ULB-TH 95/06
UMH-MG 95/02
LPTENS 95/18
May 1995

Hawking Radiation Without Transplanckian Frequencies.

R. Brout[1], S. Massar[1,2],
Service de Physique Théorique, Université Libre de Bruxelles,
Campus Plaine, C.P. 225, Bd du Triomphe, B-1050 Brussels, Belgium

R. Parentani[3]
Laboratoire de Physique Théorique de l'Ecole Normale Supérieure
24 rue Lhomond 75.231 Paris CEDEX 05, France

Ph. Spindel[4]
Mécanique et Gravitation, Université de Mons-Hainaut,
Faculté des Sciences, 15 avenue Maistriau, B-7000 Mons, Belgium

PACS numbers: 04.60.+n and 03.70.+k

HEP-TH-9506121

[1]e-mail: smassar @ ulb.ac.be
[2]Boursier IISN
[3]Unité propre de recherche du C.N.R.S. associee a l' l'Ecole Normale Supérieure et a l'Universite de Paris Sud. e-mail: parenta@physique.ens.fr
[4]e-mail: spindel @ umh.ac.be

Abstract In a recent work, Unruh showed that Hawking radiation is unaffected by a truncation of free field theory at the Planck scale. His analysis was performed numerically and based on a hydrodynamical model. In this work, by analytical methods, the mathematical and physical origin of Unruh's result is revealed. An alternative truncation scheme which may be more appropriate for black hole physics is proposed and analyzed. In both schemes the thermal Hawking radiation remains unaffected even though transplanckian energies no longer appear. The universality of this result is explained by working in momentum space. In that representation, in the presence of a horizon, the d'Alembertian equation becomes a singular first order equation. In addition, the boundary conditions corresponding to vacuum before the black hole formed are that the in–modes contain positive momenta only. Both properties remain valid when the spectrum is truncated and they suffice to obtain Hawking radiation.

1 Introduction

The theory of black hole evaporation [1] is sore beset with two unsolved dilemmas:

1) The transplanckian issue [2][3]: when deriving Hawking radiation in the usual framework of free field theory, one calls upon vacuum fluctuations at $\mathcal{I}^-$ [4][5] whose energies are $O(e^{M^2}/M)$. These propagate as such up to a Planckian distance of the horizon where there energy is redshifted down to $O(1)$ and then further down to a typical frequency of $O(M^{-1})$ upon reaching $\mathcal{I}^+$. Since gravitational interactions become strong at the Planckian scale free field theory is at best dubious.

2) The unitary issue [6]: in the semi-classical theory of back reaction, both the matter which is the source of gravity (the star), and the "partners" to the emitted Hawking photons fall into the singularity, giving rise to a density matrix description of the radiation, thus, in the last analysis, to a non-unitary description of the evolution. Can one incorporate such a quandary into quantum mechanics?

Both will require a deeper knowledge of how gravity reacts to Hawking emission in order to be resolved. But perhaps at a more preliminary stage progress may be made by introducing an effective theory which one guesses incorporates some of the features which might emerge from the fundamental theory. Such is the nature of a recent interesting contribution of W. Unruh [7], who addressed himself to the taming of the transplanckian monster. The present paper, inspired by Unruh's work, contains an analysis as well as a generalization of the taming mechanism.

Unruh's work is based on an analogy between the hydrodynamic equations of motion of sound waves in a moving fluid and those that governs s-wave emission of a massless scalar field from an incipient Schwarzschild black hole in free field theory. Thus one is led to predict the production of a thermal flux of phonons as the fluid passes from sonic to supersonic flow [8]. Through numerical computation he subsequently showed [7] that a truncation of the spectrum of sound -using a rather natural algorithm- in no way affected the thermal emission of Hawking phonons in the circumstances that this emission occurred in the untruncated theory. Carrying this lesson over to the black hole situation, the implication is that tinkering with the transplanckian part

of the photon spectrum will leave the thermal emission unaffected. We shall here follow up on Unruh's tinkering, first by introducing a truncation scheme which we believe to be more appropriate to the black hole situation, and then supply the mathematical rationale for the resistance of Hawking's result to such mutilation. Throughout we have preserved the linearity of the field equations. Whether non linear effects preserve Hawking radiation remains a moot point.

The paper is organized in seven parts. Section 2 contains a brief review of Unruh's considerations. In preparation for our analysis we present in Section 3 a variant of the technique of Damour and Ruffini [9], using momentum space considerations. This technique provides for a simple and elegant characterization of initial conditions which then rapidly leads to the understanding which we seek. In Section 4, the photon spectrum is truncated in this approach. It follows immediately from the formalism that tinkering with the transplanckian part of the spectrum does not affect the Hawking thermal emission. Section 5 incorporates the Damour-Ruffini technique into the Unruh hydrodynamic truncated model. Using similar reasoning as in Section 4, the usual thermal emission is once more recovered. Section 6 contains a comparison of the wave packet trajectories in the two cases that have been analyzed so as to draw a physical picture of the production mechanisms. Finally in Section 7 we speculate on the physics behind our phenomenological truncation procedure. Some interpretation in terms of quantum gravity is hazarded. The result of our analysis can be viewed as one of disappointment. Hawking radiation does not seems to be, in itself, an open door that leads to quantum gravity. Rather, it provokes thought in that direction, without offering, at least in a direct way, an orientation for the solutions.

2 Unruh's Hydrodynamic Model

The equation that governs the propagation of sound (amplitude $\equiv \phi$) in a perfect fluid in $1+1$ dimensions which flows with a stationary background velocity field $V(\xi)$ is

$$[(\partial_\eta + \partial_\xi V(\xi))\ (\partial_\eta + V(\xi)\partial_\xi) - \partial_\xi^2]\phi = 0 \qquad . \tag{1}$$

This equation can be rewritten in d'Alembertian form $\Box\phi = 0$ with metric $g^{00} = 1,\ g^{01} = g^{10} = V,\ g^{11} = -1 + V^2$. It is readily diagonalized by the

transformation $dt = d\eta + Vd\xi/(1-V^2)$; $d\xi = dr$ to read

$$\Box\phi = [(1-V^2)^{-1}\partial_t^2 - \partial_r(1-V^2)\partial_r]\,\phi = 0 \qquad . \tag{2}$$

In this form one recognizes the s-wave part of $\Box$ constructed from the Schwarzschild metric with the identifications: t, r = Schwarzschild time and radial coordinate respectively; $V^2 = 2M/r$. The horizon is at $|V| = 1$. Thus a fluid, whose flow rate approaches $V = -1$, will emit a thermal distribution of phonons whose temperature is given by $(1/2\pi)dV/dr|_{V=-1}$. At this point we simply use the form given by eq. (1) to motivate, with Unruh [7], the truncation procedure used for this model.

In the rest frame ($V = 0$), the spectrum of sound determined from eq. (1) is $\omega = |k|$. But since the fluid has atomic structure, the spectrum has the property $\omega = |k|$ when $k < k_0$ and $\omega \to \omega_0$ for $k >> k_0$. For example $\omega = k_0$ tanh (k/k_0) might be expected to approximate the spectrum. In general for $\omega = F(k)$, we are then led to modify eq. (1) to

$$[(-i\partial_\eta - i\partial_\xi V)\,(-i\partial_\eta - Vi\partial_\xi) - F^2\,(-i\partial_\xi)]\phi = 0 \qquad . \tag{3}$$

This equation has been the starting point of Unruh's numerical computations. Propagating backwards in time an outgoing wave packet centered about a given negative frequency, he determines the Bogoljubov coefficients by decomposing the packet at early times into its positive and negative frequency components. Setting up a vacuum state at early times (See Section 5), the β term in the Bogoljubov transformation (i.e. the weight of the positive frequency part at early times of a negative frequency mode at late times) encodes the presence of outgoing phonons. This latter conforms to the existence of the outgoing thermal Hawking flux - quite independently of the truncation function, F.

An amusing aside is provided by Landau's theory of superfluid critical velocity, V_s. Were vortices and rotons absent, the approach to critical superfluid flow would be accompanied by a thermal shower of Hawking phonons.

3 The Damour - Ruffini Method

To explain and generalize Unruh's result we shall use the Damour-Ruffini technique [9], using a variant based on momentum space introduced in ref.

[10]. It is especially convenient for our present purpose because the wave equation near the horizon and the boundary conditions defining in-modes take a particularly simple form.

Equation (2) is valid outside a collapsing spherical star. First we transform it through use of the advanced Eddington-Finkelstein system, v and r, where $v = t + r^*, r^* = r + 2M\ell n[(r-2M)/2M]$ to give

$$\Box\psi = -[\partial_r(1-2M/r)\partial_r + 2\partial_v\partial_r]\psi(v,r) = 0 \tag{4}$$

The solutions, ψ, are connected smoothly to those which are inside the star. Since the details of the star's trajectory are irrelevant in the interesting asymptotic region where the star's surface approaches the horizon ($r = 2M$), we shall for simplicity of presentation take a very simple model: the star is defined by a thin shell radially falling inward with the speed of light. We choose its trajectory to be $v = v_{st} = 0$ where st labels the star's surface.

Inside the star, for $v < 0$, the geometry is flat and described by the metric $ds^2 = dv^2 - 2dvdr$ (In terms of usual Minkowski time T one has $v = T + r$). In this coordinate system waves obey

$$\Box\psi = -\partial_r(\partial_r + 2\partial_v)\psi = 0 \tag{5}$$

The usual Minkowski modes inside the star hereafter called in-modes, are

$$\chi_\lambda^{in(v)}(v) = e^{-i\lambda v}/\sqrt{4\pi\lambda} \quad , \tag{6}$$

$$\chi_\lambda^{in(u)}(v,r) = e^{-i\lambda(v-2r)}/\sqrt{4\pi\lambda} \quad . \tag{7}$$

The u-sector of the field operator is $\hat{\phi} = \int_0^\infty d\lambda \left(a_\lambda \chi_\lambda^{in(u)} + a_\lambda^\dagger \chi_\lambda^{in(u)*}\right)$ i.e. a superposition of in-modes given by eq. (7) inside the star. The Heisenberg state is the vacuum state $|0_{in}\rangle$ which is annihilated by the destruction operators a_λ, i.e. there are no quanta inside the star. For a full description one should consider v-modes $\chi_\lambda^{(v)}$ as well. But these do not encode particle creation and will not be considered here. Henceforth we drop the label (u). (To be complete we also point out that in order to describe the s-wave sector of a 3 + 1 dimensional theory one should impose the additional boundary condition that the modes vanish at $r = 0$. This complicates the notations without modifying the result and will not be taken into account.)

Hawking radiation is encoded in the history of the outgoing modes χ_λ^{in} as they propagate through the star into the space exterior to it. In particular we shall display their form outside the star both near and far from the horizon. In the first instance, the equation that governs their behavior is

$$-\partial_x[(x/2M)\partial_x + 2\partial_v]\psi(x,v) = 0 \tag{8}$$

where $x \equiv r - 2M$. This equation holds in the domain $|x\omega| << 1$ where ω is the eigenvalue of $i\partial_v$. Indeed the phase of the exact solution of eq. (4) $e^{-i\omega v}e^{-i2\omega(x+2M\ln x)}$ and the phase of the solution of eq. (8) $e^{-i\omega v}e^{-i\omega 4M\ln x}$ differ by $O(1)$ when $|x\omega| = O(1)$. Henceforth we shall consider only typical $\omega = O(M^{-1})$ which implies that $x < M$.

From eq. (6), the v-modes (irrelevant for production) remain of the form $e^{-i\omega v}/\sqrt{4\pi\omega}$, whereas the u-modes are given by the linear combination

$$\mathcal{F}_\omega(x,v) = \frac{e^{-i\omega v}}{\sqrt{4\pi\omega}}[\theta(x)Ax^{4Mi\omega} + \theta(-x)B|x|^{4Mi\omega}] \tag{9}$$

Standard Klein Gordon normalization prescribes $|A|^2 - |B|^2 = 1$.

Far from the horizon ($x \gg 2M$), the wave equation reduces to

$$-\partial_x\left(\partial_x + 2\partial_v\right)\psi = 0 \tag{10}$$

and the u-modes ψ_ω^{out} associated to asymptotic quanta on $\mathcal{I}^+$ are equal to $\psi_\omega^{out} = e^{-i\omega(v-2r)}/\sqrt{4\pi\omega}$. Since the exact solutions of eq. (4) which connects to this asymptotic form is $\psi_\omega^{out} = e^{-i\omega(v-2r^*)}/\sqrt{4\pi\omega}$, the out-modes near the horizon are given by $A = 1,\ B = 0$ in eq. (9):

$$\psi_\omega^{out} = \frac{1}{\sqrt{4\pi\omega}}e^{-i\omega v}\theta(x)x^{4Mi\omega} \qquad . \tag{11}$$

As the crux of our analysis lies in a careful formulation of the matching conditions at the star's surface which define the modes outside in terms of the modes inside the star we now go into this matter in some detail. In order to determine the out-particles content of the in-vacuum, it is propitious to compute the Fourier transform of the modes at the surface of the star ($v = 0$). The matching conditions are then implemented in simple and elegant fashion.

The Fourier transform of eq. (8) is

$$[p\partial_p + 4Mi\omega + 1]\tilde{\mathcal{F}}_\omega(p) = 0 \tag{12}$$

whence

$$\tilde{\mathcal{F}}_\omega(p) = \frac{\sqrt{M}}{\sqrt{2}\pi}\frac{1}{p}\left[Cp^{-4Mi\omega}\theta(p) + D|p|^{-4Mi\omega}\theta(-p)\right] \qquad . \tag{13}$$

The domain of validity of eqs. 12) and 13) is $|p| \gg M^{-1}$ (for typical $\omega = O(M^{-1})$).

The Fourier transform of the Minkowski modes eq. (7) are proportional to $\delta(p - 2\lambda)$, and we recall that positive norm modes have $\lambda > 0$.

The object of the exercise is to use continuity to change the basis from modes inside the star ($\equiv \tilde{\chi}^{in}_\lambda(p)$) to those outside the star which for large p are of the from $\tilde{\mathcal{F}}_\omega(p)$. Since the former set have $p > 0$ (since $\lambda > 0$), continuity prescribes that $p > 0$ is valid for the latter set as well. In this way one establishes that an equivalent set of in modes is found from Eq. (13) with $D = 0$ and $C = 1$ (this set was obtained independently from Damour-Ruffini, but by working in Kruskal coordinates by Unruh [11] and Hawking [6]). We call this bases $\tilde{\psi}^{in}_\omega(p)$

$$\tilde{\psi}^{in}_\omega(p) = \frac{\sqrt{M}}{\sqrt{2}\pi}\theta(p)p^{-4Mi\omega-1} \qquad . \tag{14}$$

This gives $\tilde{\psi}^{in}_\omega(p)$ for sufficiently large p (e.g. $p \gg M^{-1}$). Thus for these values of p one finds that the expansion coefficients $\alpha_{\omega\lambda}$ which give $\tilde{\psi}^{in}_\omega(p)$ in terms of $\tilde{\chi}^{in}_\lambda(p)$ are $(M/8\pi^2\lambda)^{1/2}(2\lambda)^{-4Mi\omega}$. For smaller values of p which are relevant for the construction of packets which cross the surface of the star at large values of x, as we have stated above, $\tilde{\psi}^{in}_\omega(p)$ and $\tilde{\chi}^{in}_\lambda(p)$ coincide. In this case $\alpha_{\omega\lambda} \to \delta(\omega - \lambda)$.

To see explicitly that only the large values of p are relevant for packets which reach $\mathcal{I}^+$ at late times, one builds a wave packet with the modes eq. (14):

$$\tilde{\varphi}^{in}_{\omega_0,u_0}(v,p) = \int d\omega\ e^{i\omega u_0}\frac{e^{-(\omega-\omega_0)^2/2\sigma^2}}{(2\pi)^{1/4}\sigma^{1/2}}e^{-i\omega v}\tilde{\psi}^{in}_\omega(p) \tag{15}$$

The phase $e^{i\omega u_0}$ centers the wave packet on the light ray $v - 2r^* = u_0$ and the gaussian $e^{-(\omega-\omega_0)^2/2\sigma^2}$ centers the frequency around ω_0. Inserting the form of $\tilde{\psi}^{in}_\omega$ one obtains

$$\tilde{\varphi}^{in}_{\omega_0,u_0}(v,p) = \frac{\sqrt{M}(2\pi)^{1/4}\sigma^{1/2}}{\sqrt{2}\pi}\frac{1}{p}e^{-(v+4M\ln p-u_0)^2\sigma^2/2}e^{i\omega_0(v+4M\ln p-u_0)} \tag{16}$$

whereupon it is seen that the wave packet is centered on the trajectory $v + 4M \ln p = u_0$. The momentum p at the surface of the star $v = 0$ is given by $p = e^{u_0/4M}$. Hence for wave packets which reach $\mathcal{I}^+$ at late times u_0, p at the surface of the star is much larger than M^{-1}.

In conclusion, we have the important result that, for all values of p, only positive ones appear in the basis functions $\tilde{\psi}^{in}_{\omega}(p)$. Furthermore, for $p \gg M^{-1}$ which are those necessary to describe outgoing packets which begin their journey from the surface of the star to $\mathcal{I}^+$ at values of $x_{st} \ll M$, the content of the $\tilde{\chi}^{in}_{\lambda}(p)$ modes in terms of $\tilde{\psi}^{in}_{\omega}(p)$ contain both signs of ω. On the contrary for $p \to 2\omega$, which are the relevant values to describe outgoing packets which begin their journey to $\mathcal{I}^+$ at values of $x_{st} \gg M$, ω becomes equal to λ. Therefore it is the former set that gives rise to particle creation on $\mathcal{I}^+$, whereas the latter give rise to no creation. In this formalism, this is what expresses the well known fact that Hawking radiation sets in as the star's surface approaches the horizon exponentially closely. To derive these results all that has been used is continuity at the star's surface. From now on we shall characterize in-vacuum by $a_\omega |0_{in}\rangle = 0$ where a_ω are the annihilation operators associated with the ψ^{in}_{ω} modes and we re-emphasize that ω takes on both signs in this characterization.

To find the content of $|0_{in}\rangle$ in terms of the out-modes (defined on $\mathcal{I}^+$) one considers the Fourier transform of $\tilde{\psi}^{in}(p)$ valid for $p \gg M^{-1}$, hence $x \ll M$

$$\begin{aligned} \psi^{in}_{\omega}(x) &= \frac{\sqrt{M}}{\sqrt{2}\pi} \int_0^\infty dp\ e^{ipx}\ p^{-4Mi\omega-1} \\ &= \frac{\sqrt{M}}{\sqrt{2}\pi} \Gamma(-4Mi\omega)|x|^{4Mi\omega}[\theta(x)e^{+2\pi M\omega} + \theta(-x)e^{-2\pi M\omega}] \quad . \end{aligned} \tag{17}$$

The mode ψ^{in}_{ω} lies on both sides of the horizon. Only the piece outside the horizon (i.e. proportional to $\theta(+x)$) propagates out to $\mathcal{I}^+$. From eqs. (11) and (17) one has outside the horizon ($x > 0$)

$$\psi^{in}_{\omega}(x)\theta(+x) = \begin{cases} \alpha_\omega \psi^{out}_{\omega}(x) & \omega > 0 \\ \beta_{|\omega|} \psi^{out*}_{|\omega|}(x) & \omega < 0 \end{cases} \tag{18}$$

where we have introduced the Bogoljubov coefficients

$$\begin{aligned} \alpha_\omega &= \frac{\Gamma(-4Mi\omega)\sqrt{2M\omega}\; e^{2\pi M\omega}}{\sqrt{\pi}} \quad , \\ \beta_\omega &= \frac{\Gamma(+4Mi\omega)\sqrt{2M\omega}\; e^{-2\pi M\omega}}{\sqrt{\pi}} \quad , \\ |\beta_\omega/\alpha_\omega| &= e^{-4\pi M\omega} \quad . \end{aligned} \tag{19}$$

This implies a steady thermal flux of emitted particles at temperature $T_H = 1/8\pi M$.

The condition $p > 0$ used in eq. (14) in the present formalism is equivalent to the Damour-Ruffini requirement that $\psi^{in}_\omega(x)$ be analytic in the upper half x plane. What we have seen here is that it is a direct consequence of the existence of vacuum in the star. The description we have given here is readily adapted to take into account truncation of the transplanckian spectrum ($p > 1$).

4 Confronting the Transplanckien Hiatus

The mechanism of how Hawking photons get created is through the combined gravitational and Doppler red shift which is incurred as a wave packet voyages from small values of x to $\mathcal{I}^+$. In the steady state of radiation the star's surface is exponentially close to the horizon ($x_{st} \simeq Me^{-t/2M}$). In consequence the p values, obtained from eq. (17), which dominate the integrand in the packet in this region are exponentially large: the saddle point of the integrand in eq. (17) is at $p^* = 4M\omega/x$ [10].

To see this more precisely, recall that the locus of saddle points traces out the classical trajectories of photons. Near $x = 0$ they are obtained from the Hamiltonian constraint

$$H \equiv p\left(\frac{xp}{2M} + 2p_v\right) (= 0) \tag{20}$$

where we have used Eddington-Finkelstein coordinates x, v and their conjugate momentum p and p_v (compare with eq. (8)). The canonical equation are

$$\dot{p} = -\partial H/\partial x \;;\; \dot{x} = \partial H/\partial p \;;\; \dot{v} = \partial H/\partial\omega \;;\; \dot{p}_v = 0 \quad , \tag{21}$$

where the dot denotes derivative with respect to an affine parameter along the trajectory. These combine to yield

$$p_v = -\omega = const. \quad , \tag{22}$$

$$\frac{dp}{dv} = -\frac{p}{4M} \quad . \tag{23}$$

On mass shell, H vanishes and accordingly

$$x = \frac{4M\omega}{p} \quad . \tag{24}$$

Hence

$$p(v) = p_{st} e^{-v/4M} \quad \text{and} \quad x(v) = \frac{4M\omega}{p_{st}} e^{v/4M} \tag{25}$$

where p_{st} is a constant of integration. From this last equation, we see that a photon of fixed energy ω, erected at late time and reaching $x = O(M)$ at v_* (where v_* is typically of order the life time of the black hole $= O(M^3)$) must have crossed the surface of the star at $x_{st} = 4M\omega/p_{st} = e^{-v_*/4M} O(M)$ with an enormous momentum $p_{st} = e^{+v_*/4M} O(\omega)$. This is the transplanckian problem. At these high values of p, and concommitantly small values of x, free field theory certainly breaks down. The gravitational interaction of the s-wave modes under consideration, both with other modes and with the background field whose source is in the degrees of freedom of the star becomes enormous. Therefore the underpinnings of the free field calculation become completely fallacious.

This does not mean, however, that the derivation given in Section 3 is completely lost. This is Unruh's main point. We shall first illustrate these considerations in the context of the Damour-Ruffini formalism of Section 3 by introducing a truncation in that scheme and in Section 5 proceed to analyze Unruh's effective theory as given in Section 2. In both of these schemes one explores the hypothesis that the physics near the horizon can be mimicked by modifying the way the matter field propagates, but leaving both gravity and the linear character of the fluctuations unaffected.

First a few words of clarification are in order. Any truncation scheme can be formulated in intrinsic geometric terms. However it is convenient to work in a coordinate system which is privileged in the geometry of the incipient

black hole. We make the assumption that the truncation takes a simple form in such a privileged system. Explicitly, the origin is fixed and the angular momentum expansion is carried out with respect to it. The radius has an intrinsic geometric meaning. Furthermore the space time outside the star is static and the origin of time is given by the inception of radial infall.

Let us therefore begin by truncating in the Eddington-Finkelstein system. We suppose that the energy spectrum gets modified for modes probing space time at scales less then 1, i.e. for $p > 1$. Then eq. (8) is assumed to be modified to the form

$$g(-i\partial_x)\left[\frac{x}{2M}g(-i\partial_x) - 2i\partial_v\right]\psi(x,v) = 0 \qquad . \tag{26}$$

Hence the mode equation corresponding to Eq 12) becomes

$$[\partial_p g(p) + 4Mi\omega]\tilde{\mathcal{F}}_\omega(p) = 0 \qquad . \tag{27}$$

The physics that goes into the evaluation of $g(p)$ will be the subject of the conjectural discussion of Section 7. For the present we only require that $g(p) = p$ for $p < O(1)$ and $g(p) = 1$ for $p > 1$ since we anticipate that it is only the Planckian and transplanckian modes which will be affected by gravity. As in Unruh's model, the function $g(p)$ is now to be identified with the energy associated with the mode number p rather than p itself. Indeed in flat space the proposed modification eq. (26) reads

$$g(-i\partial_x)[g(-i\partial_x) - 2i\partial_v]\psi = 0 \tag{28}$$

so that one sees that for outgoing modes λ, the eigenvalue of $i\partial_v$, is equal to $g(p)/2$ rather than $p/2$ and is bounded by $1/2$. In the curved space outside the star, the energy at $\mathcal{I}^+$ ($\equiv \omega$) is Doppler and gravitationally red shifted when compared to the energy measured by the free falling observer (for a discussion of the red shift see ref [12] Chapter 3 and Appendix D). This red shift is represented by the factor $x/4M$ in eq. (26) and the energy measured by the free falling observer is $4M\omega/x = g(p)$ which is now bounded by $1/2$.

It is instructive to see how the truncation deforms the classical trajectories from the geodesic associated with the free field. The truncated version is described by the Hamiltonian

$$\mathcal{H} \equiv -g(p)[\frac{x}{2M}g(p) + 2p_v] \ (= 0) \tag{29}$$

and eqs (23) and (24) now read

$$\begin{aligned} p_v &= -\omega \quad , & (30)\\ dp/dv &= -g(p)/4M \quad . & (31)\\ x &= 4M\omega/g(p) \quad , & (32) \end{aligned}$$

Integrating, one sees that $|p|$ decreases with v, initially (when $|p| > 1$) linearly and then exponentially (when $|p| < 1$). For $|p| > 1$, one has $|x| = |4M\omega|$ and for $|p| < 1$, $|x|$ is proportional to $|p|^{-1}$, hence increases exponentially with v. Thus at the same time as one avoids transplanckian energies, one ceases to approach the horizon to within transplanckian distances (typically $\omega M = O(1)$). Section 6 contains a sketch of the production process based on these classical trajectories (see Fig. 4).

To establish Hawking radiation in the truncated theory we once more have to characterize the in-modes exterior to the star. Subsequently it must be shown that these in-modes evolve towards $\mathcal{I}^+$ so as to give the required radiation.

From the truncated equation inside the star (eq. (28)) it is seen that positive λ implies positive p. Thus continuity across $v = 0$ once more implies that in-modes on the outside also have positive p. (This crucial result, which is independent of the details of $g(p)$, results from our assumption of postulating a linear field equation which keeps the d'Alembertian factorized into a u and v piece. That this is sufficient is obvious, but that it is not necessary will become apparent in Section 5). Thus our in-modes, solution of eq. (27) with $p > 0$ are

$$\tilde{\psi}^{in}_{\omega}(p) = \frac{\sqrt{M}}{\sqrt{2}\pi}\theta(p)g(p)^{-1}e^{-4iM\omega\int^p dp/g(p)} \quad . \tag{33}$$

The Fourier transform of $\psi^{in}_{\omega}(x)$ has contributions from $p > 1$ for $x = O(1)$ and $p < 1$ for $x > O(1)$. This is seen by examining the saddle point condition in the Fourier transform which in fact reconstructs the classical trajectories

eq. (32). The piece which is relevant for Hawking production is at $x >> 1$, hence concerned with cisplanckian values of p and one has

$$\begin{aligned}
\psi^{in}_{\omega}(x)|_{\rm cis} &\simeq \int_{M^{-1}}^{1} dp \frac{\sqrt{M}}{\sqrt{2}\pi} e^{ipx} \tilde{\psi}^{in}_{\omega}(p) \\
&= \int_{M^{-1}}^{1} dp \frac{\sqrt{M}}{\sqrt{2}\pi} e^{ipx} |p|^{-4Mi\omega-1} \\
&\simeq \int_{0}^{\infty} dp \frac{\sqrt{M}}{\sqrt{2}\pi} e^{ipx} |p|^{-4Mi\omega-1} \quad ; \; x >> 1 \qquad (34)
\end{aligned}$$

thereby recovering eq. (17)[1].

It should be noted that the norm of the cisplanckian modes is the same as in the free field case. This results from the fact that the truncated problem is possessed of a conserved norm ($= 2\pi \int dp \; \tilde{\psi}^{*}_{\omega}(p) \; g(p) \; \tilde{\psi}_{\omega'}(p) = \delta(\omega - \omega')$) which reduces to the free field form for $|p| < 1$. Thus the decomposition of $\psi^{in}_{\omega}(x)|_{\rm cis}$ into pieces localized at $x > 0$ and $x < 0$ as in eq. (17) has its usual interpretation in terms of unitarity.

When translated into the variable x, what we have shown is that the free field vacuum is maintained at $x \geq O(1)$ since one only requires the characterization of the in-state in terms of its cisplanckian content in this region. Most succinctly, Unruh vacuum is still a valid concept just outside a Planckian skin of the horizon. This is every bit as valid as the statement that Unruh vacuum is the correct description of the in-state in free field theory outside the star. A little bit of Planckian fuzz around the horizon does no injury to the physics since the conversion to Hawking photons on $\mathcal{I}^{+}$ occurs outside the star due to the redshift which the outgoing modes feel on the scale of $x = O(M)$ and not $x = O(1)$ (this has been pointed out by Jacobson [13] who however did not derive the insensitivity of Hawking radiation to planckian tinkering).

[1] We point out that eq. (28) has in general a large number of linearly independent solutions (an "infinity" if $g(\partial_x)$ is not polynomial in ∂_x). These linearly independent solutions are recovered from the function $\tilde{\psi}^{in}_{\omega}(p)$ by specifying which integration contour is used to take the Fourier transform. According to how the singularities of $\tilde{\psi}^{in}_{\omega}(p)$ are encircled, different solutions are obtained. We have chosen the contour to coincide with the path in the absence of truncation, namely to lie on the real axis. Other prescriptions would presumably give rise to runaway solutions and the properties of the theory at low momentum would not coincide with the free field theory.

The picture that emerges is that fluctuations within this skin steadily develop into outgoing pairs. Note that from eq. (31), p in this region grows linearly in v. The interpretation is that there is a conversion of modes within the skin to become the usual outgoing modes of free field theory, thereby guaranteeing a steady state. Thus at large values, p has become converted from an energy to a mode counting parameter. Note that the total time of evaporation is $O(M^3)$, so that the total number of modes which boil off from those initially inside the skin is $O(M^2)$ which is proportional to the usual estimate for the entropy of the black hole.

5 The Truncated Unruh Model

In this section we analyze the production of Hawking phonons based on Unruh's truncation eq. (3). As in Section 4, the analysis is based on the momentum representation of modes. Complications occur because the equation for the modes near the horizon is second order and it is necessary to control that the transplanckian sector does not contaminate the cisplanckian physics. This was obvious in Section 4 once the boundary conditions were set.

To have a clear idea of the mechanism of production from the modes, it is first worthwhile to go into the classical motion along trajectories. Whilst this part of the analysis does not give the production per se, it does give the motion of wave packets before and after production. In this way one has a guide to the portion of the mode analysis which is relevant.

The Hamiltonian, which generates the classical trajectories corresponding to the wave equation eq. (3) is

$$H = \left[p_\eta + pV(\xi)\right]^2 - F^2(p)\Big] \ (=0) \tag{35}$$

In the context of eq. (35), p_η is the momentum conjugate to η; p being conjugate to ξ. From the canonical equations one finds

$$p_\eta = -\omega = const. \quad , \tag{36}$$

$$dp/d\eta = -pV'(\xi) \quad , \tag{37}$$

$$d\xi/d\eta = \begin{cases} V(\xi) + F'(p) \text{ on trajectories of type II or III} \\ V(\xi) - F'(p) \text{ on trajectories of type I} \end{cases} \tag{38}$$

Since the fluid flows to the left with V decreasing to the left (so that near the horizon $V = -1 + \xi/4M$), it is seen that $|p|$ always decreases in η. As a consequence we can, and shall, describe evolution with respect to η in terms of p.

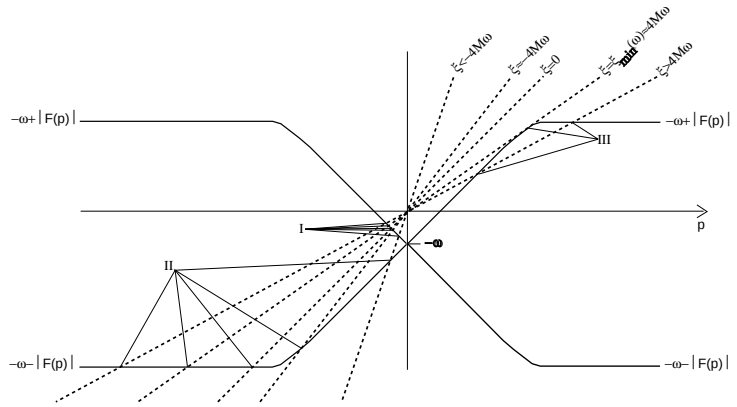

Figure 1: *The two curves $-\omega \pm |F(p)|$ (solid curves) and the family of lines $-V(\xi)p$ (doted lines) are plotted as functions of p for some representative values of ξ. Their intersections give the trajectories $\xi(p)$. Their are three trajectories labeled I, II, III. The set I corresponds to a $v = const$ trajectory. The set II crosses the horizon when $\xi = 0$ and then starts to propagate for $\xi < O(-4M\omega)$ whereupon it corresponds to a $u = const$ trajectory. The points in class III never reach the horizon: there are no solutions in this class for $\xi < \xi_{min}(\omega) = O(4M\omega)$ but their are two solutions for $\xi < \xi_{min}$ corresponding to a trajectory which bounces.*

To analyze the trajectories we use the on mass shell condition $-pV(\xi) = -\omega \pm |F(p)|$. In Fig. 1 is plotted as functions of p, the family of curves $-V(\xi)p$, and the two curves $-\omega \pm |F(p)|$. The intersections give the trajectories $\xi(p)$ at fixed ω. It is seen that there are three types of trajectories:

I The sequence of points near the origin all have small negative p, hence correspond to a usual null geodesic in Schwarzschild geometry which, in E F coordinates is $v=$ constant. They are therefore uninteresting for the production of particles.

II These are points which, for $\xi \gg -4M\omega$, have $p \ll 1$, hence $F(p)=1$. A particle on this part of the trajectory is at rest with respect to the fluid. For $\xi < -4M\omega$, p is < 1 and the particles then propagate with respect to the fluid. This trajectory stops at $p = 0$ owing to the monotonous character of $p(\eta)$ (eq. (37)).

III The third trajectory has $p > 0$. Once more for large positive p it is non propagating. As $|p|$ decreases, it starts to propagate for $p = O(1)$ whereupon it reaches a minimum value of $\xi(\equiv \xi_{min}(\omega))$ which is $O(4M\omega)$. As $|p|$ continues to decrease, ξ now increases so as to describe the propagating part of the trajectory for negative values of p.

Trajectories II and III are plotted in Fig.4 where the functions $\xi(v)$ are displayed (these are very similar to the functions $\xi(\eta)$ since the coordinate transformation used to go from eq. (1) to eq. (2), shows that near the horizon η differs from v by a regular function of ξ.). Production is concerned with the mixing of trajectories say III into II, i.e. a wave packet localized on trajectory III at large ξ and large p (that is at early times, η small) has an amplitude β to be localized on trajectory II at small p (η large). Note that had we taken $\omega < 0$ rather than $\omega > 0$, trajectory II would then have positive p and trajectory III negative p rather than the reverse.

We now study eq. (3) near the horizon where $V \simeq -1 + \xi/4M$. In momentum space, after some elementary manipulation, the equation takes the form

$$\left(\frac{p^2}{4M^2} \left[\partial_p + \frac{i\omega 4M + 1}{p} + i4M \right]^2 + F^2(p) \right) \tilde{\phi}_\omega(p) = 0 \qquad . \qquad (39)$$

For $|p| < 1$ where $F(p) = p$, the solutions are

$$\begin{aligned} \text{I} \quad &: \quad \tilde{\phi}_\omega(p) = \sqrt{\frac{M}{\pi}} \theta(\pm p) |p|^{-4Mi\omega - 1} e^{-i8Mp} \quad ; \quad |p| < 1 \quad , \\ \text{II, III} \quad &: \quad \tilde{\phi}_\omega(p) = \sqrt{\frac{M}{\pi}} \theta(\pm p) |p|^{-4Mi\omega - 1} \qquad ; \quad |p| < 1 \quad . \end{aligned} \qquad (40)$$

Solutions in class I are associated with the trajectories of class I (v-modes) and are not involved in production (the singular behavior of these modes is

due to the presence of the (spurious) Cauchy horizon at $\xi = 8M$). For $p \gg 1$, where $F(p) = 1$, the solutions are

$$\tilde{\phi}_\omega(p) = \sqrt{\frac{M}{\pi}}\theta(\pm p)|p|^{-4Mi\omega-1}e^{+i4Mp}|p|^{i\alpha_\pm} \quad ; \quad p \gg 1 \tag{41}$$

where $\alpha_\pm$ are the roots of $\alpha(\alpha-1) - (4M)^2 = 0$. We take $M \gg 1$ so that $\alpha_\pm \simeq \pm 4M - \frac{i}{2}$. These modes serve as a basis of second quantization. Their positive and negative frequency parts will then correspond to the bases of the quantized field wherein the annihilation part (positive frequency) annihilates the in-vacuum (i.e. vacuum at large ξ, hence large p). To determine this we refer to the conserved scalar product which we norm in the conventional way

$$i\int d\xi \left[\phi_{\omega'}(\xi)^* \left(\partial_\eta + V(\xi)\partial_\xi\right)\phi_\omega(\xi) - \phi_{\omega'}(\xi)^* \leftrightarrow \phi_\omega(\xi)\right] \;=\; \pm\delta(\omega-\omega') \tag{42}$$

or in momentum space

$$i2\pi\int dp \Big[\tilde{\phi}_{\omega'}(p)^* \left(\partial_\eta - ip - 1/4M - p/4M\partial_p\right)\tilde{\phi}_\omega(p) \\ -\tilde{\phi}_{\omega'}(p)^* \leftrightarrow \tilde{\phi}_\omega(p)\Big] \;=\; \pm\delta(\omega-\omega') \quad . \tag{43}$$

Inserting eq. (41) into eq. (43) it is seen that the sign of the scalar product is determined by the sign of $\alpha_\pm$ in eq. (41). The condition of positive frequency ($\alpha_+ = +4M$) corresponds to taking wave packets localized along the trajectories $-pV(\xi) = -\omega + |F(p)|$ rather than $-pV(\xi) = -\omega - |F(p)|$. The additional requirement that the modes be localized along trajectories II or III rather than I then imposes $p > 0$. The condition of positive p which played an essential rôle in the preceding sections is thereby recovered. In summary the in-modes appropriate for second quantization which give rise to Hawking radiation are

$$\tilde{\phi}^{in}_\omega(p) = \sqrt{\frac{M}{\pi}}\theta(p)p^{-4Mi\omega-1}e^{-i4Mp}p^{i4M+1/2} \quad ; \quad p \gg 1 \quad . \tag{44}$$

It remains to establish how these in-modes which have been characterized at large p evolve into their forms eq. (40) at small p. To this end we use

the WKB approximation whose validity we justify subsequently. Thus we approximate the full solution of the in-mode for positive p by

$$\tilde{\varphi}^{in}_{\omega}(p) = \theta(p) p^{-4Mi\omega-1} \frac{e^{-i4M[p-\int^p dp\ F(p)/p]}}{\sqrt{F(p)/p}} \quad . \tag{45}$$

Equation 45) is exact for both $|p| < 1$ and $|p| \gg 1$. The validity of the WKB expansion is assured owing to the inequality

$$\frac{d(p/4MF(p))}{dp} \ll 1 \quad . \tag{46}$$

This condition can be given a geometric interpretation. To this end one reexpresses the solution $\xi(p)$ of the on mass shell condition $H = 0$ (eq. (35) as $\xi(p) = \xi_F(p) + \Delta\xi(p)$ where $\xi_F(p)$ corresponds to a point which moves with the fluid $\omega + pV(\xi_F) = 0$ and $\Delta\xi$ describes motion with respect to the background $(pV'(\xi_F)\Delta\xi)^2 - F^2(p) = 0$. Then the validity of the WKB approximation takes the form

$$\frac{d(1/\Delta\xi)}{dp} \ll 1 \quad . \tag{47}$$

which expresses that the motion with respect to the background fluid is sufficiently "slow". The l.h.s. of this inequality is $O(M^{-1})$ which by assumption is $\ll 1$. Any back scattering due to the second order character of the differential equation will be a non perturbative effect (typically of $O(e^{-M})$) which would result in the mixing of u and v type modes. It is negligible.

Once having established that the low p part of the in-modes is of the form eq. (40), types II and III, with only positive p, the results of Section 3 follow forthwith: to wit Fourier transform gives a thermal distribution in ω arising from packets built from the low momentum part of the modes.

In any truncation scheme, be it linear or nonlinear, one may hope that a condition of adiabaticity similar to eq. (47) will be applicable. It will then imply that the creation of particles at a Planckian distance from the horizon is strongly suppressed. Thus Unruh vacuum will once more be a valid concept on scales $x > O(1)$ and the usual spectrum of Hawking radiation will be recovered. Were there any Planckian particles created at $x = O(1)$ they would severely disrupt the Hawking flux. Their absence can be taken to

be an expression of the general principle (very well verified experimentally) that flat space is stable against creation of Planckian particles. Since the curvature in Schwarzschild geometry acts on scales $\Delta x = O(M)$, at a Planck distance from the horizon space looks almost flat and the principle can be applied with confidence. Thus the absence of created Planckian particles at the horizon should come as no surprise.

6 The classical trajectories

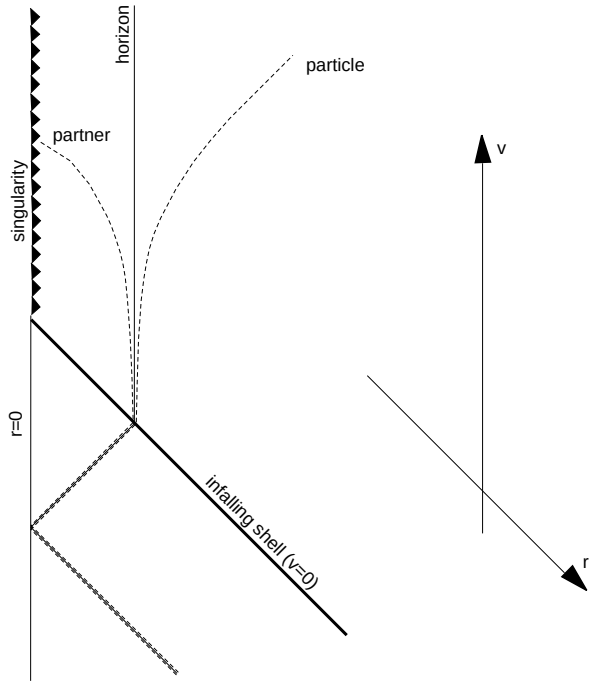

Figure 2: *The classical trajectories of outgoing null geodesics (thin dotted curves) given by the Hamiltonian eq. 20) are displayed in the Eddington-Finkelstein coordinate system. The light ray which generates the horizon, the infalling shell, the origin* $r = 0$ *and the singularity are also represented.*

We now have on hand three schemes on how to get Hawking radiation. It is interesting to display in an Eddington-Finkelstein graph the classical

trajectories corresponding to the modes in the three cases. These encode vacuum fluctuations in the past which are converted into quanta at $x = O(M)$.

In Fig 2 is displayed the usual free field model. A vacuum fluctuation emerges from the star, it is a pair that straddles the horizon. The fluctuation hugs the horizon at an exponentially small distance. Outside the star it starts to propagate (outwards for the observed Hawking photon and towards the singularity for its partner -unobservable in Schwarzschild time but taking form on the other side of the horizon in finite Kruskal time). Upon reaching $x = O(M)$ the fluctuation has become an on mass shell quanta which now propagates along the trajectory $v - 2r = const.$

In Fig 3 the effect of the simple truncation of Section 4 is shown. We have taken $g(p)$ to be unity for $p > 1$ (and of course $g(p) = p$ for $p < 1$). The propagation on either side of the horizon begins at the edges of the Planckian zone $|x| = O(1)$. The Planckian zone $|x| < 1$ is thus steadily solicited to give out radiation at a steady rate.

Finally in Fig. 4 the trajectories in Unruh's truncation are sketched. Here they begin from $\mathcal{I}^-$, but unlike the free field case they do not go into the star to reflect and then come out as in Fig 2. Rather they reach the horizon region directly whereupon at $x = \pm 1$ they reflect. The amplitude of the reflected wave is augmented; it is accompanied by a partner which appears on the other side with amplitude β and the production is encoded in $\alpha^2 - 1 = \beta^2 = (e^{\beta\omega} - 1)^{-1}$. It is fairly difficult to interpret this structure in terms of the black hole. In the fluid it comes about due to the "riding in" of the modes on the background flow.

More important, from the figures it is seen that in all cases, at large distance from the horizon $|x| \gg 1$, $|p| \ll 1$, one recovers the usual free particle trajectories. Therefore, independently of the truncation we have used, Hawking radiation remains a pair production phenomenon: the outgoing quanta are accompagned by partners on the other side of the horizon.

7 Conclusion

It appears to us that the scenario based on eq. (26) could well turn out to reflect some part of the truth. The function $g(p)$ deviates from its free field value, p, for $p > 1$ so as to give rise to a sort of quasi-particle description.

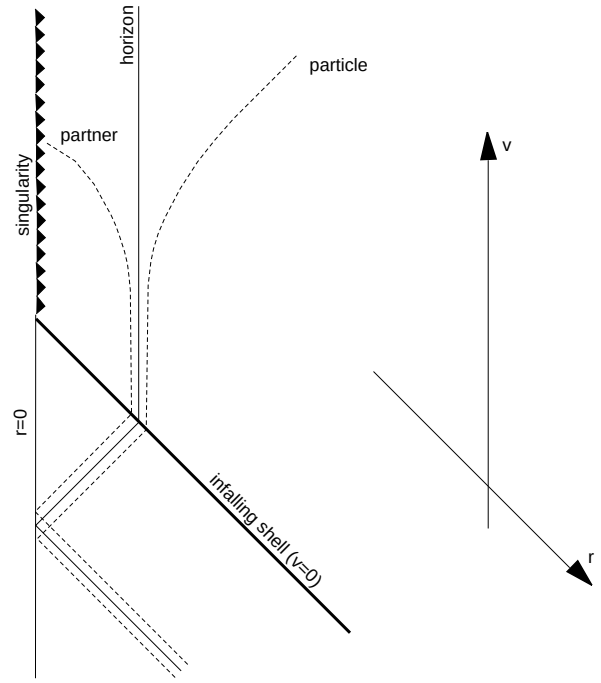

Figure 3: *The trajectories of outgoing light rays given by the truncated Hamiltonian eq. 29) are displayed in the Eddington-Finkelstein coordinate system. In this case the trajectories no longer approach the horizon exponentially but rather they stick at a planckian distance.*

The strong gravitational interaction among the modes will result in continuous mixing of angular momenta, so in the rigorous theory, restriction to s-waves will no longer be possible. The whole medium is to be regard as a matter gravitational soup, which has some s-wave content. For values of x greater than unity this content becomes that of the usual free field and there should be a turn over from a quasi to true particle theory. The transplanckian soup steadily feeds into the free field sector. It should be possible to display the transition region ($x = O(1)$, $p = O(1)$) by perturbative methods, for example by expanding the gravitational action to quadratic terms in fluctuations around the background geometry. This will show how the

modes start interacting with each other as they move into the Planckian region. We might expect $g(p)$ to become complex for $p > 1$, corresponding to a Planckian lifetime of the s-wave quasi-particle. Such behavior might encode that fact that an s-wave gets swallowed up in the extrapolation backwards in time towards the Planckian skin. Then instead of the extrapolation through the star drawn in Fig 3, the modes just peter out within the skin. This is indicated by some shading in Fig 5.

In addition there is the interaction of the modes with the gravitational field emanating from the sources which constitute the star (possibly taken together with the mean effect due to the other modes, as in the semiclassical theory). Here it would be "recoil" effects of the gravitational field which would be responsible for deviations from free field theory. That such effects can be important has recently been shown by one of us (RP) in the context of the accelerated detector [14], and more recently corroborated in a study of accelerating mirrors [15]. What the incidence of these two types of effects on the modes is for the unitarity problem remains to be seen.

Finally one must be prepared to encounter the very strong coupling problem which will arise well into the transplanckian region where conceptual problems will arise concerning the nature of space–time.

Whatever, on the basis of the above considerations, we conjecture strongly that Hawking radiation is protected from the vicissitudes of quantum gravity. It appears as an essentially kinematic response to the presence of the event horizon encountered in the collapse of a macroscopic black hole.

Added note After this manuscript was completed, C. Bouchiat and F. Englert called our attention to the following conceptual problem. The truncation we have used treats u-modes and v-modes asymmetrically, thereby explicitly breaking the invariance of the theory under local Lorentz transformations. However one should recall that the formation of a black hole by the collapse of a star induces an asymmetry between u-ness and v-ness: only a future horizon is formed but no past horizon exists. Concommitantly the field state is asymmetric, the v-part being Schwarzschild (Boulware vacuum) in character whereas the u-part is Kruskal (Hartle-Hawking) in character (i.e. the state is Unruh vacuum).

We beleive that it is not unreasonable to envisage that a phenomenological truncation of the field equations is state dependent since it should encode

the dynamics of the matter field state induced by the quantum gravitational interactions. In Unruh vacuum the truncation would then treat u and v modes differently. What would be the effect of a symmetric truncation in a symmetric state such as the eternal black hole situation remains to be seen.

References

[1] Hawking S.W., Nature **248** (1974) 30; Commun. Math. Phys., 43 (1975) 199

[2] 't Hooft G., Nucl. Phys. B **256** (1985) 727

[3] Jacobson T., Phys. Rev. D **44** (1991) 1731

[4] Englert F., Massar S. and Parentani R., Class. Quantum. Grav. **11** (1994) 2919

[5] Massar S. and Parentani R., *From Vacuum Fluctuations to Radiation: Accelerated Detectors and Black Holes*, preprint ULB-TH 94/02, gr-qc/9404057

[6] Hawking S.W., Phys. Rev. D **14** (1976) 2460

[7] Unruh W. G., Phys. Rev. D **51** (1995) 2827

[8] Unruh W. G., Phys. Rev. Lett. **21** (1981) 1351

[9] Damour T. and Ruffini R., Phys. Rev. D **14** (1976) 332

[10] Parentani R. and Brout R., Nucl. Phys. B388 (1992) 474

[11] Unruh W. G., Phys. Rev. D **14** (1976) 870

[12] Brout R., Massar S., Parentani R. and Spindel Ph. , *A Primer for Black Hole Quantum Physics*, preprint ULB-TH 95/02, UMH-MG 95/01, to appear in Phys. Reports.

[13] Jacobson T., Phys. Rev. D **48** (1993) 728

[14] Parentani R., *The Recoils of the Accelerated Atom and the Decoherence of its Fluxes*, preprint (1995) LPTENS 95/02 gr-qc 9502030

[15] Parentani R., in preparation

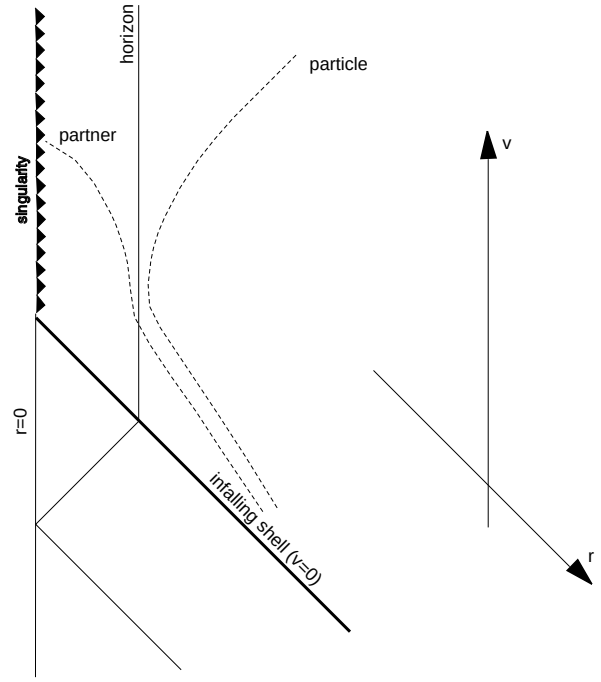

Figure 4: *The trajectories of light rays in Unruh's truncated hydrodynamic model are displayed in the Eddington-Finkelstein coordinate system. Only class II and III light rays have been displayed as it is these which are responsible for production. Note that since v and η differ by a regular function of x, the trajectories in the coordinate system η, ξ given by the Hamiltonian eq. 35) would be very similar to those depicted in the figure.*

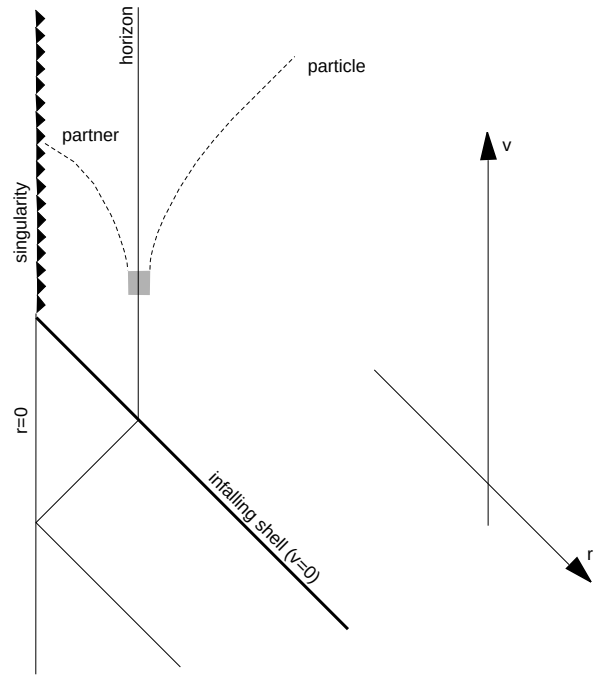

Figure 5: *The trajectories of light rays in the Eddington-Finkelstein coordinate system if the truncation $g(p)$ where complex. In this case the trajectories disapear into some quantum fuzz which is represented by some shading in the figure.*