the bilinear ones, $c_{\alpha,\alpha+1}$, with reference to eq.(2.1). To simplify things further we set from now on

$$c_{\alpha,\alpha+1}=(-1)^{\alpha}$$

without loss of generality (one can obtain the same results by suitable rescaling the couplings of the discrete states). Finally we replace N by a continuous variable x (i.e. we pass to a continuous formalism by suitably rescaling all the quantities and taking $N\to\infty$; x is the renormalized quantity that replaces N).

After these preliminaries let us concentrate on the 4–matrix models. Among the discrete states, our candidates for primary states are $\{\psi_{a,b},Q,\omega_{c,d}\}$, where

$$\psi_{a,b}=\chi_{a,0,b,0},\qquad \psi_{c,d}=\chi_{0,c,0,d}$$

The relevant genus 0 correlators to study the TFT properties can be computed from (2.27) and (2.28)

$$\begin{aligned}<\psi_{a_1,b_1}\psi_{a_2,b_2}\omega_{c,d}> &= \Big((a_1+b_1)(a_2+b_2)(c+d)-c(a_1b_2+a_2b_1+b_1b_2)\\ &\quad -\frac{b_1b_2cd}{c+d-1}\Big)x^{c+d-1}\delta_{a_1+a_2+b_1+b_2,c+d} \qquad (5.1a)\\ <\psi_{a,b}\omega_{c_1,d_1}\omega_{c_2,d_2}> &= \Big((a+b)(c_1+d_1)(c_2+d_2)-b(c_1d_2+c_2d_1+c_1c_2)\\ &\quad -\frac{abc_1c_2}{a+b-1}\Big)x^{a+b-1}\delta_{a+b,c_1+c_2+d_1+d_2} \qquad (5.1b)\end{aligned}$$

and

$$<Q\psi_{a,b}\omega_{c,d}>=(ac+ad+bd)x^{a+b-1}\delta_{a+b,c+d} \qquad (5.2)$$

We will also need $<QQQ>=x^{-1}$, which follows from the fact that the correlators involving only Q are the same as in the 2–matrix model, see [8].

Now, at any level $r = a + b$ let us select an arbitrary state among the $\psi_{a,b}$'s and call it ψ_r, $r > 0$. Let us call $\mathcal{C}$ the collection of such choices for any r. Moreover, let us set $\omega_s \equiv \omega_{0,s}$. Then the states $\{\psi_r, Q, \omega_s\}$ constitute the set of primary states of a TFT with puncture operator either Q or ψ_1 or ω_1. This can be seen as follows. The non–vanishing structure constants are

$$
\begin{aligned}
&C_{r_1,r_2,\bar{s}} \equiv < \psi_{r_1}\psi_{r_2}\omega_s > = r_1 r_2 s x^{s-1}\delta_{s,r_1+r_2} \\
&C_{r,\bar{s}_1,\bar{s}_2} \equiv < \psi_r\omega_{s_1}\omega_{s_2} > = r s_1 s_2 x^{r-1}\delta_{r,s_1+s_2} \\
&C_{0,r,\bar{s}} \equiv < Q\psi_r\omega_s > = r s x^{r-1}\delta_{r,s}, \qquad C_{0,0,0} \equiv < QQQ > = x^{-1}
\end{aligned}
$$

together with the ones obtained from these by permutation of the indices. Now, if the puncture operator is Q, the metric is

$$
\eta_{r,\bar{s}} = \eta_{\bar{s},r} \equiv < Q\psi_r\omega_s > = r s x^{r-1}\delta_{r,s}, \qquad \eta_{0,0} = x^{-1}, \tag{5.3}
$$

If the puncture operator is ψ_1, the metric is

$$
\eta_{r,\bar{s}} = \eta_{\bar{s},r} \equiv < \psi_1\psi_r\omega_s > = r s x^{r}\delta_{s,r+1}, \qquad \eta_{0,\bar{1}} = \eta_{\bar{1},0} = 1 \tag{5.4}
$$

The case when the puncture operator is ω_1 is exactly specular to the latter. These three cases, with exactly the same formulas for structure constants and metric, were met in [9], where it was proven that the inverse metric exists and the associativity conditions are satisfied. The TFT obtained with a definite choice $\mathcal{C}$ will be denoted $\mathcal{T}_{\mathcal{C}}$. If necessary one can specify the symbol of the relevant puncture operator.

Similarly, among the states $\omega_{c,d}$, $c + d = s$ let us choose an arbitrary one and let us call it $\bar{\omega}_s$, $s > 0$. Let us call $\bar{\mathcal{C}}$ such a choice for any level s. Moreover, let us set $\psi_{r,0} \equiv \bar{\psi}_r$, $r > 0$. Once again the states $\{\bar{\psi}_r, Q, \bar{\omega}_s\}$ constitute the primary states of a TFT with puncture operator either Q or ψ_1 or ω_1. We do not need to explicitly prove this since the formulas for the structure constants and the metrics are the same as the previous ones with the substitutions $\psi_r \to \bar{\psi}_r$ and $\omega_s \to \bar{\omega}_s$. The TFT obtained with a definite choice $\bar{\mathcal{C}}$ will be denoted $\mathcal{T}_{\bar{\mathcal{C}}}$.

We can think of $\mathcal{T}_{\mathcal{C}}$ and $\mathcal{T}_{\bar{\mathcal{C}}}$ as unperturbed TFT's to which we couple topological gravity. Therefore we are going to have puncture equations and recursion relations. The latter are the same as in [9] and will not be repeated here. The former can be derived from the W constraints. Instead of writing the most general formula, we write down the simplest one for the puncture operator ψ_1

$$
< \psi_1\chi_{a_1,a_2,a_3,a_4} > = a_2 < \chi_{a_1,a_2-1,a_3,a_4} > + a_4 < \chi_{a_1,a_2,a_3,a_4-1} >
$$

from which one can infer the action of the puncture: ψ_1 lowers the even indices by 1. Therefore, when ψ_1 is the puncture operator, the descendants of $\psi_{a,b}$ are going to be $\chi_{a,n,b,m}$ for positive n and m, while any $\omega_{c,d}$ may be simultaneously primary and descendant, or an isolated primary.

We notice that the situation here is an interesting generalization of the situation in 2–matrix model, [9], where we have an infinity of primary states denoted $\{T_n, Q, T_{-m}\}$, with nonnegative integer n and m, where T_n, T_{-m} are the discrete tachyonic states. Here we have ∞^2 primary states, which depend on two integral indices and could be referred to as colored tachyons. In 2–matrix model, via reduction, one obtains an infinite set of TFT models (the n–KdV models) coupled to topological gravity, whose primary and descendants are to be

found among the T_n's (or, symmetrically, among the T_{-n}'s), [11]. Similarly here we expect that, via reduction (see next section), the set $\psi_{a,b}$ with a and b positive, may support a series of matter TFT's coupled to topological gravity (i.e. primaries and descendants). Due to its characteristics – triangular structure of the primaries and relation with the product of two n–KdV models – possible candidates (certainly not the only ones) are, for example, the W_3 topological minimal models coupled to topological gravity, [18],[19].

In general, if we pass to $2q$–matrix models, the set of primaries will be represented by the states $\{\chi_{a_1,0,a_3,0,\ldots,a_{2q-1},0}\}$, by Q and by $\{\chi_{0,a_2,0,\ldots,0,a_{2q}}\}$: the primaries are ∞^q. This q should be related to the target space dimension in a string interpretation. In analogy with our previous discussion we are lead to speculate that one of the two sets above can accomodate the states of the W_{q+1} minimal models coupled to topological gravity or anlogous TFT's.

6 Non–Gaussian matrix models.

The Gaussian version of q–matrix models is sufficient to study many properties, in particular it is enough to identify the TFT character of these models. From this point of view adding new interaction terms amounts to switching on new perturbations, which is not a very interesting complication in itself. However, if we come to reductions, i.e. if we try to extract TFT's with a finite number of primaries from the infinite TFT's that characterize the multi–matrix models, we are obliged to introduce non–Gaussian interaction terms. In this paper we limit ourselves to an example: our purpose is to show both the complexity inherent in non–Gaussian perturbations and a possible way to circumvent it.

The example consists of switching on a cubic potential in the 4–matrix model. More precisely we study the model $\mathcal{M}_{3,2,2,2}$. The coupling conditions are the same as (3.4), except that in the first equation a term $3v_1Q_1^2$ must be added, where $v_1 \equiv t_{1,3}$ is the coupling of the cubic term in the potential. Consequently the coupling conditions become a non–linear system of equations for the Q_α's. Eliminating Q_2 and Q_3 we obtain:

$$\begin{aligned} &P_1 + 2(t_1 - \frac{c_1^2 t_3}{4t_2t_3 - c_2^2})Q_1 + (u_1 + c_1\frac{2t_3u_2 - u_3c_2}{4t_2t_3 - c_2^2}) - \frac{c_1c_2c_3}{4t_2t_3 - c_2^2}Q_4 + 3v_1Q_1^2 = 0 \\ &\overline{P_4} + 2(t_4 - \frac{c_3^2 t_2}{4t_2t_3 - c_2^2})Q_4 + (u_4 - c_3\frac{2t_2u_3 - u_2c_2}{4t_2t_3 - c_2^2}) - \frac{c_1c_2c_3}{4t_2t_3 - c_2^2}Q_1 = 0 \end{aligned} \tag{6.5}$$

We can therefore write this system in a simplified form as follows:

$$\begin{aligned} &P_1 + 3v_1Q_1^2 + 2\tilde{t}_1Q_1 + \tilde{u}_1 + \tilde{c}Q_4 + = 0 \\ &\overline{P_4} + 2\tilde{t}_4Q_4 + \tilde{u}_4 - \tilde{c}Q_1 = 0 \end{aligned} \tag{6.6}$$

These are formally the coupling conditions of the model $\mathcal{M}_{3,2}$ with Q_2 replaced by Q_4, the couplings being suitably renormalized, [11].

With reference to the coordinatization (2.13) the equations (6.6) in genus 0 become

$$\begin{aligned} a_1(n) &= -\frac{2\tilde{t}_4}{\tilde{c}}R(n), \qquad b_0(n) = -\frac{\tilde{u}_4 + \tilde{c}a_0(n)}{2\tilde{t}_4} \\ b_1(n) &= -\frac{n + \tilde{c}R(n)}{2\tilde{t}_4}, \qquad b_2(n) = -\frac{3v_1}{\tilde{c}}R(n)^2 \end{aligned}$$

and the recursion relations

$$2a_0(n) = -\frac{2\tilde{t}_1}{3v_1} + \frac{\tilde{c}}{6\tilde{t}_4 v_1}\Big(\tilde{c} + \frac{n}{R(n)}\Big) \tag{6.7}$$

$$2R(n) = \frac{\tilde{c}}{2\tilde{t}_4}a_0(n)^2 + \Big(\frac{2\tilde{c}\tilde{t}_1}{6\tilde{t}_4 v_1} - \frac{\tilde{c}^3}{12\tilde{t}_4^2 v_1}\Big)a_0(n) - \frac{\tilde{c}^2\tilde{u}_4}{12\tilde{t}_4^2 v_1} + \frac{\tilde{c}\tilde{u}_1}{6\tilde{t}_4 v_1} \tag{6.8}$$

We have therefore to solve a cubic equation. Once we have done this all the unknowns can be determined and the explicit form of the matrices Q_α can be calculated. The correlators (in genus 0) can be obtained as integrals of algebraic equations. Writing down such expressions is not very interesting. We can however ask ourselves whether in some region of the coupling space we can find some interesting solution. This is actually the case.

Let us first simplify the formulas by imposing $6v_1 = -1$, $\tilde{t}_1 = \tilde{u}_4 = 0$. It can be shown, [11], that this is no loss of generality. Then we impose the constraint

$$a_0 = 0 \tag{6.9}$$

The above equations then imply

$$a_1 = \tilde{u}_1, \qquad R(n) = -\frac{n}{\tilde{c}}, \qquad b_0 = b_1 = 0, \qquad b_2(n) = \frac{n^2}{2\tilde{c}^3}$$

and, therefore,

$$\tilde{u}_1 = \frac{2\tilde{t}_4}{\tilde{c}^2}n \tag{6.10}$$

In general, when we impose constraints on the fields or on the coupling space, we are not allowed to use the flow equations of the integrable hierarchy (in this case the Toda lattice hierarchy), because such constraints might deform the dynamics in a non–integrable manner. However one can prove that, in the case of the constraint (6.9) or, which is the same, (6.10), the constrained dynamics is still integrable and coincides with the Toda flows constrained by eq.(6.9): the resulting hierarchy is the KdV hierarchy, [11]. In other words, if we look at our 4–mm in the submanifold of the parameter space specified by eq.(6.10), the correlators are those of the KdV model. As is well known this model has only one primary field, ψ_1, and, in particular, $< \psi_1\psi_1 >= a_1 = \tilde{u}_1$.

By considering the model $\mathcal{M}_{p+2,2,2,2}$ one can find the p-th critical point of the KdV series. One can also identify higher KdV models. These are all (trivial) generalizations of the 2–mm. However, as we have remarked above, the structure of the q–matrix models allows for more complex and interesting reductions which will be the object of future research.

7 The discretized 1D string

It is interesting to return to the problem of discretized 1D string theory within the present formalism and see how we recover the results already obtained with other methods.

Let us start by studying and solving the following Gaussian partition function:

$$Z = \int DM_i \exp\left[Tr\left(\frac{\tilde{t}}{2}\sum_{i=1}^{2q} M_i^2 + c\sum_{i=1}^{2q-1} M_i M_{i+1}\right)\right] \tag{7.1}$$

The corresponding coupling conditions are ($t = \tilde{t}/c$):

$$\begin{aligned} P_1 &+ ctQ_1 + cQ_2 = 0, \\ tQ_i &+ Q_{i-1} + Q_{i+1} = 0, \quad i = 2, \ldots 2q-1 \\ \overline{\mathcal{P}_{2q}} &+ ctQ_{2q} + cQ_{2q-1} = 0 \end{aligned} \tag{7.2}$$

We express all Q_α matrices $\alpha = 2, \ldots 2q-1$ in terms of Q_1 and Q_{2q}. For this purpose we introduce the determinant of $n \times n$ matrix:

$$D_n = \begin{vmatrix} t & 1 & 0 & \ldots & \ldots & 0 \\ 1 & t & 1 & 0 & \ldots & 0 \\ 0 & 1 & t & \ddots & . & . \\ . & 0 & 1 & \ddots & 1 & 0 \\ 0 & \ldots & 0 & \ddots & t & 1 \\ 0 & & \ldots & 0 & 1 & t \end{vmatrix}$$

The determinant D_n satisfies the recursion relation:

$$\begin{aligned} D_{n+1} &= tD_n - D_{n-1} \\ D_0 &= 1, D_1 = t \end{aligned}$$

The expression of D_n is :

$$D_n = \sum_{k=0}^{[n/2]} t^{n-2k} (-1)^k \begin{pmatrix} n-k \\ k \end{pmatrix}$$

or:

$$D_n = \frac{r_1^{n+1} - r_2^{n+1}}{r_1 - r_2}$$

where r_1, r_2 are the roots of the second order equation $r^2 - tr + 1 = 0$.

Solving the system (7.3) we get ($D_0 = 1$):

$$Q_i = \frac{(-1)^{i+1}}{D_{2q-2}} (Q_1 D_{2q-i-1} - Q_{2q} D_{i-2}), i = 2, \ldots 2q-1$$

For Q_1, Q_{2q} we get the usual 2-matrix model with quadratic potential coupling conditions:

$$\begin{aligned} P_1 + \left(ct - \frac{D_{2q-3}}{D_{2q-2}} \right) Q_1 + \frac{c}{D_{2q-2}} Q_{2q} = 0 \\ \overline{\mathcal{P}_{2q}} + \left(ct - \frac{D_{2q-3}}{D_{2q-2}} \right) Q_{2q} + \frac{c}{D_{2q-2}} Q_1 = 0 \end{aligned}$$

The Q_1, Q_{2q} matrices are:

$$\begin{aligned} Q_1 &= I_+ + g_1(n)\epsilon_-, \\ Q_{2q} &= h_{2q}(n) I_+ + R_{2q}(n)\epsilon_- \end{aligned}$$

where:

$$I_+ = \sum_{n=0}^{\infty} E_{n,n+1}, \epsilon_- = \sum_{n=0}^{\infty} nE_{n,n-1},$$

with

$$g_1(n) = h_{2q}(n)R_{2q}(n) = -2\frac{(ctD_{2q-2} - D_{2q-3})D_{2q-2}}{B}, \quad R_{2q}(n) = \frac{cD_{2q-2}}{B}$$

and

$$B = c^2(4t^2-1)D_{2q-2}^2 - 8ctD_{2q-2}D_{2q-3} + D_{2q-3}^2$$

The form of Q_α matrices is:

$$Q_\alpha = h_\alpha(n)I_+ + g_\alpha(n)\epsilon_-, \quad \alpha = 2\ldots 2q-1$$

with

$$h_\alpha(n) = \frac{(-1)^{\alpha+1}}{cB}[cD_{2q-\alpha-1} - (ctD_{2q-2} - D_{2q-3})D_{\alpha-2}]$$
$$g_\alpha(n) = \frac{(-1)^{\alpha+1}}{B}[(ctD_{2q-2} - D_{2q-3})D_{2q-\alpha-1} - cD_{\alpha-2}]$$

Using (2.26), (2.27), we can now calculate the 1-point correlation functions:

$$< \tau_{\alpha,2r} >= \mathrm{Tr}(Q_\alpha^{2r}) = \sum_{k=0}^{r} \frac{(-1)^k 2^{-k}(2r)!}{k!(r-k)!} \begin{pmatrix} N \\ r-k-1 \end{pmatrix} (h_\alpha(n)g_\alpha(n))^r$$

and 2-point correlation functions:

$$< \tau_{1,2r}\tau_{\alpha,2s} >= \mathrm{Tr}[(Q_1^{2r})_+, (Q_\alpha^{2s})_-]$$

The 2-point correlation function is :

$$< \tau_{1,2r}\tau_{\alpha,2s} >= \sum_{\substack{0 \le k \le r \\ 0 \le k' \le s}} \sum_{\substack{0 \le i \le r-k \\ s-k' \le j \le 2(s-k') \\ i+j = r+s-k-k'}} L_{rs}(k,k',i,j|N)f(\alpha)$$

with:

$$L_{r,s}(k,k',i,j|N) = \frac{(2r)!}{(r-2l)!(2l-2k-i)!k!}\frac{(2s)!}{(s-2l')!(2s-2k'-j)!k'!}(-1)^{k+k'}2^{-k-k'}$$
$$\sum_{n=0}^{N-1}\left[\begin{pmatrix} n+2l-2k \\ i \end{pmatrix}\begin{pmatrix} n+2i+2j \\ j \end{pmatrix} - \begin{pmatrix} n+2l'-2k' \\ j \end{pmatrix}\begin{pmatrix} n+2i+2j \\ i \end{pmatrix}\right]$$

and where $f(\alpha)$ is:

$$f(\alpha) = (g_1 h_\alpha / g_\alpha)^{i+k}(g_\alpha)^{r+s}(h_\alpha)^{s-r}, \alpha = 2\ldots 2q-1$$

We can obtain a more explicit formula for the determinant D_n when $2 \geq t \geq -2$. In this case the roots of the equation $r^2 - tr + 1 = 0$ are complex $r_{1,2} = \exp(\pm i\omega/2)$ and the determinant is :

$$D_n = \frac{\sin(n\omega/2)}{sin(\omega/2)}$$

with $\omega = (1/2)\arctan\sqrt{(2/t)^2 - 1}$. This formula permits us to single out the dependence of the 1-point correlator $< \tau_{\alpha,2r} >$ on the parameter α. First we calculate:

$$h_\alpha(n)g_\alpha(n) = (1/2)[(2A + A') + (A' - 2A\cos(\omega(q-1)))\cos(\omega(q-\alpha))]$$

with

$$A = \frac{D_{2q-3} - ctD_{2q-2}}{B}, A' = \frac{c^2 D_{2q-2}^2 - (ctD_{2q-2} - D_{2q-3})^2}{Bc}$$

Hence we have the following behaviour:

$$(h_\alpha(n)g_\alpha(n))^r = \sum_{k=-r}^{r} d_k e^{i\omega k\alpha}$$

Now we pass from the discrete variable $\alpha = 1, \ldots 2q$ to a continuous time $t \in [0, T/2]$. We introduce the puncture operator:

$$O^{(2p)} = \int_0^{t/2} dt \mathrm{Tr}(Q^{2r}(t))e^{ipt} = K_r \sum_{k=-r}^{r} d_k \delta(p + \omega k)$$

where K_r behaves like N^{r+1}.

In the pure chain models (no potentials V_α), $Q(t)$ is proportional to either I_+ or I_-. From this it follows that $\mathrm{Tr}(Q^r(t))$ is independent of t. Hence:

$$O^{(2p)} \sim \delta(p)$$

The conclusion is that for quadratic models we have, apart from the fundamental state with zero momentum, also other excited states (discrete states) with integer momenta $p = n\omega$, n integer.

We can study the 2-point correlation functions in the same framework. We take the particular case:

$$< \tau_{1,2r}\tau_{\alpha,2r} >= \sum_{n=0}^{r} M_n (g_\alpha)^n (h_\alpha)^{2r-n}$$

with $n = i + k$ and:

$$M_n = \sum_{0 \leq k,k' \leq r} L_{2r,2r}(k, k', n-k, 2r - k' - n|N)(g_1)^n$$

and look at the dependence on the parameter α. Using the dependence on α for :

$$(g_\alpha)^n (h_\alpha)^{2r-n} = \sum_{k=-r}^{r} K_r^{(k)} \exp(i\omega\alpha)$$

we can calculate the 2-point correlation function. Passing to the continuous time and using the symmetry of $K_r^{(k)}K_{-r}^{(k)}$ we can write the expression of the 2-point correlation function, as:

$$< \tau_{1,2r}\tau_{\alpha,2r} > (\alpha) = \sum_{k=0}^{r} K_r^{(k)} \sin \omega k(\alpha - \alpha_1)$$

We can now evaluate 2-point correlators of puncture operators in the momentum space:

$$G^{(2r)}(p) = \int_0^{T/2} \frac{dt}{2\pi} e^{-p(t-t_1)} < \tau_{1,2r}\tau_{\alpha,2r} > (t) = \sum_{k=0}^{r} \frac{K_r^{(k)}}{p^2 + (k\omega)^2} \tag{7.3}$$

We have intermediate states at all integer momenta $p = k\omega$, k integer. The pulsation ω depends on the scaling we use when passing from discrete values of $\alpha = 1, \ldots 2q$ to continuous time $t \in [0, T/2]$.

We can now apply all this to the $c = 1$ string theory model model, [16][17]. The $c = 1$ model with discrete time can be formulated as a multi-matrix model with the partition function:

$$Z = \int dM_i \exp\left[-\frac{\beta}{2} Tr\left(\sum_{i=1}^{n-1} \frac{(M_{i+1} - M_i)^2}{\epsilon} + \epsilon \sum_{i=1}^{n} V(M_i)\right)\right]$$

with a quartic potential $V(M) = M^2 - gM^4$. However, only the contribution near saddle point $V'(M_c) = 0$, where the potential is quadratic in the fluctuation ΔM, is essential

$$V(M) = \frac{1}{4g} - \frac{2(\Delta M)^2}{\beta}, M = M_c + \frac{\Delta M}{\sqrt{\beta}} \tag{7.4}$$

The new partition function is (up to the constant $\exp(-N\beta\epsilon/(8g))$:

$$Z = \int dM_i \exp\left[Tr\left(\sum_{i=1}^{n} \Delta M_i^2 (2\epsilon - \frac{1}{\epsilon}) + \frac{1}{\epsilon} \sum_{i=1}^{n-1} \Delta M_i \Delta M_{i+1}\right)\right]$$

It represents a string theory on circle with radius $R \sim \frac{1}{\epsilon}$. This is exactly our initial partition function (7.1) with the identifications $\tilde{t} = 2(2\epsilon - 1/\epsilon)$, $c = 1/\epsilon$ than $t = 2(2\epsilon^2 - 1)$ and we have the following limiting cases for the determinant D_n:

$$\begin{aligned} D_n &\sim t^n \qquad for \quad \epsilon \to \infty, t \to \infty \\ D_{3n+k} &\sim (-1)^n(1+k), \quad k = 0, -1, -2 \quad for \quad \epsilon \to 0, t \to 2(2\epsilon^2 - 1) \end{aligned}$$

From the results before (7.3) we have found that our model describes particles with energy levels equal to $(n + 1/2)\omega(\epsilon)/\beta$. For small ϵ, $\omega \sim \epsilon$ and changing ϵ (lattice spacing) means a linear change of energy scale. In this limit the model describes a string with the discretized real line as target space.

Because $sin(\omega/2) = 2\epsilon\sqrt{1 - \epsilon^2}$, for $\epsilon \leq 1$ the pulsation becomes ω complex, which is a sign of instability of the model. For $\epsilon \to \infty$ the model decouples in q noninteracting 1-matrix models. The instability is due to the liberation of the vortices and give rise to the Kosterlitz-Thouless transition for ϵ near 1.

Our method allows us to calculate (at least in principle) all n-point functions at any genus and what is more important it permits calculations in the vortex region where $\epsilon \leq 1$, $t \geq 2$.

Appendix A. The W–constraints

This Appendix is devoted to the derivation of the W–constraints in q–matrix models. From both the coupling equations (2.12c) and consistency conditions (2.15), we get the W-constraints in the form: $Tr(Q^{n+r}(\alpha)\partial^r_{\lambda_\alpha}(*)) = 0$ where $*$ are the relations (2.9) *.

W-constraints have the form:

$$W^{[r]}_n(\alpha)Z_N(t,c) = 0,\ r \geq 0, n \geq -r; \quad \alpha = 1,\dots q. \tag{7.5a}$$

or

$$(\mathcal{L}^{[r]}_n(\alpha) - (-1)^r T^{[r]}_n(\alpha))Z_N(t,c) = 0.$$

involving the interaction operator $T^{[r]}_n$ which depends only on all the couplings $g_{a_1\dots a_q}$, except $g_{0,\dots,0,a_\alpha,0,\dots,0} = t_{\alpha,a_\alpha}$.

For example $T^{[1]}_n$ and $T^{[2]}_n$ are:

$$\begin{aligned} T^{[1]}_n(\alpha) &= a_\alpha g_{a_1\dots a_q}\frac{\partial}{\partial g_{a_1\dots,a_\alpha+n,\dots a_q}} \\ T^{[2]}_n(\alpha) &= a_\alpha a'_\alpha g_{a_1\dots a_q} g_{a'_1\dots a'_q}\frac{\partial}{\partial g_{a_1+a'_1\dots,a_\alpha+a'_\alpha+n,\dots a_q+a'_q}} + \\ &+ a_\alpha(a_\alpha - 1)g_{a_1\dots a_q}\frac{\partial}{\partial g_{a_1\dots,a_\alpha+n,\dots a_q}} \end{aligned} \tag{7.5b}$$

The operator $\mathcal{L}^{[r]}_n(1)$ has the same form as that of the two-matrix model:

$$\mathcal{L}^{[r]}_n(1) = \int dz : \frac{1}{r+1}(\partial_z + J)^{r+1} : z^{r+n} \tag{7.5c}$$

where :: is the normal ordering and $J(z)$ is the $U(1)$ current:

$$J(z) = \sum_{k=1}^{p_1} k t_{1,k} z^{k-1} + N z^{-1} + \sum_{k=1}^{\infty} z^{-k-1}\frac{\partial}{\partial t_{1,k}} \tag{7.5d}$$

The same expression holds for $\mathcal{L}^{[r]}_n(q)$.

The expression of $\mathcal{L}^{[r]}_n(\alpha), \alpha = 2,\dots q-1$ is different due to the absence of the P-matrix term:

$$\mathcal{L}^{[r]}_n(\alpha) = \int dz : \frac{1}{r}(\partial_z + V'_\alpha)^r P_\alpha : z^{r+n} \tag{7.5e}$$

with

$$\begin{aligned} V'_\alpha &= \sum_{k=1}^{p_\alpha} k t_{\alpha,k} z^{k-1} + N z^{-1}, \\ P_\alpha &= N z^{-1} + \sum_{k=1}^{\infty} z^{-k-1}\frac{\partial}{\partial t_{\alpha,k}} \end{aligned} \tag{7.5f}$$

* For another approach see [20].

The explicit expression of the first terms is:

$$\begin{aligned}
\mathcal{L}_n^{[1]}(\alpha) &= \sum_{k=1}^{\infty} k t_{\alpha,k} \frac{\partial}{\partial t_{\alpha,k+n}} + N t_{\alpha,1}\delta_{n,-1} \\
\mathcal{L}_n^{[2]}(\alpha) &= \sum_{k=1}^{\infty} k(k-1) t_{\alpha,k} \frac{\partial}{\partial t_{\alpha,k+n}} + \sum_{k_1,k_2} k_1 k_2 t_{\alpha,k_1} t_{\alpha,k_2} \frac{\partial}{\partial t_{\alpha,k+n}} + \\
&+ N^2 t_{\alpha,1}\delta_{n,-1} + N(t_{\alpha,1}^2 + 2t_{\alpha,2})\delta_{n,-2}
\end{aligned}$$

As an example we write down the $W_{-1}^{[1]}, W_0^{[1]}$ and $W_1^{[1]}$ constraints for the three matrix model.

$W^{[1]}$:

$$\begin{aligned}
\sum k t_k \langle \tau_{k-1} \rangle + N t_1 + c_{12} \langle \lambda_1 \rangle + c_{13} \langle \sigma_1 \rangle &= 0 \\
\sum k u_k \langle \lambda_{k-1} \rangle + N u_1 + c_{12} \langle \tau_1 \rangle + c_{23} \langle \sigma_1 \rangle &= 0 \\
\sum k s_k \langle \sigma_{k-1} \rangle + N s_1 + c_{23} \langle \lambda_1 \rangle + c_{13} \langle \tau_1 \rangle &= 0
\end{aligned} \tag{7.5g}$$

$$\begin{aligned}
\sum k t_k \langle \tau_k \rangle + c_{12} \langle \chi_{110} \rangle + c_{13} \langle \chi_{101} \rangle &= -\frac{N(N+1)}{2} \\
\sum k u_k \langle \lambda_k \rangle + c_{12} \langle \chi_{110} \rangle + c_{23} \langle \chi_{011} \rangle &= 0 \\
\sum k t_k \langle \sigma_k \rangle + c_{13} \langle \chi_{101} \rangle + c_{23} \langle \chi_{011} \rangle &= -\frac{N(N+1)}{2}
\end{aligned} \tag{7.5h}$$

$$\begin{aligned}
\sum k t_k \langle \tau_{k+1} \rangle + (N+1) \langle \tau_1 \rangle + c_{12} \langle \chi_{210} \rangle + c_{13} \langle \chi_{201} \rangle &= 0 \\
\sum k u_k \langle \lambda_{k+1} \rangle + c_{12} \langle \chi_{120} \rangle + c_{23} \langle \chi_{021} \rangle &= 0 \\
\sum k t_k \langle \sigma_{k+1} \rangle + (N+1) \langle \sigma_1 \rangle + c_{13} \langle \chi_{102} \rangle + c_{23} \langle \chi_{021} \rangle &= 0
\end{aligned} \tag{7.5i}$$

One easily sees from the second group of identities that the limit of pure chain models (cosmological point) does not exists for three–mm. The same thing holds for odd–q matrix models. However, writing down the W constraints for even–q matrix models, one can see that such a limit exists. This confirms the results obtained with other methods.

Appendix B. Explicit derivation of 1p correlators.

In this Appendix we give the derivation of the 1–point functions promised in section 4. For this we need to know Q^p where $Q = I_+ + a\epsilon_-$;we express it in terms of the normal ordered quantities $: Q^l :$ $(QQ =: QQ : +[QQ])$

$$Q^p = \sum_{k=0}^{[p/2]} : Q^{p-2k} : ([QQ])^k A_k^{(p)} \tag{7.5j}$$

where the contractions are $[QQ] = -aI_0$ because $[I_+, \epsilon_-] = I_0$;the normal ordering means that we have expressions with I_+ on left side and ϵ_- on the right side.

The coefficient $A_k^{(p)}$ is the number of ways in which we can choose k pairs from p identical objects:

$$A_k^{(p)} = \frac{1}{k!} \begin{pmatrix} p \\ 2 \end{pmatrix} \begin{pmatrix} p-2 \\ 2 \end{pmatrix} \cdots \begin{pmatrix} p-2k+2 \\ 2 \end{pmatrix} = \frac{p! 2^{-k}}{(p-2k)! k!}$$

Using :

$$: Q^{p-2k} := \sum_{i=0}^{p-2k} \left(\begin{array}{c} p-2k \\ i \end{array} \right) a^i I_+^{p-2k-i} (\epsilon_-)^i \tag{7.5k}$$

We have the result:

$$Q^{2p} = \sum_{k=0}^{2p} \sum_{i=0}^{2p-2k} \left(\begin{array}{c} 2p-2k \\ i \end{array} \right) (-1)^k a^{i+k} A_k^{(2p)} I_+^{2p-2k-i} (\epsilon_-)^i \tag{7.5l}$$

We define the Q_1 matrix:

$$\begin{aligned} Q_1 = I_+ + a_0 I_0 + a_1(\epsilon_-) = Q(a_1) + a_0 I_0 \\ Q_1^r = \sum_{2l=0}^{r} \left(\begin{array}{c} r \\ 2l \end{array} \right) Q(a_1)^{2l} a_0^{r-2l} \end{aligned} \tag{7.5m}$$

Using (7.5l) the expression of Q_1^r is:

$$Q_1^r = \sum_{2l=0}^{r} \sum_{k=0}^{2l} \sum_{i=0}^{2l-2k} \left(\begin{array}{c} r \\ 2l \end{array} \right) \left(\begin{array}{c} 2l-2k \\ i \end{array} \right) (-1)^k A_k^{(2l)} a_1^{i+k} a_0^{r-2l} I_+^{2l-2k-i} (\epsilon_-)^i \tag{7.5n}$$

The same expressions are for :

$$\begin{aligned} Q_\alpha^r = \sum_{2l=0}^{r} \sum_{k=0}^{2l} \sum_{i=0}^{2l-2k} \left(\begin{array}{c} r \\ 2l \end{array} \right) \left(\begin{array}{c} 2l-2k \\ i \end{array} \right) (-1)^k A_k^{(2l)} f_1(\alpha) I_+^{2l-2k-i} (\epsilon_-)^i \\ f_1(\alpha) = (g_\alpha / h_\alpha)^{i+k} h_\alpha^{2l} s_\alpha^{r-2l}, \alpha = 1 \ldots q \end{aligned} \tag{7.5o}$$

Because we have the summation:

$$\mathrm{Tr}(I_+^k (\epsilon_-)^k) = k! \sum_{N=0}^{N-1} \left(\begin{array}{c} n+k \\ k \end{array} \right) = k! \left(\begin{array}{c} N+k \\ k+1 \end{array} \right)$$

the 1-point correlation function is:

$$< \tau_r >= \mathrm{Tr} Q_1^r = \sum_{2l=0}^{r} \sum_{k=0}^{l} \frac{(-1)^k 2^{-k} r!}{(r-2l)! k! (l-k)!} \left(\begin{array}{c} N+l-k \\ l-k+1 \end{array} \right) (h_\alpha g_\alpha)^l s_\alpha^{r-2l} \tag{7.5p}$$

References

[1] L.Bonora and C.S.Xiong, Nucl.Phys.**B405**(1993)191.

[2] M. Douglas, Phys.Lett.**B238** (1990) 176.

[3] P.di Francesco and D.Kutasov, Phys.Lett. **265B** (1991) 385.

[4] J.De Boer, Nucl.Phys. **B336** 602.

[5] I.Kostov, Phys.Lett. **297B** (1992) 74.

[6] C.Ahn and K.Shigemoto, *Multi–matrix model and 2D Toda multi–component hierarchy* CLNS 91/1054.

[7] S.Kharchev,A.Marshakov,A.Mironov,A.Morozov, Nucl.Phys.**B397**(1993)339

[8] L.Bonora and C.S.Xiong, Nucl.Phys. **B434** (1995) 408.

[9] L.Bonora and C.S.Xiong, *Topological field theory interpretation and LG representation of $c = 1$ string theory*, hep-th/9505140.

[10] S.Aoyama and Y.Kodama, Mod.Phys.Lett. **A9** (1994) 2481. *Topological Landau–Ginzburg theory with a rational potential and the dispersionless KP hierarchy*, hep-th/9505122.

[11] L.Bonora C.P.Constantinidis and C.S.Xiong, *Exact correlators of two–matrix models*, SISSA 189/94/EP, UTHEP-695, hep-th/9412006.

[12] K.Ueno and K.Takasaki, Adv. Studies in Pure Math. 4 (1984) 1.

[13] D. Bessis, C. Itzykson, and J.-B. Zuber, Adv. Appl. Math.**1**(1980) 109.

[14] C.Itzykson and B.Zuber, J.Math.Phys. **21** (1980) 411.

[15] M.L.Mehta, Comm.Math.Phys. 79 (1981) 327. S. Chedha, G. Mahoux and M. Mehta, J. Phys. A: Math. Gen. **14**(1981)579.

[16] D.J.Gross,Lectures at Jerusalem Winter School, 1990

[17] D.J.Gross,I.Klebanov,M.Newman,Nucl.Phys.**350 B**(1990) 621

[18] W.Lerche, *Chiral Rings and Integrable Systems for Models of Topological Gravity*, hep-th/9401121

[19] M.Bershadsky, W.Lerche,D.Nemeshansky, N.P.Warner,*Extended $N = 2$ superconformal structure of gravity and W-gravity coupled to matter*, hep-th/9211040

[20] H.Itoyama and Y.Matsuo, Phys.Lett. **B262** (1991) 233.