To Appear: *Phys. Rev.* **D15**

Quantum cosmology of generalized two-dimensional dilaton-gravity models

James E. Lidsey

Astronomy Unit, School of Mathematical Sciences,
Queen Mary & Westfield, Mile End Road, LONDON, E1 4NS, U.K.

Abstract

The quantum cosmology of two-dimensional dilaton-gravity models is investigated. A class of models is mapped onto the constrained oscillator-ghost-oscillator model. A number of exact and approximate solutions to the corresponding Wheeler-DeWitt equation are presented. A wider class of minisuperspace models that can be solved in this fashion is identified. Supersymmetric extensions to the induced gravity theory and the bosonic string theory are then considered and closed-form solutions to the associated quantum constraints are derived. The possibility of applying the third-quantization procedure to two-dimensional dilaton-gravity is briefly discussed.

PACS: 04.60.Ds, 04.60.Kz, 98.80.Hw

1 Introduction

It is widely thought that quantum gravitational effects become important on scales of the order of the Planck length. It follows, therefore, that insight into the nature of quantum gravity might be gained by considering the very early Universe and the end-point of black hole evaporation. However, there remain many unresolved technical and conceptual difficulties with $(3+1)$-dimensional quantum gravity and to make progress one must consider simplified toy models.

Recently there has been considerable interest in two-dimensional theories of gravity. The Einstein action is a topological invariant in two dimensions and must therefore be modified if the theory is to be non-trivial. The simplest extension is to include a non-minimally coupled, self-interacting scalar 'dilaton' field. These models are closey related to string theory in non-critical dimensions [1] and may provide a solvable framework in which some of the questions raised in quantum gravity can be studied. Indeed, black hole evaporation has been extensively investigated following the introduction of the Callan, Giddings, Harvey and Strominger (CGHS) model [2].

Quantization of two-dimensional models has been performed by employing a number of techniques such as the BRST and Dirac operator methods [3, 4]. However, it is possible that lower-dimensional gravity may also shed light on some of the issues raised in quantum cosmology, such as the problem of extracting physical predictions from the wavefunction of the Universe. In this paper we investigate the quantum cosmology of a generalized class of two-dimensional models.

A number of approaches to quantum cosmology may be taken. In the path integral formalism, for example, the wavefunction of the Universe is expressed as a path integral over a certain class of metrics, matter distributions and manifolds [5, 6]. This formalism has recently been investigated within the context of two-dimensional quantum gravity by Ishikawa [7].

Alternatively, one may follow the Dirac quantization procedure [8]. The wavefunction of the Universe is annihilated by the Hamiltonian operator and, in principle, its functional form can be determined by solving the zero-energy Schrödinger equation. This equation decomposes into the Wheeler-DeWitt and momentum constraint equations that describe, respectively, the invariance of the theory under reparametrizations of time and spatial diffeomorphisms [9].

In general, the Wheeler-DeWitt equation is a functional differential equation and is very difficult to solve. Solutions can be found in four-dimensional theories by invoking the 'minisuperspace' approximation and freezing out the inhomogeneous modes. It is not yet established that such an approach can lead to meaningful results in higher dimensions, although arguments have been developed to suggest that it may be relevant [10]. On the other hand, the existence of a Killing vector in a wide class of spatially closed, two-dimensional cosmologies implies that all classical solutions to the field equations are spatially homogeneous [3]. It can therefore be argued that the minisuperspace approach is exact for these models [11, 12].

Adi and Solomon have adopted a geometrical approach and found a new solution of the Wheeler-DeWitt equation [13]. Navarro-Salas *et al.* [11] have quantized the induced gravity theory [14] via the covariant phase-space and reduced ADM phace-space methods. They also derived and solved the Wheeler-DeWitt equation for this model. Henneaux, on the other hand, performed the quantization of this theory in the functional Schrödinger representation by first solving the supermomentum constraint at the classical level [15]. This technique was subsequently generalized to other models [16].

In this paper we follow the approach normally employed in four-dimensional quantum cosmology. We consider spatially closed cosmologies within the generalized class of two-dimensional dilaton-gravity models, where the dilaton field is assumed to be constant on the surfaces of homogeneity. The models are defined by the functional form of the dilaton potential and specific models have been considered previously in this fashion by a number of authors [11, 17, 18, 19]. We find that the Wheeler-DeWitt equation is exactly solvable for a wide class of potentials.

The paper is organized as follows. The classical dynamics of these models is considered in Section 2 and a subset is mapped onto the constrained oscillator-ghost-oscillator model. In Section 3, these models are quantized and a number of exact solutions to the Wheeler-DeWitt equation are presented. An interpretation of these solutions is discussed within the context of a specific model. Two classes of approximate solutions are presented in Section 4. The first is a power series solution to the Wheeler-DeWitt equation derived by employing a modification of the Picard iteration scheme [20]. The second is the class of WKB solutions derived by means of a Legendre transformation [21]. In Section 5, a wider class of exactly solvable two-dimensional minisuperspaces is identified. If appropriate conditions are satisfied, these models can also be mapped onto the constrained oscillator-ghost-oscillator model. The superpotential of the wavefunction must be a separable function of the minisuperspace null coordinates. We employ this observation to transform the Wheeler-DeWitt equation derived from a renormalizable dilaton-gravity model into a solvable form [19]. In Section 6, a supersymmetric extension is considered. It is found that the superspace Hamiltonians derived from the induced gravity theory and the string effective action may each be viewed as the bosonic component of a supersymmetric Hamiltonian. This symmetry is preserved at the quantum level and the associated quantum constraints are solved exactly in closed form for both models. We conclude in Section 7 with a brief discussion on the possibility of applying the third quantization procedure to two-dimensional cosmologies.

Units are chosen such that $\hbar = c = 1$ unless otherwise stated.

2 Two-dimensional dilaton-gravity models

2.1 The generalized action

The most general action for two-dimensional dilaton-gravity that is invariant under local reparametrizations and does not contain third or higher-order derivatives is [3, 22]

$$S = -\frac{1}{2\pi} \int d^2x \sqrt{-g} \left[c_2(\Phi) R + c_1(\Phi) g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + U(\Phi) \right], \tag{2.1}$$

where $g_{\mu\nu}$ is the metric on the two-dimensional space-time manifold, g is its determinant, $c_1(\Phi)$ and $c_2(\Phi)$ are functions of the dilaton scalar field Φ, R is the curvature scalar and $U(\Phi)$ is the dilaton potential.

Specific forms for the functions $\{c_1, c_2, U\}$ correspond to different two-dimensional models. For example, the induced gravity action is a special case of Eq. (2.1) with $c_1 = 1$, $c_2 = 2\Phi$ and $U = 4\lambda^2$ [23]. The constant curvature condition may be derived from this action and provides a suitable analogue to the Einstein field equations in two dimensions [14]. Also of interest is the spherically-symmetric four-dimensional Einstein-Hilbert action. This is equivalent to Eq. (2.1) if $c_2 = e^{-2\Phi}$, $c_1 = 2c_2$ and $U = 2(1 - \Lambda e^{-2\Phi})$, where Λ is the four-dimensional cosmological constant. In this example the dilaton field is related to the radius of the two-sphere.

Actions of the form (2.1) also arise in string theory [1]. To leading order in the inverse string tension α', the tree-level effective action for the closed bosonic string in two dimensions is given by

$$S = -\frac{1}{2\pi} \int d^2x \sqrt{-g} e^{-2\Phi} \left[R + 4 (\nabla\Phi)^2 + D(\Phi) \right], \tag{2.2}$$

where $D(\Phi) = c = 16/\alpha'$ is proportional to the effective central charge and the tachyon field is assumed to be zero [24]. The field strength $H_{\mu\nu\lambda}$ of the antisymmetric tensor field vanishes identically in two dimensions. This action is closely related to the gravitational sector of the CGHS model [2].

Closed string loop corrections introduce additional terms into the beta functions and therefore modify the effective action (2.2) [25]. If field derivatives are neglected, only the dilaton potential $D(\Phi)$ is altered and, in general, the loop corrections have the form

$$D(\Phi) = \sum_{n \geq 0} a_n e^{2n\Phi}, \tag{2.3}$$

where $a_0 = c$ and n represents the number of handles inserted [26]. The values of the other coefficients a_n are determined by the string theory.

McGuigan, Nappi and Yost have considered two-dimensional string theories containing gauge fields [27]. They showed that open strings are governed by the Born-Infeld action for non-linear electrodynamics [28]. Orientable open strings may couple to $SU(N)$ and non-orientable strings to $SO(N)$ or $Sp(N)$, where N must be even for non-orientable strings [29]. When hole and crosscap corrections [30] are included, it is found that the modified equations of motion may be derived from an effective action of the form (2.2), where

$$D(\Phi) = c - \kappa(N + 2\eta)e^{\Phi}, \tag{2.4}$$

and $\eta = -1, 0, +1$ when the gauge group is $SO(N)$, $SU(N)$, or $Sp(N)$, respectively. κ is a positive-definite open string coupling constant.

Mignemi has recently investigated action (2.2) with $D = \Lambda e^{-2h\Phi}$ for some arbitrary constant h, and has found black hole solutions [31]. This action reduces to the string effective action in the limit $h \to 0$ and is also conformally equivalent to a two-dimensional higher-order, pure gravity theory with a Lagrangian given by $L = R^{h/(h-1)}$, where $h \neq 1$ [32].

The general action (2.1) may be simplified after suitable redefinitions of the dilaton and graviton fields [22]. If c_1 and c_2 are positive-definite functions, we may define a new scalar field

$$\Sigma \equiv \sqrt{2}\int d\Phi \sqrt{c_1(\Phi)} \tag{2.5}$$

and perform the conformal transformation

$$\tilde{g}_{\mu\nu} \equiv \Omega^2 g_{\mu\nu}, \qquad \Omega^2 \equiv e^{-2\rho}, \tag{2.6}$$

where

$$\rho = \frac{c_2}{q^2} - \frac{1}{4}\int^{\Sigma} d\Sigma' \left(\frac{dc_2}{d\Sigma'}\right)^{-1} \tag{2.7}$$

and q is a constant. The action (2.1) transforms to

$$S = -\frac{1}{2\pi}\int d^2x\sqrt{-\tilde{g}}\left[\frac{1}{2}q\psi\tilde{R} + \frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\psi\partial_\nu\psi + V(\psi)\right], \tag{2.8}$$

where

$$q\psi \equiv 2c_2[\Phi(\Sigma)], \qquad V(\psi) = e^{2\rho}U(\Phi). \tag{2.9}$$

Thus, the models are defined uniquely by the functional form of the dilaton potential. Unless otherwise stated, we shall view theory (2.8) as our starting action and we therefore drop the tildes for notational simplicity.

2.2 Two-dimensional cosmologies

In the canonical framework the topology of space-time is $\Sigma \times \Re$, where the real line $\Re$ corresponds to the time dimension and the spatial section Σ is either a real line or a circle S^1. The former case applies to two-dimensional black hole solutions, whereas the compact spatial topology is relevant for cosmological models. In this case the world-interval has the form

$$ds^2 = -N^2(t)dt^2 + a^2(t)dx^2, \tag{2.10}$$

where $a(t)$ is the radius of the spatial hypersurfaces and $N(t)$ is the lapse function. The spatial sections represent surfaces of constant ψ. When the line element is given by Eq. (2.10), the Ricci curvature satisfies $\sqrt{-g}R = -2\partial_t(aK)$, where the extrinsic curvature scalar is given by $K = -\dot{a}/(aN)$ and a dot denotes differentiation with respect to t. It follows, therefore, that the action (2.8) takes the form

$$S = \int dt \left[\frac{1}{N} \left(q\dot{\psi}\dot{a} + \frac{1}{2} a\dot{\psi}^2 \right) - NaV \right] \tag{2.11}$$

after integration over the spatial sections.

We now proceed to express the kinetic terms of Eq. (2.11) in canonical and diagonal forms by introducing new variables. This will allow the classical field equations to be solved for appropriate choices of lapse function and leads to simple forms for the Hamiltonian constraint. We begin by defining the new coordinate pair

$$u \equiv \sqrt{2}qae^{\psi/2q}, \qquad v \equiv \sqrt{2}qe^{-\psi/2q}. \tag{2.12}$$

The range of these variables is determined by physical considerations. The physically interesting region of parameter space is $0 < \{a, q\psi\} < +\infty$ and this corresponds to the range $0 < u < +\infty$ and $0 < v < \sqrt{2}q$. In terms of these variables the action (2.11) has the form

$$S = -\int dt \left[\frac{1}{N}\dot{u}\dot{v} + NaV \right] \tag{2.13}$$

and the Hamiltonian constraint, derived by functionally differentiating the action with respect to the non-dynamical lapse function, is given by

$$\frac{1}{N^2}\dot{u}\dot{v} - \frac{uv}{2q^2} V[\psi(v)] = 0. \tag{2.14}$$

The second term on the left-hand side is the 'superpotential'. Its direct dependence on the dynamical degrees of freedom may be eliminated by introducing the rescaled variables

$$\alpha \equiv \frac{u^2}{4q^2} \tag{2.15}$$

and

$$\beta \equiv \int^v dv' v' V[\psi(v')] = -q \int^\psi d\psi' e^{-\psi'/q} V(\psi'), \tag{2.16}$$

where β represents a rescaled dilaton field. It follows that $0 < \alpha < +\infty$, but the range of values spanned by β is model-dependent. The Jacobian of this transformation vanishes if $V(\psi)$ vanishes and we therefore restrict our discussion to potentials that are either positive- or negative-definite. The action (2.13) now takes the form

$$S = -\int dt \left[\frac{1}{NaV}\dot{\alpha}\dot{\beta} + NaV \right] \tag{2.17}$$

and the corresponding Hamiltonian constraint (2.14) becomes

$$H = aV(p_\alpha p_\beta - 1) = 0, \tag{2.18}$$

where $p_\alpha = -\dot{\beta}/(NaV)$ and $p_\beta = -\dot{\alpha}/(NaV)$ are the momenta conjugate to α and β, respectively.

In the gauge $N^{-1} = aV$, the field equations take the simple form $\ddot{\alpha} = \ddot{\beta} = 0$ and have the general solution

$$\begin{aligned} \alpha &= \alpha_0 + b(t - t_0) \\ \beta &= \beta_0 + b^{-1}(t - t_0), \end{aligned} \tag{2.19}$$

where $\{\alpha_0, \beta_0, t_0, b\}$ are constants. We conclude, therefore, that the classical dynamics of these two-dimensional Universes is equivalent to that of a non-interacting point particle propagating on two-dimensional Minkowski space. The variables α and β may be viewed as null coordinates and the regime of Minkowski space accessible to the 'particle' depends on the dilaton potential.

We now impose the additional restriction that β remains positive- or negative-definite for all physically relevant values of the scale factor and dilaton field. In this case we may introduce a third pair of variables defined by[1]

$$w \equiv \sqrt{\gamma\beta} - \sqrt{\alpha}, \qquad z \equiv \sqrt{\gamma\beta} + \sqrt{\alpha}, \tag{2.20}$$

where $\gamma = \beta/|\beta|$. For real values of α and β, $z \geq |w|$. In the gauge

$$N = \frac{1}{aV}(\alpha\gamma\beta)^{1/2}, \tag{2.21}$$

action (2.17) transforms to

$$S = -\int dt \left[\gamma\left(\dot{z}^2 - \dot{w}^2\right) + \frac{1}{4}\left(z^2 - w^2\right)\right]. \tag{2.22}$$

This is the action for the constrained oscillator-ghost-oscillator system when $\beta < 0$ and the model corresponds to a constrained hyperbolic system if $\beta > 0$. In this latter case, the field equations have the general solution

$$\begin{aligned} w &= Ae^{t/2} + Be^{-t/2} \\ z &= Ce^{t/2} + De^{-t/2}, \end{aligned} \tag{2.23}$$

where the constants of proportionality satisfy $AB = CD$ and are chosen to ensure $z \geq |w|$ for all t. The trajectory of this solution is a central conic section in this sector of the (w, z) plane. If $\beta < 0$, however, the general solution is

$$w = \epsilon E \cos\left(t/2 + \theta_1\right), \qquad z = E\cos\left(t/2 + \theta_2\right), \tag{2.24}$$

[1] This restriction on β ensures that the corresponding Wheeler-DeWitt equation will not be of the elliptic type when expressed in terms of w and z.

where $\{E, \theta_1, \theta_2\}$ are arbitrary, real constants and $|\epsilon| = 1$. These solutions lie on the family of ellipses [17]

$$w^2 + z^2 - 2\epsilon wz \cos\theta = E^2 \sin^2\theta, \tag{2.25}$$

where the eccentricity is determined by $\theta \equiv \theta_1 - \theta_2$ and the major axis lies along the line $w = \epsilon z$.

This correspondence between the constrained oscillator-ghost-oscillator and a specific two-dimensional dilaton-gravity model was recently observed by Önder and Tucker [17] in the synchronous gauge $N = 1$. Their model corresponds to the choice $c_1 = c$, $c_2 = \frac{1}{2}\Phi$ and $U = \Lambda + \lambda e^{c\Phi}$ in action (2.1), where $\{c, \lambda, \Lambda\}$ are constants. Within the context of this model, they employed such a correspondence to investigate the connection between the classical and quantum cosmologies. They identified appropriate linear superpositions of quantum states that highlighted the classical orbits (2.25) and were therefore able to conclude that a definite correlation between classical and quantum solutions exists in this model. This is interesting because the question of how a classical space-time emerges from a quantum theory of the Universe is currently unresolved. The above analysis generalizes the results of Ref. [17] and shows that the correspondence between the oscillator model and two-dimensional cosmologies arises in a wide class of dilaton-gravity models. This suggests that two-dimensional theories may provide valuable insight into the problems associated with quantum cosmology in higher dimensions. In view of this we proceed in the next Section to investigate the quantum cosmological behaviour of these models.

3 Exact quantum wavefunctions

3.1 The Wheeler-DeWitt equation

The cosmological models defined by Eq. (2.17) are quantized with the algebra $[\alpha, p_\alpha]_- = i$ and $[\beta, p_\beta]_- = i$. The Wheeler-DeWitt equation is the operator form of the Hamiltonian constraint (2.18) and is realized by identifying p_α with $-i\partial/\partial\alpha$ and p_β with $-i\partial/\partial\beta$. The physical states, Ψ, of the Universe are annihilated by this Hamiltonian operator. We shall not consider the ambiguities that arise in operator ordering, so the Wheeler-DeWitt equation has the form

$$\left[\frac{\partial^2}{\partial\alpha\partial\beta} + 1\right]\Psi = 0. \tag{3.1}$$

This equation admits a number of exact solutions [33]. One family is given by

$$\Psi_b = e^{-ib\alpha - i\beta/b}, \tag{3.2}$$

where b is an arbitrary, complex constant. $|\Psi_b|$ is bounded everywhere when $Imb = 0$ and is bounded for $Imb < 0$ if $\beta < 0$.

A natural generalization of this solution is to include a variable amplitude $\Delta(\alpha, \beta)$. Substitution of the ansatz $\Psi = \Delta\Psi_b$ into Eq. (3.1) implies that Δ satisfies

$$\frac{\partial^2 \Delta}{\partial\alpha\partial\beta} - \frac{i}{b}\frac{\partial\Delta}{\partial\alpha} - ib\frac{\partial\Delta}{\partial\beta} = 0 \tag{3.3}$$

and one non-trivial solution to this equation is

$$\Delta = b\alpha - \frac{\beta}{b}. \tag{3.4}$$

In terms of the coordinate pair (2.20) the Wheeler-DeWitt equation (3.1) transforms into [17, 33, 34, 35, 36, 37]

$$\left[\frac{\partial^2}{\partial w^2} - \frac{\partial^2}{\partial z^2} + \gamma\left(w^2 - z^2\right)\right]\Psi = 0. \tag{3.5}$$

This has separable solutions of the form $\Psi = \sum_n c_n \Psi_n$, where

$$\Psi_n = H_n\left[\sqrt{c}w\right] H_n\left[\sqrt{c}z\right] e^{-c\left[w^2+z^2\right]/2}, \tag{3.6}$$

c_n are arbitrary complex coefficients, H_n is the Hermite polynomial of order n and $c = i$ $(c = 1)$ for $\gamma = +1$ $(\gamma = -1)$.

If β does not change sign, a third class of solution is generated by defining the variables

$$\begin{aligned} s &= \frac{1}{6}\ln\left(4\alpha\gamma\beta\right) \\ r &= \frac{1}{6}\ln\left(\frac{\alpha}{\gamma\beta}\right) \end{aligned} \tag{3.7}$$

and the Wheeler-DeWitt equation becomes

$$\left[\frac{\partial^2}{\partial s^2} - \frac{\partial^2}{\partial r^2} + 9\gamma e^{6s}\right]\Psi = 0. \tag{3.8}$$

The wavefunction is an eigenstate of the momentum operator $\partial/\partial r$ and has the separable form

$$\Psi_p = e^{ipr} Z_{\pm ip/3}\left(\sqrt{\gamma}e^{3s}\right), \tag{3.9}$$

where p is a separation constant and $Z_{\pm ip/3}$ represents a linear combination of ordinary Bessel functions of order $\pm ip/3$.

It should be noted that technical questions arise when quantizing with variables that are restricted to a finite range, as is the case in the derivation of Eqs. (3.1) and (3.5). However, these issues are beyond the scope of the present work [38]. On the other hand, the variables (3.7) are unrestricted and Eq. (3.8) can be derived from the corresponding classical action with an appropriate choice of factor ordering. Eqs. (3.1) and (3.5) may then be derived directly from this equation by a change of variables.

An additional class of exact solutions may be generated by defining new variables

$$\mu \equiv \frac{\alpha}{2} + \sqrt{2\gamma\beta}, \qquad \nu \equiv \frac{\alpha}{2} - \sqrt{2\gamma\beta}. \tag{3.10}$$

The Wheeler-DeWitt equation transforms into

$$\left[\frac{\partial^2}{\partial\mu^2} - \frac{\partial^2}{\partial\nu^2} + \gamma(\mu-\nu)\right]\Psi = 0 \tag{3.11}$$

and has the separable solution

$$\Psi = \left[c_1 Ai(m-\gamma\mu) + c_2 Bi(m-\gamma\mu)\right]\left[c_3 Ai(m-\gamma\nu) + c_4 Bi(m-\gamma\nu)\right] \tag{3.12}$$

in terms of Airy functions, where $\{m, c_j\}$ are arbitrary constants [39].

3.2 Exponential dilaton interactions

It is useful to consider a specific model in order to discuss these solutions. We shall investigate an action of the form (2.2), where the dilaton potential is given by $D = \Lambda e^{-2h\Phi}$ for some constants $\{h, \Lambda\}$ [31]. This action is conformally equivalent to Eq. (2.8), where

$$q\psi = 2e^{-2\Phi}, \qquad V(\psi) = \frac{\Lambda}{8}\left(\frac{q\psi}{2}\right)^h e^{\psi/q}. \tag{3.13}$$

The kinetic terms in the action (2.11) may be diagonalized by introducing the variables

$$\begin{aligned} T &= \frac{q}{\sqrt{2}}\left(ae^{\psi/2q} + e^{-\psi/2q}\right) \\ X &= \frac{q}{\sqrt{2}}\left(ae^{\psi/2q} - e^{-\psi/2q}\right) \end{aligned} \tag{3.14}$$

and this implies that the minisuperspace metric is transformed into the Minkowski metric, where T is the timelike coordinate and X is the spacelike coordinate. Since $T \geq |X|$, only the interior region of the future light cone of the origin is covered by this coordinate system. We deduce, therefore, that the variables (2.12) represent the null coordinates $u = T + X$ and $v = T - X$ in this region. It should be noted that only a finite region of the interior of the future light cone is represented because $v < v_{max} = \sqrt{2}q$ is bounded from above.

For this model the (α, β) coordinates are given by

$$\alpha = \frac{1}{2}a^2 e^{\psi/q} \tag{3.15}$$

and

$$\beta = -\frac{\Lambda}{4}\frac{1}{1+h}\left(\frac{q\psi}{2}\right)^{1+h}, \qquad h \neq -1$$
$$\beta = -\frac{\Lambda}{4}\ln(\psi), \qquad h = -1, \tag{3.16}$$

respectively, and when $h \neq -1$, the sign of β is uniquely determined by the sign of $\Lambda/(1+h)$.

The parameters $\{\alpha, \beta\}$ may also be viewed as null coordinates over a region of Minkowski space spanned by the timelike coordinate $\tilde{p} = \alpha + \beta$ and spacelike coordinate $\tilde{q} = \alpha - \beta$. Thus, if $\beta > 0$, it follows that $\tilde{p} \geq |\tilde{q}|$ and the analysis is again restricted to the interior of the future light cone of the origin. Since $\alpha, \beta \in (0, \infty)$, the whole of the interior is now covered. On the other hand, if $\beta < 0$, $\tilde{q} \geq |\tilde{p}|$, and the propagation of the wavefunction is restricted to the Rindler wedge of Minkowski space.

The simplest interpretation of the wavefunction identifies an oscillating solution to the Wheeler-DeWitt equation as a Lorentzian geometry and a cosmological singularity is associated with an infinite number of oscillations [40]. A non-oscillating solution represents a classically forbidden Euclidean geometry. Let us consider the case where $\beta < 0$ and $h \neq -1$. Eq. (3.6) represents the basis for a discrete spectrum of Euclidean solutions, where the parameter n determines the excitation level of the wavefunction [36]. The ground state is associated with $n = 0$ and excited states with $n > 0$. This ground state is identical to solution (3.2) when $b = -i$. Hence, we may view the solution $\Psi_{b=-i}$ as the ground state of a continuous spectrum of excited states (3.2) that are parametrized by the separation constant b with $Reb = 0$ and $Imb < 0$.

Although these classes of Euclidean solution appear to correspond to classically forbidden behaviour, Lorentzian wavefunctions may be generated from appropriate linear combinations of the excited states. A more general solution to Eq. (3.1) is given by [33]

$$\Psi = \int_C dc M(c) e^{-c\alpha - \gamma\beta/c}, \tag{3.17}$$

where $M(c)$ is an arbitrary function of the parameter $c \equiv ib$ and C represents some contour of integration in the complex plane. If $M(c) = \frac{1}{2}c^{(ip-3)/3}$ and the contour of integration is over the positive half of the real axis, Eq. (3.17) may be evaluated exactly in terms of the modified Bessel function:

$$\Psi_p = \frac{1}{2}\int_0^\infty dc c^{(ip-3)/3} e^{-c\alpha - |\beta|/c} = e^{ipr} K_{ip/3}\left(2\sqrt{\alpha\gamma\beta}\right). \tag{3.18}$$

We recognize this superposition as solution (3.9) with $Z = K(e^{3s})$. This solution may also be generated from a linear combination of harmonic oscillator wavefunctions and, in general, solutions (3.2), (3.6) and (3.9) may be expressed as linear combinations of one other for positive and negative β [33, 36, 37].

The modified Bessel function has the asymptotic form $K_q(x) \propto x^{-q}$ for $|x| \ll 1$ and $q \neq 0$. Thus, the wavefunction has the form $\Psi_p \propto e^{ip(r \pm s)}$ for small spatial geometries $(s \to -\infty)$ and these represent plane waves in the variables (r, s). The wavefunction oscillates an infinite number of times when the spatial volume of the Universe vanishes and we identify this point as a cosmological singularity. However, the wavefunction is exponentially damped for $e^{3s} > |p|/3$ and this region of minisuperspace is classically forbidden. It is interesting to relate this solution to the classical solution in terms of the variables (3.7). The gravitational field equations derived from action (2.17) in the gauge $N = 9e^{6s}/(4aV\gamma)$ are given by

$$\ddot{r} = 0, \qquad \ddot{s} = \frac{27}{4\gamma} e^{6s} \tag{3.19}$$

and the general solution satisfying the Hamiltonian constraint is

$$\begin{aligned} r &= At + B \\ \pm t &= \frac{1}{3A} \ln \left[e^{-3s} + \left(e^{-6s} - \frac{9}{4A^2} \right)^{1/2} \right], \end{aligned} \tag{3.20}$$

where $\{A, B\}$ are arbitrary constants. (We have perfomed a linear translation on t without loss of generality). It follows that the value of s is bounded from above by the constraint $e^{3s} < 2A/3$ and we may therefore identify the eigenvalue of the momentum operator $\partial/\partial r$ with the integration constant A, i.e. $|p| = 2A$.

In general, it is difficult to evaluate Eq. (3.17) exactly. However, it can be related directly to solution (3.12) by performing a trivial rescaling $\tilde{c} = -2c$ and specifying

$$M(\tilde{c}) = \frac{1}{\tilde{c}^{1/2}} \exp\left[-\frac{\tilde{c}^3}{12} + m\tilde{c} \right], \tag{3.21}$$

where m is an arbitrary constant. Substitution of variables (3.10) into Eq. (3.17) therefore implies that

$$\Psi = \int_C \frac{d\tilde{c}}{\tilde{c}^{1/2}} \exp\left[-\frac{\tilde{c}^3}{12} + (\mu + \nu + 2m)\frac{\tilde{c}}{2} + \frac{1}{4\tilde{c}}(\mu - \nu)^2 \right]. \tag{3.22}$$

An integral of this form has been evaluated previously by Halliwell and Louko within the context of the path integral quantization of the four-dimensional de Sitter Universe [41]. They showed that it may be expressed in terms of products of Airy functions, where the specific combination is determined by the contour of integration. In our example, we wish to construct wavefunctions from a linear combination of bounded wavefunctions of the form (3.2) and therefore require $Re\tilde{c} < 0$. If the contour of integration is chosen to lie along the line $\tilde{c} = i\eta - \epsilon$, where η is real and $\epsilon > 0$, it will pass to the left of the origin in the complex plane. In this case, Eq. (3.22) is given by [41]

$$\Psi = Ai(\mu + m)Ai(\nu + m). \tag{3.23}$$

Up to a numerical constant, this solution corresponds to Eq. (3.12) with $c_1 = c_3$ and $c_2 = c_4 = 0$. It oscillates if either $\nu + m < 0$ or $\mu + m < 0$ and is exponentially damped when both arguments are positive.

Thus far, we have derived exact solutions to the Wheeler-DeWitt equation. However, it is useful to search for approximate solutions as well. Although such solutions are not exact, they can provide insight into the nature of the wavefunction. In the following Section, we shall discuss two classes of approximate solutions.

4 Approximate wavefunctions

4.1 Power series solutions

Power series solutions to the Wheeler-DeWitt equation (3.1) may be derived by expanding the wavefunction as the infinite sum of functions

$$\Psi = \sum_{m=0}^{\infty} \Psi_m. \tag{4.1}$$

This ansatz is a consistent solution to Eq. (3.1) if

$$\frac{\partial^2 \Psi_0}{\partial \alpha \partial \beta} = 0 \tag{4.2}$$

and

$$\frac{\partial^2 \Psi_m}{\partial \alpha \partial \beta} = -\Psi_{m-1}, \qquad m \geq 1. \tag{4.3}$$

Eq. (4.2) is the canonical, one-dimensional wave equation and has the general solution

$$\Psi_0 = P(\alpha) + Q(\beta), \tag{4.4}$$

where P and Q are arbitrary, twice continuously differentiable functions. A modification of the Picard iteration scheme [20, 34] may now be established by expressing Ψ_m in terms of quadratures with respect to the null variables $\{\alpha, \beta\}$. When $m = 1$, Eq. (4.3) admits the separable solution

$$\Psi_1 = -\left[\beta \int^{\alpha} d\alpha_1 P(\alpha_1) + \alpha \int^{\beta} d\beta_1 Q(\beta_1)\right] \tag{4.5}$$

and this result may then be substituted back into Eq. (4.3) to derive Ψ_2 and so on. The general pattern is easy to deduce after a few iterations and we conclude, therefore, that

$$\begin{aligned} \Psi = \Psi_0 + \sum_{m=1}^{\infty} \frac{(-1)^m}{m!} \Bigg[& \beta^m \int^{\alpha} d\alpha_m \ldots \int^{\alpha_3} d\alpha_2 \int^{\alpha_2} d\alpha_1 P(\alpha_1) \\ & + \alpha^m \int^{\beta} d\beta_m \ldots \int^{\beta_3} d\beta_2 \int^{\beta_2} d\beta_1 Q(\beta_1) \Bigg] \end{aligned} \tag{4.6}$$

is also a solution to the Wheeler-DeWitt Eq. (3.1).

It should be emphasized that we have not assumed a semi-classical approximation in deriving these power series solutions. The advantage of this scheme is that the wavefunction is given in terms of *arbitrary* functions of α and β. In many minisuperspace models these variables are related to the spatial volume of the Universe, where small values of α or β typically correspond to small spatial volumes. Indeed, this is the case for two-dimensional dilaton-gravity cosmologies, since α is proportional to the square of the scale factor. Consequently, for $P(\alpha) = 0$, we may view solution (4.6) in the region of the origin as an expansion in powers of a small parameter α.

4.2 Semi-classical wavefunctions

Within the context of four-dimensional cosmologies, the nature of space-time is accurately described by classical physics when the spatial volume of the Universe is significantly larger than the Planck scale. It follows, therefore, that classical behaviour from the quantum regime should be predicted by the quantum theory. Presently, the problem of how such a transition might occur is an unresolved one. However, it is reasonable to suppose that the nature of semi-classical wavefunctions may provide some insight.

In the WKB approximation, corresponding to the limit $\hbar \to 0$, one treats some of the degrees of freedom $\{c\}$ as classical variables and the remainder $\{q\}$ quantum-mechanically. The wavefunction is then viewed as a linear superposition of waves of the form $\Psi \approx e^{-iS/\hbar}$, where S is the classical action satisfying the Hamilton-Jacobi equation. This equation is derived by identifying the conjugate momenta in Eq. (2.18) with $p_\alpha = \partial S/\partial\alpha$ and $p_\beta = \partial S/\partial\beta$. It takes the form of a non-linear, first-order partial differential equation:

$$\frac{\partial S}{\partial \chi}\frac{\partial S}{\partial \beta} = \chi, \tag{4.7}$$

where a new variable $\chi \equiv \sqrt{2\alpha}$ has been introduced. It is well known that there exists a one-to-one correspondence between congruences of classical solutions and solutions to the Hamilton-Jacobi equation in two-dimensional minisuperspaces [42]. In principle, therefore, an arbitrary solution to Eq. (4.7) may be generated once the classical solutions are known.

However, parametric solutions may be found more directly by employing a Legendre transformation [21]. We define new variables

$$\xi \equiv \frac{\partial S}{\partial \chi}, \qquad \eta \equiv \frac{\partial S}{\partial \beta} \tag{4.8}$$

and a new function

$$\rho(\xi, \eta) \equiv \chi\xi + \beta\eta - S(\chi, \beta). \tag{4.9}$$

Partial differentiation with respect to ξ implies that

$$\chi = \frac{\partial \rho}{\partial \xi} = \xi \eta, \tag{4.10}$$

where the second equality follows from Eq. (4.7). Eq. (4.10) has the *general* solution

$$\rho = \frac{1}{2}\eta \xi^2 + f(\eta), \tag{4.11}$$

where $f(\eta)$ is an arbitrary function of η.

A parametric solution may now be found by transforming back into the old variables. After differentiation of Eq. (4.11) with respect to η, we find that

$$\begin{aligned} S &= \frac{2\alpha}{\eta} - f(\eta) + \eta \frac{df}{d\eta} \\ \beta &= \frac{\alpha}{\eta^2} + \frac{df}{d\eta}. \end{aligned} \tag{4.12}$$

It should be noted that this Legendre transformation is only self-consistent if the Jacobian

$$J = \frac{\partial^2 S}{\partial \chi^2}\frac{\partial^2 S}{\partial \beta^2} - \left(\frac{\partial^2 S}{\partial \chi \partial \beta} \right)^2 \tag{4.13}$$

is non-vanishing. Solutions are said to be 'developable' if $J \neq 0$ and 'non-developable' if $J = 0$. All developable solutions can be written in the parametric form of Eq. (4.12). In principle, we can determine $\eta = \eta(\alpha, \beta)$ from the second equation in (4.12) once the functional form of $f(\eta)$ has been specified. Substituting this result into the first equation yields the action in terms of the canonical variables, or equivalently, in terms of the original variables via Eqs. (2.12), (2.15) and (2.16).

For example, the Jacobian is non-vanishing if

$$f(\eta) = \frac{\alpha_i}{\eta} + \beta_i \eta, \tag{4.14}$$

where $\{\alpha_i, \beta_i\}$ are finite constants, and this ansatz leads to the action

$$S = 2\sqrt{(\alpha - \alpha_i)(\beta - \beta_i)}. \tag{4.15}$$

When $f = 0$, this solution is closely related to the exact solution of Eq. (3.8) that is given by $\Psi = H_0^{(2)}\left(2\sqrt{\alpha\beta}\right)$, where $H_0^{(2)}(x)$ is the Hankel function. For small arguments this function has the asymptotic form $2i\pi^{-1}\ln(x)$, so the wavefunction does not oscillate. On the other hand, the Hankel function has the form $H_0^{(2)}(x) \propto x^{-1/2}e^{\pm ix}$ at large arguments and this does have oscillatory behaviour. In this example, a large argument corresponds to a large value of the scale factor. Consequently, the argument of this solution may be identified with the action (4.15) and represents a classically allowed solution that has tunneled from the Euclidean regime.

Exact solutions can also be found if $f \propto \eta^{\pm 3}$ and $f \propto \ln \eta$. Furthermore, it is interesting to note that solution (3.2) to the full Wheeler-DeWitt equation (3.1) is of the WKB form $\Psi = e^{-iS}$, where $S = b\alpha + b^{-1}\beta$. This is an exact, non-developable solution to the Hamilton-Jacobi equation (4.7) and in this sense the WKB approximation is exact for this solution.

This concludes our discussion on approximate solutions. In the next Section we shall investigate whether other minisuperspace models can be solved in the manner discussed above.

5 A class of integrable minisuperspaces

It is interesting to investigate whether a wider class of models leads to the Wheeler-DeWitt equation (3.1). To proceed, we investigate an equation of the form

$$\left[\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + 4m^2(x,y)\right]\Psi = 0, \tag{5.1}$$

where the superpotential, $m^2(x,y)$, is some function of the minisuperspace coordinates (x,y).

We introduce new variables $\alpha = \alpha(\sigma)$ and $\beta = \beta(\tau)$ that are *arbitrary* functions of the minisuperspace null coordinates $\sigma \equiv x + y$ and $\tau \equiv x - y$. These new variables satisfy the boundary conditions $\partial\alpha/\partial x = \partial\alpha/\partial y$ and $\partial\beta/\partial x = -\partial\beta/\partial y$ and these constraints ensure that the derivative terms in Eq. (5.1) are transformed into the canonical form:

$$\left[\frac{\partial\alpha}{\partial x}\frac{\partial\beta}{\partial x}\frac{\partial^2}{\partial\alpha\partial\beta} + m^2\right]\Psi = 0. \tag{5.2}$$

It follows that Eq. (5.2) reduces to Eq. (3.1) if the new variables α and β are themselves solutions to the equation

$$m^2 = \frac{\partial\alpha}{\partial x}\frac{\partial\beta}{\partial x} = \frac{d\alpha}{d\sigma}\frac{d\beta}{d\tau}. \tag{5.3}$$

In principle, therefore, Eq. (5.1) may be solved if a solution to the constraint equation (5.3) can be found. Effectively, the problem of solving the linear, second-order partial differential equation (5.1) has been reduced to finding a solution to the non-linear, *first-order* equation (5.3) and in many cases it is considerably easier to solve this latter equation. Indeed, it is clear from the second equality in Eq. (5.3) that when the superpotential has the generic form

$$m^2(x,y) = m_+(\sigma)m_-(\tau), \tag{5.4}$$

where $m_\pm$ are some known analytical functions, Eq. (5.3) admits the general separable solution

$$\alpha = \lambda \int^{\sigma} d\sigma' m_+(\sigma'), \qquad \beta = \lambda^{-1} \int^{\tau} d\tau' m_-(\tau'), \tag{5.5}$$

where λ is an arbitrary separation constant. The region of minisuperspace covered by these coordinates is determined by the specific form of the superpotential.

The Jacobian of the transformation leading to Eq. (5.2) vanishes whenever the null variables $\alpha = \alpha(\sigma)$ or $\beta = \beta(\tau)$ have turning points and these will occur at the zero points of the superpotential if Eq. (5.3) is satisfied. Thus, the Wheeler-DeWitt equation can be mapped onto the unit-mass Klein-Gordon equation if the superpotential is positive- or negative-definite over the whole region of minisuperspace covered by $\{\alpha, \beta\}$ and, in addition, is a separable function of these null coordinates. For example, if $\alpha > 0$ and $\beta < 0$, the solutions discussed in Section 3.2 are also solutions to Eq. (5.1). If, on the other hand, the superpotential does vanish at some point in minisuperspace, equivalent transformations to those discussed above may be performed on both sides of the zero-point. The two solutions in the different regions may then be matched at the boundary by requiring that the wavefunction and its first derivative are continuous [43].

There are a number of interesting minisuperspaces for which Eq. (5.3) can be solved exactly. In many cases the superpotential of the wavefunction is independent of one of the null coordinates, i.e. it is a single function of either σ or τ. This is the case for the Wheeler-DeWitt equation derived from a renormalizable, two-dimensional dilaton-gravity theory. One-loop quantum corrections to the CGHS action have been calculated by Russo, Susskind and Thorlacius [44]. In the conformal gauge $g_{+-} = -e^{2\rho}/2$, $g_{\pm\pm} = 0$, the one-loop effective action has the form

$$S = \frac{1}{\pi} \int d^2x \left[-\frac{1}{\kappa}\partial_+\chi\partial_-\chi + \frac{1}{\kappa}\partial_+\Omega\partial_-\Omega + \mu^2 e^{2(\chi-\Omega)/\kappa} + \frac{1}{2}\sum_{j=1}^{N} \partial_+ f_j \partial_- f_j \right], \tag{5.6}$$

where

$$\chi = \kappa\rho - \frac{\kappa}{2}\Phi + e^{-2\Phi} \tag{5.7}$$

represents a Liouville-type field,

$$\Omega = \frac{\kappa}{2}\Phi + e^{-2\Phi} \tag{5.8}$$

is a rescaled version of the dilaton field Φ, f_j are conformal scalar fields and the constants $\kappa = (N-24)/12$ and μ^2 are assumed to be positive-definite[2].

[2]The reader is referred to Ref. [44] for the details of the derivation. The numerical value of κ is determined by including the one-loop contributions from the reparametrization ghosts, dilaton and conformal fields. The theory is one-loop finite if $\kappa = (N-24)/12$.

The Wheeler-DeWitt equation corresponding to this renormalizable model of dilaton-gravity has been derived by Mazzitelli and Russo [19]. It has the form

$$\left[\frac{\kappa}{4}\frac{\partial^2}{\partial\chi_0^2} - \frac{\kappa}{4}\frac{\partial^2}{\partial\Omega_0^2} - \frac{1}{2}\sum_{j=1}^{N}\frac{\partial^2}{\partial f_{j0}^2} - 4\mu^2 e^{2(\chi_0-\Omega_0)/\kappa} - \kappa - 2\right]\Psi = 0, \tag{5.9}$$

where χ_0, etc., represent the zero modes of the harmonic expansion of the fields on the cylinder. In this analysis it is assumed that the coupling between the zero and higher-order modes is negligible and this is equivalent to invoking the minisuperspace approximation. This represents an improvement over the approximation employed to derive Eq. (3.1), however, since this latter equation follows from the classical action (2.2), whereas Eq. (5.9) follows directly from the one-loop effective action (5.6).

Eq. (5.9) is solved by separating the wavefunction into its gravitational and matter components with the ansatz $\Psi = \Phi(\chi_0, \Omega_0)\varphi(f_{j0})$. The plane waves $\varphi = \exp\left[i\sum_j Z_j f_{j0}\right]$ form a basis for the solutions, where Z_j are arbitrary constants. By identifying $(x, y) \equiv (\chi_0, \Omega_0)$, it is readily seen that Φ statisfies Eq. (5.1), where the superpotential is given by

$$\kappa m^2 = \frac{Z^2}{2} - \kappa - 2 - 4\mu^2 e^{2(\chi_0-\Omega_0)/\kappa} \tag{5.10}$$

and $Z^2 \equiv \sum_j Z_j^2$ represents the total momentum eigenvalue of the matter sector. If $Z^2 < 2\kappa + 4$, m^2 is negative-definite and a function of $(\chi_0 - \Omega_0)$ only. We may therefore choose $m_+ = 1$ in Eq. (5.5) and this implies that Φ satisfies an equation of the form $\partial^2\Phi/\partial\alpha\partial\beta = -\Phi$, where

$$\alpha = \lambda(\chi_0 + \Omega_0)$$

$$\beta = \frac{1}{\lambda\kappa}\left[\left(\frac{Z^2}{2} - \kappa - 2\right)(\chi_0 - \Omega_0) - 2\kappa\mu^2 e^{2(\chi_0-\Omega_0)/\kappa}\right]. \tag{5.11}$$

On the other hand, the superpotential vanishes along the null line

$$\chi_0 - \Omega_0 = \frac{\kappa}{2}\ln\left[\frac{Z^2 - 2\kappa - 4}{8\mu^2}\right] \tag{5.12}$$

if $Z^2 > 2\kappa + 4$ and in this case different coordinate representations must be employed on either side of this line.

We conclude, therefore, that the wavefunctions discussed in earlier sections also apply to this renormalizable model of dilaton-gravity. In particular, Lorentzian solutions to Eq. (5.9) may be generated from linear superpositions of Euclidean solutions and vice-versa when $\alpha > 0$ and $\beta < 0$. It follows immediately from Eq. (5.11) that these conditions are satisfied for all $\chi_0 > \Omega_0 > 0$ when $\lambda > 0$ and $Z^2 < 2\kappa + 4$.

This concludes our discussion on exact bosonic wavefunctions. In the following Section we shall investigate whether supersymmetric extentions to the quantum models discussed above can be performed.

6 Supersymmetric quantum cosmology

Graham discovered that a hidden symmetry exists in the Bianchi IX Universe by showing how the classical superspace Hamiltonian may be viewed as the bosonic part of a supersymmetric Hamiltonian [45]. It has now been shown that this hidden symmetry exists in all Bianchi class A models [46, 47, 48, 49]. This implies that a supersymmetry can be introduced at the quantum level. This supersymmetric extension of the quantum theory has significant consequences for quantum cosmology, as shown by calculations for the Bianchi II Universe [49]. It is thought that these extensions may provide valuable insight into some of the questions relevant to a complete theory of quantum gravity. In particular, they may resolve the problems encountered when one attempts to construct a conserved probability from the wavefunction [48]. It is therefore of interest to investigate whether hidden supersymmetries exist in two-dimensional dilaton-gravity models.

We begin by briefly reviewing the 'hidden symmetry' method of Graham [45]. In the minisuperspace approximation the classical Hamiltonian constraint takes the form

$$2H_0 = G^{\mu\nu} p_\mu p_\nu + W(q) = 0, \tag{6.1}$$

where $G_{\mu\nu}$ is the metric with signature $(-,+,+,\dots)$ on the $(D+1)$-dimensional minisuperspace spanned by the finite number of degrees of freedom q^μ $(\mu = 0, 1, \dots, D)$. The momenta conjugate to these variables are p_μ and W represents the superpotential. This Hamiltonian is the bosonic component of a supersymmetric Hamiltonian [50, 51] if there exists a function $I(q)$ that respects the same symmetries as H_0 and is itself a solution to the Euclidean Hamilton-Jacobi equation

$$W = G^{\mu\nu} \frac{\partial I}{\partial q^\mu} \frac{\partial I}{\partial q^\nu}. \tag{6.2}$$

Fermionic degrees of freedom $\varphi^\mu, \bar{\varphi}^\nu$ obeying the spinor algebra

$$[\varphi^\mu, \varphi^\nu]_+ = [\bar{\varphi}^\mu, \bar{\varphi}^\nu]_+ = 0, \qquad [\varphi^\mu, \bar{\varphi}^\nu]_+ = G^{\mu\nu} \tag{6.3}$$

are then introduced. It follows that the supercharges

$$Q \equiv \varphi^\mu \left(p_\mu + i \frac{\partial I}{\partial q^\mu} \right), \qquad \bar{Q} \equiv \bar{\varphi}^\mu \left(p_\mu - i \frac{\partial I}{\partial q^\mu} \right) \tag{6.4}$$

satisfy

$$Q^2 = \bar{Q}^2 = 0 \tag{6.5}$$

and, if Eq. (6.2) is satisfied, the Hamiltonian (6.1) may be written as

$$2H_0 = [Q, \bar{Q}]_+, \qquad [H_0, Q]_- = [H_0, \bar{Q}]_- = 0. \tag{6.6}$$

These equations represent the algebra for a single, complex supersymmetry charge Q and the model therefore exhibits an $N = 2$ supersymmetry [51].

This symmetry is preserved at the quantum level by choosing the representation $\bar{\varphi}^\mu = \theta^\mu$ and $\varphi^\mu = G^{\mu\lambda}\partial/\partial\theta^\lambda$ for the fermionic degrees of freedom, where θ^μ are Grassmann variables [45, 47]. The bosonic degrees of freedom have the usual representation $p_\mu = -i\hbar\partial/\partial q^\mu$. Eqs. (6.5) and (6.6) now represent the operator realizations of the supersymmetric algebra. The quantized superspace Hamiltonian is given by

$$H = H_0 + \frac{\hbar}{2}\frac{\partial^2 I}{\partial q^\mu \partial q^\nu}[\bar{\varphi}^\mu, \varphi^\nu]_- \tag{6.7}$$

and has an additional term that vanishes in the classical limit. The existence of this term suggests that suitable imaginary or complex solutions to Eq. (6.2) will be difficult to find. It follows that the supersymmetric wavefunctions are annihilated by the supercharge operators

$$Q\Psi = \bar{Q}\Psi = 0 \tag{6.8}$$

and it is these constraints that represent the 'square roots' of the Wheeler-DeWitt equation.

6.1 Induced gravity theory

To investigate whether the two-dimensional cosmological models considered in Section 2 exhibit a hidden supersymmetry of the form discussed above, we must first identify the symmetries of the classical Hamiltonian (2.14). The kinetic part is invariant under the simultaneous interchanges $u \leftrightarrow \pm v$. However, the full Hamiltonian is not necessarily invariant under this interchange because of the generality of the dilaton potential. On the other hand, it is symmetric under both simultaneous interchanges if the potential is constant, i.e. if $V \equiv \lambda^2$. This form of the potential in Eq. (2.8) corresponds to the induced gravity action for $q = \sqrt{8}$ and it is straightforward to verify that the above symmetries are equivalent to an invariance under $\alpha \leftrightarrow \pm\beta$.

Evaluation of Eq. (2.16) implies that $\beta = q^2\lambda^2 e^{-\psi/q}$ and comparing Eq. (2.18) with Eq. (6.1) implies that the non-zero components of the minisuperspace metric are $G_{\alpha\beta} = G_{\beta\alpha} = a\lambda^2$. The superpotential is therefore given by $W = -2a\lambda^2$ and it follows that one solution to Eq. (6.2) that respects the symmetries of the Hamiltonian is

$$I = -2i(\alpha\beta)^{1/2}. \tag{6.9}$$

Since we require this 'Euclidean' action to be real, we must assume that $\lambda^2 < 0$.

The functional form of the supersymmetric wavefunction may now be determined by solving the constraints (6.8). Due to the anticommutation relations obeyed by the Grassmann variables θ^μ, the general supersymmetric wavefunction may be expanded as

$$\Psi = A_+ + B_0\theta^0 + B_1\theta^1 + C_2\theta^0\theta^1, \tag{6.10}$$

where the bosonic functions $\{A_+, B_0, B_1, C_2\}$ are functions of $\{\alpha, \beta\}$ only. The annihilation of the wavefunction by the supercharge operators then translates into a set of coupled, first-order partial differential equations:

$$
\begin{aligned}
\left[\frac{\partial}{\partial\alpha}+\frac{\partial I}{\partial\alpha}\right]A_+ &= 0\\
\left[\frac{\partial}{\partial\beta}+\frac{\partial I}{\partial\beta}\right]A_+ &= 0\\
\left[\frac{\partial}{\partial\alpha}+\frac{\partial I}{\partial\alpha}\right]B_1-\left[\frac{\partial}{\partial\beta}+\frac{\partial I}{\partial\beta}\right]B_0 &= 0\\
\left[\frac{\partial}{\partial\alpha}-\frac{\partial I}{\partial\alpha}\right]B_1+\left[\frac{\partial}{\partial\beta}-\frac{\partial I}{\partial\beta}\right]B_0 &= 0\\
\left[\frac{\partial}{\partial\alpha}-\frac{\partial I}{\partial\alpha}\right]C_2 &= 0\\
\left[\frac{\partial}{\partial\beta}-\frac{\partial I}{\partial\beta}\right]C_2 &= 0.
\end{aligned}
\tag{6.11}
$$

The solution to these equations is given by

$$
\begin{aligned}
A_+ &= e^{-I}\\
C_2 &= e^{I}\\
B_0 &= \frac{\partial F}{\partial\alpha}+F\frac{\partial I}{\partial\alpha}\\
B_1 &= \frac{\partial F}{\partial\beta}+F\frac{\partial I}{\partial\beta},
\end{aligned}
\tag{6.12}
$$

where the function $F=F(\alpha,\beta)$ is itself a solution to the equation

$$
\frac{\partial^2 F}{\partial\alpha\partial\beta}+\left(\frac{\partial^2 I}{\partial\alpha\partial\beta}-\frac{\partial I}{\partial\alpha}\frac{\partial I}{\partial\beta}\right)F=0.
\tag{6.13}
$$

When I is given by Eq. (6.9), Eq. (6.13) simplifies to

$$
\left[\frac{\partial^2}{\partial w^2}-\frac{\partial^2}{\partial z^2}-w^2+z^2+2\right]F=0,
\tag{6.14}
$$

where $\{w,z\}$ are defined by Eq. (2.20). Hence, F may be interpreted physically as the wavefunction for a quantum system describing two coupled harmonic oscillators that have identical frequencies but a difference in energy of 2. The solution to Eq. (6.14) has the separable form

$$
F=H_n\left(\sqrt{\gamma\beta}+\sqrt{\alpha}\right)H_{n+1}\left(\sqrt{\gamma\beta}-\sqrt{\alpha}\right)e^{-(\alpha+\gamma\beta)}.
\tag{6.15}
$$

The functions A_+ and C_2 represent the empty and filled fermion sectors of the Hilbert space. Both may be interpreted as lowest-order, WKB approximations to exact solutions of the bosonic Wheeler-DeWitt equation (3.8). This equation may be written formally as $\hat{H}_{(0)}\Psi = \hat{H}_{(1)}\Psi$, where we have split the Wheeler-DeWitt operator into the two components

$$\hat{H}_{(0)} = -\frac{\partial^2}{\partial s^2} - 9\gamma e^{6s}, \qquad \hat{H}_{(1)} = -\frac{\partial^2}{\partial r^2}. \tag{6.16}$$

Application of $\hat{H}_{(1)}$ implies that $\hat{H}_{(1)}\Psi = E\Psi$, where $E = p^2$. If the eigenvalue p is real, this equation may be interpreted as the Schrödinger equation, where E represents the energy associated with $\hat{H}_{(1)}$ [52]. Therefore, the state with $E = 0$ corresponds to the state of lowest energy. When $\beta < 0$ and $p = 0$, the general form of the bosonic wavefunction (3.9) is given by a linear combination of modified Bessel functions $I_0(x)$ and $K_0(x)$, where $x = 2\sqrt{\alpha\gamma\beta}$. For large x these functions have the asymptotic forms $I_0 \propto e^x$ and $K_0 \propto e^{-x}$, respectively, and these limits correspond to the solutions C_2 and A_+. The supersymmetric vacua are therefore closely related to their semi-classical limits and correspond to the pure bosonic states of lowest energy. These features appear to be generic properties of supersymmetric ground state wavefunctions [51].

6.2 A string inspired model

String theory exhibits a symmetry known as target space duality [53]. (For a recent review see, e.g., Ref. [54]). In two-dimensional space-times, a string cannot tell if it is propagating on a circle of radius a or of radius a^{-1}. In effect, this allows one to transform between theories of radii a and a^{-1} after a suitable translation on the dilaton field [55]. It is convenient to consider this symmetry within the context of the action (2.2) in the absence of loop corrections, i.e. $D = c > 0$. The Hamiltonian derived from this action for the cosmological space-time (2.10) is given by

$$H = e^{-2\Phi}\left[\frac{4}{N^2}\left(\dot{a}\dot{\Phi} - a\dot{\Phi}^2\right) - ca\right] \tag{6.17}$$

and it is straightforward to verify that Eq. (6.17) is invariant under the duality transformation

$$a = \frac{1}{A}, \qquad \Phi = \phi + \ln a. \tag{6.18}$$

If we introduce the coordinate pair

$$X \equiv ae^{-\Phi}, \qquad Y = e^{-\Phi}, \tag{6.19}$$

the Hamiltonian takes the form

$$H = -\frac{4}{N^2}\dot{X}\dot{Y} - cXY \tag{6.20}$$

and invariance under the duality transformation (6.18) is therefore equivalent to an invariance under the simultaneous interchange $X \leftrightarrow Y$.

The momenta conjugate to X and Y are $p_X = 4\dot{Y}/N$ and $p_Y = 4\dot{X}/N$, respectively. It is convenient to perform a rescaling of these degrees of freedom by defining $\alpha \equiv X^2$ and $\beta \equiv Y^2$. The classical Hamiltonian (6.20) is then given by Eq. (6.1), where the non-vanishing components of the minisuperspace metric are $G_{\alpha\beta} = G_{\beta\alpha} = (\alpha\beta)^{1/2}$ and the superpotential $W = 2c(\alpha\beta)^{1/2}$. There exists a hidden supersymmetry if I satisfies

$$\frac{\partial I}{\partial \alpha}\frac{\partial I}{\partial \beta} = c \tag{6.21}$$

and also respects the duality symmetry $\alpha \leftrightarrow \beta$. One solution satisfying the necessary conditions is $I = 2\sqrt{c\alpha\beta}$ and since $\{\alpha, \beta\}$ are positive-definite, $c > 0$ is necessary for the solution to be real.

We conclude, therefore, that the general supersymmetric wavefunction may also be found in closed form for this theory. The bosonic functions in the Grassmann basis expansion of the wavefunction are again given by Eq. (6.12), but F has the slightly different form

$$F = H_n(\eta)H_{n+1}(\xi)e^{-(\xi^2+\eta^2)/2}, \tag{6.22}$$

where $\xi \equiv c^{1/4}(\alpha^{1/2} + \beta^{1/2})$ and $\eta \equiv c^{1/4}(\alpha^{1/2} - \beta^{1/2})$.

7 Conclusions and discussion

In this paper we have investigated the quantum cosmology of a generalized class of two-dimensional dilaton-gravity models. If the dilaton potential contains no roots, i.e. if $V(\psi) \neq 0$ for all physically interesting ψ, the classical dynamics of these Universes is equivalent to that of a non-interacting, point particle propagating over a portion of two-dimensional Minkowski space. A large subset of this class of models is dynamically equivalent to the isotropic, constrained oscillator-ghost-oscillator system. This suggests that the relationship between quantum configurations and classical space-times, as discussed in Ref. [17], could be generalized to these models.

Furthermore, these correspondences imply that the Wheeler-DeWitt equation can be expressed as the unit-mass Klein-Gordon equation if a suitable choice of factor ordering is made. This allows a number of exact and approximate solutions to be found. Quantum states corresponding to Lorentzian geometries may be generated from an infinite sum of Euclidean solutions and vice-versa. The Hamilton-Jacobi equation can be solved by employing a Legendre transformation and all developable solutions to this equation were found in parametric form.

We proceeded to identify a wider class of integrable two-dimensional minisuperspaces that can be solved exactly by mapping the Wheeler-DeWitt equation onto the unit-mass Klein-Gordon equation. This mapping is possible if the superpotential of the wavefunction is a separable function of the null coordinates over minisuperspace. We applied this result to the Wheeler-DeWitt equation derived from a renormalizable, two-dimensional dilaton-gravity model [19].

One of the main problems with the quantum cosmology program is the construction of a non-negative norm from solutions to the Wheeler-DeWitt equation. This equation is a hyperbolic, second-order partial differential equation, so the conserved current associated with it is not necessarily semi-positive definite. Consequently, it is not clear that such a current will provide a suitable measure of probability. A similar problem is encounted when the Klein-Gordon equation for a scalar field is solved. In this case, however, the ambiguity is resolved by taking the 'square root' and in view of the close correspondence between the Wheeler-DeWitt and Klein-Gordon equations, it has been suggested that a similar technique might solve the corresponding problem in quantum cosmology [48, 56].

This suggests that one should search for supersymmetric extensions to quantum cosmology. It was shown in Section 6 that the classical Hamiltonians derived from the induced gravity theory and a string-inspired model may be viewed as the bosonic components of a supersymmetric Hamiltonian. In the latter case, the origin of this symmetry may be traced to the invariance of string theory under duality transformations. The hidden symmetry method was employed to derive the corresponding quantum constraints for the two models. This method differs from other approaches to supersymmetric quantum cosmology because it does not start from a field theory of supergravity [49]. The quantum constraints can be solved exactly and closed-form expressions for the general supersymmetric wavefunction were found. It would be interesting to investigate whether this method can be applied to more general models.

An alternative approach to quantum cosmology is the third quantization procedure [57, 58]. The aim of this approach is to construct a consistent probabilistic measure in quantum gravity by promoting the wavefunction of the Universe to a quantum field operator that acts on a Hilbert space of states. The 'vacuum' state in this space is identified as the state where the Universe does not exist. Topology changing processes can then be described by including self-interactions of the Universe field. Moreover, in the minisuperspace approximation a suitable combination of the dynamical degrees of freedom may be associated with a time variable in the Wheeler-DeWitt equation. It then follows that the superpotential of the wavefunction may be viewed as a 'time-dependent' function. In ordinary quantum field theory it is well known that particles are created from the vacuum by a time-varying external potential and this suggests that Universes could be created via a similar process. In practice the Universe field is expanded into positive frequency in- and out-mode functions and their hermitian conjugates. The in- and out-modes are related to one another by the Bogoliubov coefficients and these determine the number of Universes in a given mode [59]. The creation of Universes in this picture arises because the two Hilbert spaces generated by the in- and out-mode functions are inequivalent and this results in non-zero Bogoliubov coefficients.

Recently Vilenkin [60] has argued against this picture of Universe creation. His main objection is that the time variable constructed in minisuperspace models is generally not a monotonically increasing function since Universes can contract as well as expand. He then interprets the creation of a pair of Universes in terms of a contracting Universe that begins reexpanding at a finite radius. On the other hand, he does suggest that third quantization might be appropriate for describing topology changing processes in two-dimensional Universes.

Since there is currently no generally accepted interpretation of third quantization, it is of interest to investigate its consequences further. The procedure can be applied to the class of two-dimensional models (2.8) for which β, as defined in Eq. (2.16), is positive-definite for all ψ. The variables (3.7) take all values in the range $(s, r) \in (-\infty, +\infty)$ and we may therefore view s as the time variable in the Wheeler-DeWitt equation (3.8). The scale factor of the Universe vanishes as $s \to -\infty$ and infinite spatial volume corresponds to the limit $s \to \infty$. Formally, this model is identical to the one considered previously by Hosoya and Morikawa [58], so their results will apply here. The appropriately normalized positive-frequency in- and out-mode functions are given by

$$\begin{aligned} u_p^{in}(s,r) &= \left(\frac{\pi}{6}\right)^{1/2} \left(sinh\frac{\pi|p|}{3}\right)^{-1/2} e^{ipr} J_\nu\left(e^{3s}\right) \\ u_p^{out}(s,r) &= \frac{1}{2}\left(\frac{\pi}{3}\right)^{1/2} e^{-\pi|p|/6} e^{ipr} H_\nu^{(2)}\left(e^{3s}\right), \end{aligned} \tag{7.1}$$

respectively, where $\nu = -i|p|/3$. As $s \to \infty$, $u_p^{out} \propto e^{iS}$ and this is the WKB solution given by Eq. (4.15) with $f = 0$. It follows that the Bogoliubov coefficients are given by $c_1(p,q) = b_1 \delta_{pq}$ and $c_2(p,q) = b_2 \delta_{pq}$, where

$$b_1 = \left(1 - e^{-2\pi|p|/3}\right)^{-1/2}, \qquad |b_2| = \left(e^{2\pi|p|/3} - 1\right)^{-1/2} \tag{7.2}$$

and the average number of Universes with 'energy' p therefore has a Planckian distribution

$$N_p = |c_2(p,p)|^2 = \left(e^{2\pi|p|/3} - 1\right)^{-1}. \tag{7.3}$$

Hosoya and Morikawa [58] extended this free field theory by including a Ψ^3 interaction that describes the spliting of a 'mother' Universe into two 'baby' Universes of identical topology. By treating the mother Universe in a classical fashion, they showed that the quantized baby Universes also have a Planckian distribution. The formal equivalence of their model with those studied in this work suggests that similar conclusions should apply for a wide class of two-dimensional cosmologies. It would be of interest to investigate these possibilities further.

The author is supported by the Particle Physics and Astronomy Research Council (PPARC), UK.

References

1. M. B. Green, J. H. Schwarz, and E. Witten, *Superstring Theory* (Cambridge University Press, Cambridge, 1988).

2. C. G. Callan, S. B. Giddings, J. A. Harvey, and A. Strominger, Phys. Rev. D **45**, R1005 (1992).

3. T. Banks and M. O'Loughlin, Nucl. Phys. B **362**, 649 (1991).

4. K. Isler and J. M. Lina, Nucl. Phys. B **358**, 713 (1991); E. Abdalla, M. C. B. Abdalla, J. Gamboa, and A. Zadra, Phys. Lett. B **273**, 222 (1991); A. Miković, Phys. Lett. B **291**, 19 (1992).

5. J. B. Hartle and S. W. Hawking, Phys. Rev. D **28**, 2960 (1983).

6. J. J. Halliwell, Int. J. Mod. Phys. **A5**, 2473 (1990); and references therein.

7. A Ishikawa, Phys. Rev. D **50**, 2609 (1994).

8. P. A. M. Dirac, Proc. R. Soc. **A246**, 326 (1958).

9. B. S. DeWitt, Phys. Rev. **160**, 1113 (1967); J. A. Wheeler, *Battelle Rencontres* (Benjamin, New York, 1968).

10. L. J. Garay, J. J. Halliwell, and G. A. Mena Marugán, Phys. Rev. D **43**, 2572 (1991).

11. J. Navarro-Salas, M. Navarro, C. F. Talavera, and V. Aldaya, Phys. Rev. D **50**, 901 (1994).

12. M. Seiberg, Prog. Theor. Phys. Suppl. **102**, 319 (1990); G. Moore, N. Seiberg, and M. Staudacher, Nucl. Phys. B **362**, 665 (1991); J. Navarro-Salas, M. Navarro, and V. Aldaya, Phys. Lett. B **318**, 293 (1993).

13. E. Adi and S. Solomon, Phys. Lett. B **336**, 152 (1994).

14. R. Jackiw, in *Quantum Theory of Gravity*, ed. S. Christensen (Adam Hilger, Bristol, 1984) p. 403; C. Teitelboim, in *Quantum Theory of Gravity*, ed. S. Christensen (Adam Hilger, Bristol, 1984) p. 327.

15. M. Henneaux, Phys. Rev. Lett. **54**, 959 (1985).

16. T. Thiemann and H. A. Kastrup, Nucl. Phys. B **399**, 211 (1993); J. Gegenberg and G. Kunstatter, Phys. Rev. D **47**, R4192 (1993); T. Hori, Prog. Theor. Phys. **90**, 743 (1993); D. Loius-Martinez, J. Gegenberg, and G. Kunstatter, Phys. Lett. B **321**, 193 (1994).

17. M. Önder and R. W. Tucker, Phys. Lett. B **311**, 47 (1993); Class. Quantum Grav. **11**, 1243 (1994).

18. J. Navarro-Salas and C. F. Talavera, Nucl. Phys. B **423**, 686 (1994); J. Navarro-Salas, M. Navarro, and C. F. Talavera, "Diffeomorphisms, Noether charges and canonical formalism in 2D dilaton gravity", Preprint hep-th/9411105 (1994).

19. F. D. Mazzitelli and J. G. Russo, Phys. Rev. D **47**, 4490 (1992).

20. D. Zwillinger, *Handbook of Differential Equations* (Academic Press, New York, 1989).

21. C. R. Chester, *Techniques in Partial Differential Equations* (McGraw-Hill, New York, 1970).

22. J. G. Russo and A. A. Tseytlin, Nucl. Phys. B **382**, 259 (1992).

23. A. M. Polyakov, Phys. Lett. B **163**, 207 (1981); Mod. Phys. Lett. **A2**, 899 (1987).

24. C. Callan, C. Lovelace, E. Martinec, and M. J. Perry, Nucl. Phys. B **262**, 593 (1985).

25. C. Lovelace, Nucl. Phys. B **273**, 413 (1986); W. Fischler and L. Susskind, Phys. Lett. B **173**, 262 (1986).

26. C. Callan, C. Lovelace, C. Nappi, and S. Yost, Nucl. Phys. B **288**, 525 (1987).

27. M. D. McGuigan, C. R. Nappi, and S. A. Yos, Nucl. Phys. B **375**, 421 (1992).

28. M. Born and L. Infeld, Proc. R. Soc. **144**, 425 (1934); E. Fradkin and A. A. Tseytlin, Phys. Lett. B **163**, 123 (1985); A. Abouelsaood, C. Callan, C. Nappi, and S. Yost, Nucl. Phys. B **280**, 599 (1987).

29. N. Marcus and A. Sagnotti, Phys. Lett. B **119**, 97 (1982).

30. C. Callan, C. Lovelace, C. Nappi, and S. Yost, Nucl. Phys. B **293**, 83 (1984).

31. S. Mignemi, Phys. Rev. D **50**, R4733 (1994).

32. G. Magnano, M. Ferrais, and M. Francaviglia, Gen. Rel. Grav. **19**, 465 (1987); H. J. Schmidt, J. Math. Phys. **32**, 1562 (1991).

33. D. N. Page, J. Math. Phys. **32**, 3427 (1991).

34. J. E. Lidsey, Class. Quantum Grav. **11**, 1211 (1994).

35. S. W. Hawking, Phys. Rev. D **37**, 904 (1988); Mod. Phys. Lett. **A5**, 453 (1990); A. Zhuk, Phys. Rev. D **45**, 1192 (1992); L. J. Garay, Phys. Rev. D **48**, 1710 (1993); T. Dereli and R. W. Tucker, Class. Quantum Grav. **10**, 365 (1993); T. Dereli, M. Önder and R. W. Tucker, Class. Quantum Grav. **10**, 1425 (1993).

36. S. W. Hawking and D. N. Page, Phys. Rev. D **42**, 2655 (1990).

37. L. M. Campbell and L. J. Garay, Phys. Lett. B **254**, 49 (1991).

38. J. Klauder, in *Relativity*, edited by M. Carmeli, S. Fickler, and L. Witten (Plenum, New York, 1970); M. Carreau, E. Fahri, and S. Gutmann, Phys. Rev. D **42**, 1194 (1990); E. Fahri and S. Gutmann, Int. J. Mod. Phys. **A5**, 3029 (1990).

39. *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun, Natl. Bur. Stand. Appl. Math. Ser. No. 55 (U.S. GPO, Washington, D.C., 1965).

40. S. W. Hawking, Nucl. Phys. B **239**, 257 (1984); S. Wada, Prog. Theor. Phys. **75**, 1365 (1986); Mod. Phys. Lett. **A3**, 645 (1988).

41. J. J. Halliwell and J. Louko, Phys. Rev. D **39**, 2206 (1989).

42. C. Lanczos, *The Variational Principles of Mechanics* (University of Toronto Press, Toronto, 1966).

43. J. Uglum, Phys. Rev. D **46**, 4365 (1992).

44. J. G. Russo, L. Susskind, and L. Thorlacius, Phys. Rev. D **46**, 3444 (1992).

45. R. Graham, Phys. Rev. Lett. **67**, 1381 (1991).

46. G. F. R. Ellis and M. A. H. MacCallum, Commun. Math. Phys. **12**, 108 (1969).

47. P. D. D'Eath, S. W. Hawking, and O. Obregón, Phys. Lett. B **302**, 183 (1993); M. Asano, M. Tanimoto, and N. Yoshino, Phys. Lett. B **314**, 308 (1993); O. Obregón, J. Pullin, and M. P. Ryan, Phys. Rev. D **48**, 5642 (1993); J. Bene and R. Graham, Phys. Rev. D **49**, 799 (1994); R. Capovilla and J. Guven, Preprint gr-qc/9402025, (1994).

48. J. Bene and R. Graham, Phys. Rev. D **49**, 799 (1994); and references therein.

49. O. Obregón, J. Socorro, and J. Benítez, Phys. Rev. D **47**, 4471 (1993).

50. E. Witten, Nucl. Phys. B **188**, 513 (1981).

51. M. Claudson and M. B. Halpern, Nucl. Phys. B **250**, 689 (1985); R. Graham and D. Roeckaerts, Phys. Rev. D **34**, 2312 (1986).

52. A. Zhuk, Class. Quantum Grav. **9**, 2029 (1992).

53. K. Kikkawa and M. Yamasaki, Phys. Lett. B **149**, 357 (1984); M. Sakai and I. Senda, Prog. Theor. Phys. **75**, 692 (1986).

54. A. Giveon, M. Porrati and E. Rabinovici, Phys. Rep. **244**, 77 (1994).

55. P. Ginsparg and C. Vafa, Nucl. Phys. B **289**, 414 (1987); T. H. Buscher, Phys. Lett. B **201**, 466 (1988); A. A. Tseytlin, Mod. Phys. Lett. **A6**, 1721 (1991).

56. T. Dereli, M. Önder, and R. W. Tucker, Phys. Lett. B **324**, 134 (1994); and references therein.

57. V. A. Rubakov, Phys. Lett. B **214**, 503 (1988); W. Fischler, I. Klebanov, J. Polchinski, and L. Susskind, Nucl. Phys. B **327**, 157 (1989).

58. A. Hosoya and M. Morikawa, Phys. Rev. D **39**, 1123 (1989).

59. N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, 1982).

60. A. Vilenkin, Phys. Rev. D **50**, 2581 (1994).