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QUANTUM LOGIC AND DECOHERING HISTORIES *

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ABSTRACT

An introduction is given to an algebraic formulation and generalisation of the consistent histories approach to quantum theory. The main technical tool in this theory is an orthoalgebra of history propositions that serves as a generalised temporal analogue of the lattice of propositions of standard quantum logic. Particular emphasis is placed on those cases in which the history propositions can be represented by projection operators in a Hilbert space, and on the associated concept of a 'history group'.

1. Introduction

In recent years, much attention has been devoted to the so-called 'decoherent histories' approach to quantum theory. A major motivation for this scheme is a desire to replace the traditional Copenhagen interpretation of quantum theory with one that avoids any fundamental split between observer and system and the associated concept of state-vector reduction induced by a measurement. The key ingredient of the new approach is an assertion that, under certain conditions, a probability can be ascribed to a complete *history* of a quantum system without invoking any external state-vector reductions in the development of the history. Any such scheme would clearly be particularly attractive in quantum cosmology where a fundamental observer-system split seems to be singularly inappropriate.

Whether or not the new approach really does solve the conceptual challenges of quantum theory has been the subject of much recent debate; in particular, Dowker and Kent[1] have raised some serious doubts in the context of their penetrating analysis of the original programme. However, the main concern of the present paper is not conceptual issues as such but rather the possibility that the decoherent histories programme could provide a framework for solving certain technical or structural

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problems that arise in quantum gravity. An example is the 'problem of time' that features prominently in canonical quantum gravity. One plausible conclusion from the extensive discussion of this issue is that the conventional notion of time applies only in some semi-classical limit: a conclusion that, if true, must throw doubt on the entire standard quantum formalism, depending as it does on certain *prima facie* views on the nature of time. One way of tackling this issue could be with the aid of a suitably generalised notion of a space-time history.

However, of even greater importance perhaps is the question of whether quantum ideas should apply to space-time itself in addition to the metric or other fields that it carries. The inappropriateness of conventional quantum ideas becomes particularly apparent if one tries to develop non-continuum models of space-time involving, say, quantised point-set topologies. As with the conceptual problems of quantum cosmology, the challenge posed by issues of this type goes well beyond the question of which particular approach to quantum gravity (for example: superstring theory; canonical quantisation) is 'correct' by suggesting the need for a radical reappraisal of quantum theory itself. I believe that a suitably generalised version of the consistent histories programme could fulfil this role.

2. The Main Ideas

2.1. The Consistent Histories Formalism in Normal Quantum Theory

The consistent histories approach to standard quantum theory was pioneered by Griffiths[2], Omnès[3,4,5,6,7,8], and Gell-Mann and Hartle [9,10,11,12,13,14], and starts from a result in conventional quantum theory concerning the joint probability of finding each of a time-ordered sequence of properties[a] $\alpha = (\alpha_{t_1}, \alpha_{t_2}, \ldots, \alpha_{t_n})$ with $t_1 < t_2 < \cdots < t_n$ (we shall call a sequence of this type a *homogeneous history*, and refer to the sequence of times as the *temporal support* of the history). Namely, if the initial state at time t_0 is a density matrix ρ_{t_0} then the joint probability of finding all the properties in an appropriate sequence of measurements is

$$Prob(\alpha_{t_1}, \alpha_{t_2}, \ldots, \alpha_{t_n}; \rho_{t_0}) = tr_{\mathcal{H}}(C_\alpha \rho_{t_0} C_\alpha^\dagger) \tag{1}$$

where the 'class' operator C_α is given in terms of the Schrödinger-picture projection operators α_{t_i} on the Hilbert space $\mathcal{H}$ as

$$C_\alpha := U(t_0, t_n)\alpha_{t_n} U(t_n, t_{n-1})\alpha_{t_{n-1}} \ldots U(t_2, t_1)\alpha_{t_1} U(t_1, t_0) \tag{2}$$

where $U(t, t') = e^{-i(t-t')H/\hbar}$ is the unitary time-evolution operator from time t' to t. We note in passing that C_α is often written as the product of projection operators

$$C_\alpha = \alpha_{t_n}(t_n) \ldots \alpha_{t_2}(t_2)\alpha_{t_1}(t_1) \tag{3}$$

[a] A typical property is that the value of some physical quantity lies in some specified range.

where $\alpha_{t_i}(t_i) := U(t_i, t_0)^\dagger \alpha_{t_i} U(t_i, t_0)$ is the Heisenberg picture operator defined with respect to the fiducial time t_0.

The main assumption of the consistent-histories interpretation of quantum theory is that, under appropriate conditions, the probability assignment Eq. (1) is still meaningful for a *closed* system, with no external observers or associated measurement-induced state-vector reductions (thus signalling a move from 'observables' to 'beables'). The satisfaction or otherwise of these conditions (the 'consistency' of a complete set of histories: see below) is determined by the behaviour of the *decoherence function* $d_{(H,\rho)}$. This is the complex-valued function of pairs of homogeneous histories $\alpha = (\alpha_{t_1}, \alpha_{t_2}, \ldots, \alpha_{t_n})$ and $\beta = (\beta_{t'_1}, \beta_{t'_2}, \ldots, \beta_{t'_m})$ defined as

$$d_{(H,\rho)}(\alpha, \beta) := tr_{\mathcal{H}}(C_\alpha \rho C_\beta^\dagger) \tag{4}$$

where the temporal supports of α and β need not be the same. The physical interpretation of the complex number $d_{(H,\rho)}(\alpha, \beta)$ is as a measure of the extent to which the histories α and β are incompatible in the sense that it is not meaningful to assert "either α is realised *or* β is realised"[b]. A key ingredient in the formalism is the idea of finding collections of projectors that are sufficiently coarse (*i.e.,* project onto sufficiently large subspaces of $\mathcal{H}$) that the decoherence function of pairs of such can vanish.

Note that, as suggested by the notation $d_{(H,\rho)}$, both the initial state and the dynamical structure (*i.e.,* the Hamiltonian H) are coded in the decoherence function. A homogeneous history $(\alpha_{t_1}, \alpha_{t_2}, \ldots, \alpha_{t_n})$ itself is just a 'passive', time-ordered sequence of propositions that can be read as the single sequential proposition "α_{t_1} is true at time t_1, and then α_{t_2} is true at time t_2, and then $\ldots$, and then α_{t_n} is true at time t_n".

2.2. Generalised History Theory

An important suggestion of Gell-Mann and Hartle was to develop a new type of quantum theory in which the ideas of 'history' and 'decoherence function' would be fundamental in their own right. In particular, a history need no longer be just a time-ordered sequence of projection operators. They suggested that the crucial ingredients in such a theory would be (i) a 'coarse-graining' operation on the generalised histories; (ii) a mechanism for forming a logical 'or' of a pair of 'disjoint' histories (so that, in certain circumstances, one can talk about "history A or history B" being realised); and (iii) a negation operation (so that, in appropriate circumstances, one can make assertions like "history A is *not* realised").

Much of their thinking on this matter was motivated by path integrals where a typical coarse-grained history is that the path in the configuration space Q lies in

[b] A paradigmatic example of such a situation is the pair of paths that could be followed classically by a particle in the two-slit experiment.

some specified subset of paths. Thus they defined the decoherence function

$$d(\alpha, \beta) := \int_{q\in\alpha, q'\in\beta} Dq\, Dq'\, e^{-i(S[q]-S[q'])/\hbar}\delta(q(t_1), q'(t_1))\rho((q(t_0), q'(t_0)) \tag{5}$$

where the integral is over paths that start at time t_0 and end at time t_1, and where α and β are subsets of paths in Q. In this case, to say that a pair of histories α and β is disjoint means simply that they are disjoint subsets of the path space of Q, in which case d clearly possesses the additivity property

$$d(\alpha \oplus \beta, \gamma) = d(\alpha, \gamma) + d(\beta, \gamma) \tag{6}$$

for all subsets γ of the path space. Similarly, $\neg\alpha$ is represented by the complement of the subset α of path space, in which case the decoherence function satisfies

$$d(\neg\alpha, \gamma) = d(1, \gamma) - d(\alpha, \gamma) \tag{7}$$

where 1 denotes the entire path space (the 'unit' history).

2.3. An Algebraic Scheme

I would now like to summarise the algebraic scheme proposed by Isham and Linden[15,16] for placing the Gell-Mann and Hartle scheme in a precise mathematical framework that brings out the natural relation to concepts in quantum logic. In studying this rather abstract scheme it is appropriate to keep in mind that, in practice, the notion of a generalised history can include many different types of mathematical object. For example, in the case of quantum gravity, a 'history' could be a (possibly non globally-hyperbolic) space-time geometry (or subset of such), or a geometry augmented with other fields. Or it could include a specification of the space-time manifold—thereby describing a type of quantum topology—or the 'history' could even be an arbitrary topological space. Of particular importance is the idea that the class of generalised histories will generically include 'non-abelian' versions of the above. By this is meant some analogue of the fact that, in a class operator like Eq. (2) in standard quantum theory, the Schrödinger-picture projectors α_{t_i} at different times t_i may not commute (for example, they could include both position and momentum projectors), unlike the projectors onto subspaces of configuration space that arise in normal path integrals. Indeed, in some cases, the ideas that follow can be viewed as defining a non-commutative version of a path integral.

The basic rules of our version of the Gell-Mann and Hartle axioms are as follows[15,16].

1. The fundamental ingredients in the theory are (i) a space $\mathcal{U}P$ of propositions about possible 'histories' (or 'universes'); and (ii) a space $\mathcal{D}$ of *decoherence func-*

tions. A decoherence function is a complex-valued[c] function of pairs $\alpha, \beta \in \mathcal{U}P$ whose value $d(\alpha, \beta)$ is a measure of the extent to which the history propositions α and β are 'mutually incompatible'. The pair $(\mathcal{U}P, \mathcal{D})$ is to be regarded as the generalised-history analogue of the pair $(\mathcal{L}, \mathcal{S})$ in standard quantum theory where $\mathcal{L}$ is the set of propositions about the system at some fixed time, and $\mathcal{S}$ is the space of quantum states.

2. The set $\mathcal{U}P$ of history propositions is equipped with the following, logical-type, algebraic operations:

   (a) A partial order $\leq$. If $\alpha \leq \beta$ then β is said to be *coarser* than α, or a *coarse-graining* of α; equivalently, α is *finer* than β, or a *fine-graining* of β. The heuristic meaning of this relation is that α provides a more precise affirmation of 'the way the universe is' (in a transtemporal sense) than does β.

   The set $\mathcal{U}P$ possesses a *unit* history proposition 1—heuristically, the proposition about possible histories/universes that is always true—and a *null history proposition* 0—heuristically, the proposition that is always false. For all $\alpha \in \mathcal{U}P$ we have $0 \leq \alpha \leq 1$.

   (b) There is a notion of two history propositions α, β being *disjoint*, written $\alpha \perp \beta$. Heuristically, if $\alpha \perp \beta$ then if either α or β is 'realised' the other certainly cannot be.

   Two disjoint history propositions α, β can be combined to form a new proposition $\alpha \oplus \beta$ which, heuristically, is the proposition 'α *or* β'. This partial[d] binary operation is assumed to be commutative and associative, *i.e.*, $\alpha \oplus \beta = \beta \oplus \alpha$, and $\alpha \oplus (\beta \oplus \gamma) = (\alpha \oplus \beta) \oplus \gamma$ whenever these expressions are meaningful.

   (c) There is a negation operation $\neg\alpha$ such that, for all $\alpha \in \mathcal{U}P$, $\neg(\neg\alpha) = \alpha$.

A crucial question is how the operations $\leq$, $\oplus$ and $\neg$ are to be related. We shall postulate the following, minimal, requirements:[e]

i) $\quad$ *$\neg\alpha$ is the unique element in $\mathcal{U}P$ such that $\alpha \perp \neg\alpha$ with $\alpha \oplus \neg\alpha = 1$;*

[c] This is the precise point at which complex numbers enter the generalised scheme. Complex numbers are used in analogy to what occurs in standard quantum theory in the context of the class function Eq. (2) or the path integral Eq. (5). However, this does not rule out other possibilities for the space in which the decoherence functions take their values.

[d] It should be noted that the structure of an orthoalgebra is much weaker than that of a lattice. In the latter, there are two connectives $\wedge$ and $\vee$, both of which are defined on *all* pairs of elements. This contrasts with the single, partial operation $\oplus$ in an orthoalgebra. A lattice is a special type of orthoalgebra, with $a \oplus b$ being defined on disjoint lattice elements a, b as $a \vee b$. Here, 'disjoint' means that $a \leq \neg b$.

[e] Note that the second condition is manifestly true for a Boolean algebra (in which case, without loss of generality, the $\leq$ ordering can be regarded as set inclusion) , and also for the algebra of projection operators on a Hilbert space, in which $P \leq Q$ means that P projects onto a subspace of the range

ii) $\alpha \leq \beta$ *if and only if there exists* $\gamma \in \mathcal{U}P$ *such that* $\beta = \alpha \oplus \gamma$.

Both conditions are true of, for example, subsets of paths in a configuration space and, together with the other requirements above, essentially say that $\mathcal{U}P$ is an *orthoalgebra*; for a full definition see Foulis *et al*[17]. One consequence is that

$$\alpha \perp \beta \textit{ if and only if } \alpha \leq \neg\beta. \tag{8}$$

An orthoalgebra is probably the minimal useful mathematical structure that can be placed on $\mathcal{U}P$, but of course that does not prohibit the occurrence of a stronger one; in particular $\mathcal{U}P$ could be a lattice[f]. The possibility of generalising the structure of $\mathcal{U}P$ to be that of a 'difference poset' has been suggested recently by Pulmanova[18]. This allows the propositions to be extended to include 'effects': a possibility that is of some importance in quantum theory in general[19].

The next step is to formalise the notion of a decoherence function. Specifically, a *decoherence function* is a map $d : \mathcal{U}P \times \mathcal{U}P \to \mathbb{C}$ that satisfies the following conditions:

1. *Hermiticity*: $d(\alpha, \beta) = d(\beta, \alpha)^*$ for all α, β.

2. *Positivity*: $d(\alpha, \alpha) \geq 0$ for all α.

3. *Additivity*: if $\alpha \perp \beta$ then, for all γ, $d(\alpha \oplus \beta, \gamma) = d(\alpha, \gamma) + d(\beta, \gamma)$. If appropriate, this can be extended to countable sums.

4. *Normalisation*: $d(1, 1) = 1$.

In addition to the above we adopt the following definitions of Gell-Mann and Hartle: A set of history propositions $\{\alpha^1, \alpha^2, \ldots, \alpha^N\}$ is said to be *exclusive* if $\alpha^i \perp \alpha^j$ for all $i, j = 1, 2, \ldots, N$. The set is *exhaustive* (or *complete*) if it is exclusive and if $\alpha^1 \oplus \alpha^2 \oplus \ldots \oplus \alpha^N = 1$. In algebraic terms, an exclusive and exhaustive set of history propositions is simply a *partition of unity* in the orthoalgebra $\mathcal{U}P$.

It must be emphasised that, within this scheme, only *consistent* sets of history propositions are given an immediate physical interpretation. A complete set $\mathcal{C}$ of history propositions is said to be (strongly) consistent with respect to a particular decoherence function d if $d(\alpha, \beta) = 0$ for all $\alpha, \beta \in \mathcal{C}$ such that $\alpha \neq \beta$. Under these circumstances $d(\alpha, \alpha)$ is regarded as the *probability* that the history proposition α is true. The axioms above then guarantee that the usual Kolmogoroff probability rules are satisfied on the Boolean algebra generated by $\mathcal{C}$.

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of Q (one then writes Q as the sum of P and the projector onto the orthogonal complement of the range of P in the range of Q).

[f]One virtue of the weaker structure is that no one has been able to define a satisfactory tensor product for lattices whereas this *is* possible for orthoalgebras [17].

In this context, it is worth remarking that the idea of an orthoalgebra is closely related to that of a *Boolean manifold*[g]: an algebra that is 'covered' by a collection of Boolean subalgebras with appropriate compatibility conditions on any pair that overlap[21]. Being Boolean, these subalgebras of propositions carry a logical structure that is essentially classical: a feature of the decoherent histories scheme that was particularly emphasised in the seminal work of Griffiths and Omnès. In the approach outlined above, these Boolean algebras are glued together from the outset to form a universal algebra $\mathcal{U}P$ of propositions from which the physically interpretable subsets are selected by the consistency conditions with respect to a chosen decoherence function.

For a different perspective that places the emphasis on the separate Boolean algebras see the recent paper by Griffiths[22] in which he emphasises the dangers that can occur if logical deductions arising from incompatible consistent sets are mixed together. Dangers of this type are potentially present in all uses of quantum logic, and one must be very careful not to assume *a priori* that the algebraic operations employed have a logical interpretation in any semantic sense even though, mathematically speaking, they do look like logical (albeit, non-distributive) connectives.

3. Some Key Results

I will now summarise some of the main results that have been achieved in this quantum-logic like approach to the generalised histories programme. For further details the reader should consult the original papers[15,16,24,29]. It should be noted that the results discussed here all concern the important special case in which the history propositions can be represented by projectors on some Hilbert space.

3.1. Inclusion of Standard Quantum Theory in the Scheme

The similarity of the axioms above to those of conventional quantum logic motivates investigating the possibility of representing history propositions with projection operators on a Hilbert space. In particular, the question arises if it is possible to find such a representation for the homogeneous history propositions of standard quantum theory. Note that this is not a trivial matter since the product of two projection operators P and Q (such as appears in the class operator Eq. (3)) is not itself a projector unless $[\,P,Q\,]=0$.

The key to resolving this issue is the observation that what we are seeking is a quantum version of *temporal* logic rather than the logic of single-time propositions used in most discussions of physics. To this end, consider a temporal-logic sequential

[g] In turn, this is closely related to the idea of a *manual*: a concept that has been developed extensively in standard quantum logic by Foulis and Randall (see Foulis *et al*[20] and references therein). In many respects this structure seems the most appropriate of all in which to develop a generalised history theory; however, this remains a task for the future.

conjunction $A \sqcap B$ to be read as "A is true and then B is true". Then the proposition $A \sqcap B$ is false if (i) A is false and then B is true, or (ii) A is true and then B is false, or (iii) A is false and then B is false; symbolically:

$$\neg(A \sqcap B) = \neg A \sqcap B \text{ } or \text{ } A \sqcap \neg B \text{ } or \text{ } \neg A \sqcap \neg B. \tag{9}$$

Now we make the crucial observation that, unlike the simple product PQ, the *tensor* product $P \otimes Q$ of a pair of projection operators P, Q on a Hilbert space $\mathcal{H}$ is *always* a projection operator. Indeed, the product of homogeneous operators on $\mathcal{H} \otimes \mathcal{H}$ is defined as $(A \otimes B)(C \otimes D) := AC \otimes BD$, while the adjoint operation is $(A \otimes B)^\dagger := A^\dagger \otimes B^\dagger$, and hence $(P \otimes Q)^2 = P^2 \otimes Q^2 = P \otimes Q$, and $(P \otimes Q)^\dagger = P^\dagger \otimes Q^\dagger = P \otimes Q$.

Since $P \otimes Q$ is a genuine projection operator we have[h] the relation $\neg(P \otimes Q) = 1 \otimes 1 - P \otimes Q$ on $\mathcal{H} \otimes \mathcal{H}$, and so

$$\begin{aligned} \neg(P \otimes Q) = 1 \otimes 1 - P \otimes Q &= (1-P) \otimes Q + P \otimes (1-Q) + (1-P) \otimes (1-Q) \\ &= \neg P \otimes Q + P \otimes \neg Q + \neg P \otimes \neg Q \end{aligned} \tag{10}$$

which exactly models Eq. (9). This suggests representing the two-time sequential conjunction "α_{t_1} at time t_1 and then α_{t_2} at time t_2" with the tensor product[i] $\alpha_{t_1} \otimes \alpha_{t_2}$. Of course, not every projection operator in[j] $\mathcal{H}_{t_1} \otimes \mathcal{H}_{t_2}$ is of this homogeneous form. In particular, an inhomogeneous projection operator like $\alpha_{t_1} \otimes \alpha_{t_2} + \beta_{t_1} \otimes \beta_{t_2}$ can represent the proposition "(α_{t_1} at time t_1 and then α_{t_2} at time t_2) *or* (β_{t_1} at time t_1 and then β_{t_2} at time t_2)" provided that the projectors $\alpha_{t_1} \otimes \alpha_{t_2}$ and $\beta_{t_1} \otimes \beta_{t_2}$ are disjoint[k]. History propositions of this type (*i.e.,* sums of disjoint homogeneous history propositions) are called *inhomogeneous* and are an important generalisation of the idea of a history proposition.

Further investigation shows that this idea of using tensor products works very well and, in general, the homogeneous n-time history proposition $\alpha_{t_1}, \alpha_{t_2}, \ldots, \alpha_{t_n}$ can be represented by the projection operator $\alpha_{t_1} \otimes \alpha_{t_2} \otimes \cdots \otimes \alpha_{t_n}$ on the tensor product $\mathcal{H}_{t_1} \otimes \mathcal{H}_{t_2} \otimes \cdots \otimes \mathcal{H}_{t_n}$. Inhomogeneous history propositions then correspond to sums of pairwise-disjoint homogeneous history propositions. By this means the theory of discrete-time histories in standard quantum theory can be placed in the framework

[h] In standard quantum logic, the projector that represents the negation of a proposition R is $1 - R$, which projects onto the orthogonal complement of the range of R.

[i] This representation works well in capturing the essential nature of temporal propositions. However, it constitutes a striking departure from conventional thinking about the role of time in quantum theory, and therefore the idea needs to be handled carefully. For example, note that even if α_{t_1} and α_{t_2} are a pair of propositions that do *not* commute, the homogeneous history projectors $\alpha_{t_1} \otimes 1_{t_2}$ and $1_{t_1} \otimes \alpha_{t_2}$ *do* commute by virtue of the law of tensor product multiplication.

[j] Both $\mathcal{H}_{t_1}$ and $\mathcal{H}_{t_2}$ are isomorphic copies of the Hilbert space $\mathcal{H}$ on which the original quantum theory is defined: the t_1 and t_2 subscripts in $\mathcal{H}_{t_1}$ and $\mathcal{H}_{t_2}$ serve only as a reminder of the times to which the propositions α_{t_1} and α_{t_2} refer.

[k] In general, a pair of projectors P and Q is *disjoint* if $PQ = 0$.

of the axiomatic scheme above. In particular, it can be shown that the decoherence function $d_{(H,\rho)}(\alpha,\beta)$ in Eq. (4) of a pair of homogeneous history propositions $\alpha = (\alpha_{t_1},\alpha_{t_2},\ldots,\alpha_{t_n})$ and $\beta = (\beta_{t'_1},\beta_{t'_2},\ldots,\beta_{t'_m})$ can be written in terms of the associated tensor product operators as

$$d_{(H,\rho)}(\alpha,\beta) = tr_{\otimes^{n+m}\mathcal{H}}((\alpha_{t_1}\otimes\alpha_{t_2}\otimes\cdots\otimes\alpha_{t_n})\otimes(\beta_{t'_1}\otimes\beta_{t'_2}\otimes\cdots\otimes\beta_{t'_m})X) \qquad (11)$$

for a certain operator X on $\otimes^{n+m}\mathcal{H}$.

3.2. The Analogue of Gleason's Theorem

Not all projection operators in a tensor product $\mathcal{H}_{t_1}\otimes\mathcal{H}_{t_2}\otimes\cdots\otimes\mathcal{H}_{t_n}$ can be written as sums of *disjoint* homogeneous histories[l]. Nevertheless, the discussion above raises the question of whether generalised history theories exist in which the orthoalgebra $\mathcal{U}P$ is the set $\mathcal{P}(\mathcal{V})$ of projection operators on some Hilbert space $\mathcal{V}$ that is *not* just the tensor product of temporally-labelled copies of a single Hilbert space $\mathcal{H}$. Indeed, such examples could provide an extensive source of specific realisations of the axioms. However, to be viable, this suggestion requires a classification of the possible decoherence functions on $\mathcal{P}(\mathcal{V})$: a problem that is a direct analogue of that solved by Gleason[23] in his famous theorem in standard quantum logic.

In standard quantum theory on a Hilbert space $\mathcal{H}$, a state is defined to be a function $\sigma:\mathcal{P}(\mathcal{H})\to I\!R$ with the following properties:

1. *Positivity*: $\sigma(P)\geq 0$ for all $P\in\mathcal{P}(\mathcal{H})$.

2. *Additivity*: if P and R are disjoint projectors then $\sigma(P\oplus R)=\sigma(P)+\sigma(R)$. This requirement is usually extended to include countable collections of propositions.

3. *Normalisation*: $\sigma(1)=1$

where the unit operator 1 on the left hand side represents the unit proposition that is always true. Gleason's theorem asserts that, when $\dim\mathcal{H}>2$, such states are in one-to-one correspondence with density matrices ρ on $\mathcal{H}$, with

$$\sigma(P) = tr(P\rho) \qquad (12)$$

for all projection operators $P\in\mathcal{P}(\mathcal{H})$.

[l] Linden and I suspect that *any* projection operator can be obtained from the set of homogeneous history propositions by the application of the full *lattice* operations in the space of projectors, but we do not know a general proof of this. However, even if true, it is not clear what the physical significance of this would be. History propositions that are neither homogeneous or inhomogeneous have been referred to[16] as *exotic*; an example is the proposition corresponding to the projector onto an inhomogeneous vector $u_{t_1}\otimes u_{t_2}+v_{t_1}\otimes v_{t_2}$ in $\mathcal{H}_{t_1}\otimes\mathcal{H}_{t_2}$. It remains an intriguing topic for research to see if they have any role to play in our history version of standard quantum theory.

The analogous result for decoherence functions was proved by Isham, Linden and Schreckenberg[24]. Specifically, the decoherence functions $d \in \mathcal{D}$ on a space of projectors[m] $\mathcal{P}(\mathcal{V})$ (with $\dim \mathcal{V} > 2$) are in one-to-one correspondence with operators X on the tensor product $\mathcal{V} \otimes \mathcal{V}$ with

$$d_X(\alpha, \beta) = tr(\alpha \otimes \beta X) \; for \; all \; \alpha, \beta \in \mathcal{P}(\mathcal{V}), \tag{13}$$

and where X satisfies:

1. $X^\dagger = MXM$ where the operator M is defined on $\mathcal{V} \otimes \mathcal{V}$ by $M(u \otimes v) := v \otimes u$;
2. for all $\alpha \in \mathcal{P}(\mathcal{V})$, $tr(\alpha \otimes \alpha X_1) \geq 0$ where $X = X_1 + iX_2$ with X_1 and X_2 hermitian;
3. $tr(X_1) = 1$.

Note that Eq. (11) is the particular form taken by this expression in the case of standard quantum theory.

A simple illustration of this theorem has been given by Schreckenberg[25]. Let $\{\alpha^1, \alpha^2, \ldots, \alpha^N\}$ be any partition of unity in $\mathcal{P}(\mathcal{V})$, so that $\alpha^1 + \alpha^2 + \cdots + \alpha^N = 1$ and $\alpha^i \alpha^j = \delta^{ij} \alpha^i$. For any weights $w_1, w_2, \ldots, w_N$ with $w_i > 0$ and $\sum_{i=1}^N w_i = 1$, define $X := \sum_{i=1}^N w_i \, \alpha_i \otimes \alpha_i$. Then it is easy to see that the history propositions represented by $\{\alpha^1, \alpha^2, \ldots, \alpha^N\}$ are a consistent set with respect to the decoherence function d_X defined by this particular choice of X.

It should be noted that whatever analogue there may be of both dynamics and initial conditions is coded into the structure of the single operator X. In the example Eq. (11) that pertains to standard Hamiltonian quantum theory the operator X takes on a very special form. However, the theorem stated above for the general case where $\mathcal{U}P = \mathcal{P}(\mathcal{V})$ for some $\mathcal{V}$ allows for a wide range of possible operators X and hence for a wide range of generalisations of dynamics and initial conditions. This is the basis of our hope that the generalised scheme may provide a powerful tool for handling physical situations in which the notion of time is non-standard, such as that arising in canonical quantum gravity or in more exotic programmes aiming at quantising the stucture of space-time itself.

The proof of the classification of decoherence functions uses Gleason's theorem which, over the years, has been generalised to a variety of types of algebraic structure. Not surprisingly, a similar situation holds for decoherence functions and, in particular, Wright[26] has recently extended the classification theorem to the case where the history propositions are represented by projections in an arbitrary[n] von Neumann algebra. The basis of his work is an earlier result[27] detailing the conditions under which a state defined on the projectors in a von Neumann algebra $\mathcal{A}$ can be extend to a linear

[m] In the original proof only finite-dimensional Hilbert spaces $\mathcal{V}$ were discussed.
[n] Strictly speaking, the von Neumann algebra has to have no direct summand of type I_2.

functional on the entire algebra. When applied to the history situation this result can be used to show that a bounded decoherence functional can be extended to a bilinear function on $\mathcal{A}$; a Gelfand–Naimark–Segal type of construction (which is naturally suggested by the 'inner product' nature of the defining conditions of a decoherence functional) then completes the process.

3.3. The History Group

In standard canonical quantum theory an important role in constructing specific theories is played by the group of canonical commutation relations. For example, the quantum theory of a point particle moving in one dimension is specified by requiring the Hilbert space to carry an irreducible representation of the Weyl–Heisenberg group $\mathcal{W}$ whose Lie algebra is associated with the familiar commutation relation $[\,x, p\,] = i\hbar$. The famous Stone and von Neumann theorem then shows that the familiar representation on wave functions is essentially unique. More generally, if the classical configuration space is a homogeneous space G/H then the quantisation can be associated[28] with irreducible representations of a new canonical group constructed from G.

The question of interest is whether there may be an analogue of the canonical group in a history theory whose propositions are associated with projectors on some Hilbert space $\mathcal{V}$ as discussed above. To explore this issue let us question again the origin of the representation of a homogeneous history proposition (with temporal support $\{t_1, t_2, \ldots, t_n\}$) in standard quantum theory by a projection operator on the tensor product $\mathcal{H}_{t_1} \otimes \mathcal{H}_{t_2} \otimes \cdots \otimes \mathcal{H}_{t_n}$ of n copies of the original Hilbert space $\mathcal{H}$.

One answer is the temporal-logic approach that was sketched earlier. Another option is to invoke the purely algebraic fact[24] that the trace of a product of operators $A_1, A_2, \ldots, A_n$ on a Hilbert space $\mathcal{H}$ can be written as a trace on $\otimes^n\mathcal{H}$ in the form

$$tr_{\mathcal{H}}(A_1 A_2 \ldots A_n) \equiv tr_{\otimes^n\mathcal{H}}(A_1 \otimes A_2 \otimes \cdots \otimes A_n\, S) \tag{14}$$

for a certain fixed operator S on $\otimes^n\mathcal{H}$.

However, in the case where $\mathcal{H} = L^2(I\!R)$ (*i.e.,* the Hilbert space of wave functions used in elementary wave mechanics) one could also say that the history Hilbert space $L^2_{t_1}(I\!R) \otimes L^2_{t_2}(I\!R) \otimes \cdots \otimes L^2_{t_n}(I\!R)$ comes from a representation of the product $W_{t_1} \times W_{t_2} \times \cdots \times W_{t_n}$ of n copies of the Weyl–Heisenberg group, one for each time slot in the temporal support of the history proposition. Thus the tensor-product Hilbert space could be viewed as arising as a representation of the 'temporally-gauged' (!) canonical algebra[o]

$$\begin{aligned} [\,x_{t_i}, x_{t_j}\,] &= 0 && (15)\\ [\,p_{t_i}, p_{t_j}\,] &= 0 && (16)\\ [\,x_{t_i}, p_{t_j}\,] &= i\hbar\delta_{ij}. && (17) \end{aligned}$$

[o]I am are assuming here that the value of $\hbar$ is the same at each time slot in the temporal support.

This observation motivates the intriguing idea that it may be possible to specify generalised history theories by finding an appropriate 'history group' $\mathcal{G}$ whose irreducible unitary representations give the Hilbert space $\mathcal{V}$ on which the history propositions are to be defined. In particular, the projectors in the spectral decompositions of the self-adjoint generators of $\mathcal{G}$ will give a preferred class of propositions—rather as the generators of a standard canonical group provide a special class of classical observables that can be represented unambiguously in the quantum theory[28].

3.4. Continuous Histories

The use of a history group has been illustrated recently by Isham and Linden[29] in the context of *continuous* time histories in standard quantum theory. An obvious problem when handling continuous histories is to define an appropriate continuous product $\prod_{t\in I\!R}\alpha_t$ of projection operators for use in a class operator. However, we have shown[29] that this can be done for projections onto coherent states, and explicit expressions have been given for this product as well as for the associated decoherence function of a pair of such continuous histories.

On the other hand, the discussion above of a history group suggests that, in the case of continuous histories, the appropriate analogue of Eq. (15–17) is

$$[\, x_t, x_{t'}\,] \;=\; 0 \tag{18}$$

$$[\, p_t, p_{t'}\,] \;=\; 0 \tag{19}$$

$$[\, x_t, p_{t'}\,] \;=\; i\hbar\delta(t-t'). \tag{20}$$

Thus the continuous-time history version of one-dimensional wave mechanics looks like a one-dimensional quantum field theory, but with the 'fields' being labelled[p] by time rather than space! In particular, as shown in Isham and Linden[29], this history-group algebra does indeed provide the correct Hilbert space for the history theory. A key technical ingredient is the fact that Bosonic Fock space can be written as a certain continuous tensor product[30], thereby linking the representation of the history group with the idea of continuous temporal logic.

4. Conclusion

We have seen that the generalised history scheme proposed by Gell-Mann and Hartle can be given a precise mathematical form in which the roles of the space $\mathcal{U}P$ of history propositions and the space $\mathcal{D}$ of decoherence functions are analogous to those in standard quantum theory of the space $\mathcal{L}$ of single-time propositions and the space $\mathcal{S}$ of states respectively. We saw that finite-time history propositions in standard quantum theory can be fitted into this generalised algebraic framework by

[p]It is most important *not* to confuse the time-labelled operators with Heisenberg-picture operators: the one-parameter families of operators x_t and p_t, $t\in I\!R$, are in the Schrödinger picture.

identifying a homogeneous history proposition $(\alpha_{t_1}, \alpha_{t_2}, \ldots, \alpha_{t_n})$ with the projection operator $\alpha_{t_1} \otimes \alpha_{t_2} \otimes \cdots \otimes \alpha_{t_n}$ on the tensor product $\mathcal{H}_{t_1} \otimes \mathcal{H}_{t_2} \otimes \cdots \otimes \mathcal{H}_{t_n}$. The tensor-product space also provides a natural home for inhomogeneous history propositions via the disjoint 'or' operation defined on pairs of disjoint homogeneous histories. By this means we arrive at a concrete implementation of the idea of temporal quantum logic.

This result suggests that a large number of generalised history theories might be found by looking at a more general situation in which history propositions are represented by projectors on some 'history' Hilbert space $\mathcal{V}$ that is not necessarily a temporally-labelled tensor product. This lead us to consider the analogue of Gleason's theorem for decoherence functions, and hence to the representation of any such in the form $d_X(\alpha, \beta) = tr_{\mathcal{V}\otimes\mathcal{V}}(\alpha \otimes \beta X)$.

Finally, we suggested that the Hilbert space $\mathcal{V}$ that carries the generalised history propositions could arise as an irreducible representation of a history group $\mathcal{G}$—a history analogue of the canonical group of conventional quantum theory. Note that, in the context of continuous-time, standard quantum theory, paths in configuration space correspond to a certain maximal Boolean subalgebra of the orthoalgebra of history projectors. Thus the history group serves to embed this Boolean algebra in a specific non-Boolean orthoalgebra. It is in this sense that a decoherence function can sometimes be understood as a non-commutative analogue of a standard path integral.

Generalised history theories of the type discussed above offer a wide-ranging extension of standard ideas in quantum theory and are well suited for implementing some of the more exotic ideas often discussed in the context of quantum gravity. For example, it becomes quite feasible to consider a scheme in which the basic history propositions include assertions that the space-time topology belongs to some particular subset of point-set topologies on a fixed or variable set of space-time points. A less exotic example would be to study decoherence functionals and space-time metric propositions that are manifestly invariant under the action of the space-time diffeomorphism group. This would be a natural way of using the quantum history programme to find a space-time oriented approach to quantum gravity. Discussions of this and other applications will appear later.

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