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Quantum loop representation for fermions coupled to Einstein - Maxwell field.

Kirill V. Krasnov

Bogolyubov Institute for Theoretical Physics, Kiev, Ukraine

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Abstract

Quantization of the system comprising gravitational, fermionic and electromagnetic fields is developed in the loop representation. As a result we obtain a natural unified quantum theory. Gravitational field is treated in the framework of Ashtekar formalism; fermions are described by two Grassmann-valued fields. We define a C^*-algebra of configurational variables whose generators are associated with oriented loops and curves; "open" states – curves – are necessary to embrace the fermionic degrees of freedom. The quantum representation space is constructed as a space of cylindrical functionals on the spectrum of this C^*-algebra. Choosing the basis of "loop" states we describe the representation space as the space of oriented loops and curves; then configurational and momentum loop variables become in this basis the operators of creation and annihilation of loops and curves. It turnes out that the representation constructed is reducible; there is an invariant sub-space in the representation space which consists of all states containing open ends. Thus, the irreducible representation is realized on the space of all states containing fermionic "exitations". We also discuss the problem of hermiticity of operators defined. The important difference of the constructed representation from the loop representation of pure gravity is that the momentum loop operators act in our case by joining loops in the only compatible with their orientaiton way, while in the case of pure gravity this action is more complicated.

I. INTRODUCTION

Recent developement of nonperturbative quantum gravity has shown that the loop representation is quite a decent tool for dealing with generally covariant field theories. This representation allowed one to find a wide class of solutions of quantum general relativity constraints [1], [2]; an interesting interface with the knot theory [5] has also been found. The main goal of this paper is to construct the loop representation for the system which includes fermionic and two gauge fields: gravitational and electromagnetic.

The loop representaion in quantum theory is based on using the so-called loop variables which are the well-known in Yang-Mills theories Wilson loop functionals. The dynamical variables of Yang-Mills theory (as well as of general relativity in the framework of Ashtekar variables) are connection field over the spatial manifold and its conjugate momentum. Wilson loop functionals form a set of gauge invariant non-local quantities built from the connection field and the "loop" approach is to regard these quantities as basic variables. This becomes a very powerful means when one works with a generally covariant field theory. In this case the lack of background structure does not allow one to construct a renormalized operator corresponding to a local classical variable – in other words a renormalization procedure for constructing such an operator turns out to be background dependent. On the other hand, loop variables are non-local quantities; one does not need any background structure to construct a representation of these variables in terms of operators in some Hilbert space. So the general strategy we will follow is that proposed by Ashtekar and Isham [4]: it is to regard loop quantities as basic variables on the classical level and to construct quantum theory representing the Poisson algebra which loop quantities generate by a certain operator algebra.

There is also another problem which loop representation seems to be suited for – it is the problem of presence of constraints. Constraints generate symmetry transformations and because of presence of symmetries not all degrees of freedom of the Lagrange formulation are physical. The general strategy for quantizing such a system is to choose the coordinates on its phase space which have the simplest properties under the symmetry transformations and regard them as basic variables. The loop variables are just these quantities. The symmetry transformations of general relativity in the framework of Ashtekar variables are gauge transformations and spatial diffeomorphisms. As we will see, for the system including also fermionic and electromagnetic fields the symmetry group consists of two similar parts: gauge and diffeomorphism transformations. It is the advantage of using loop variables that they are gauge invariant and transform very naturally under the diffeomorphism group, namely as the geometrical objects with which they are associated. That is why their usage simplifies considerably the problem of finding solutions to the gauge and diffeomorphism constraints.

Because loop variables contain *all* gauge invariant information *any* local gauge invariant quantity can be expressed as a limit of corresponding loop variable. This means that the Hamiltonian constraint of the theory can be written in terms of loop variables with a properly chosen limit procedure. This provides us with the Hamiltonian operator regularization method as the operator corresponding to the classical "loop" expression becomes a well defined operator in the "loop" space. It has been shown by Rovelli and Smolin [11] that there exists such a way to take a limit that no divergences appear in the result. So the loop representation which is based on usage of the loop variables can reduce the problem of solving the Hamiltonian constraint to a simple combinatorical problem in the "loop" space.

As it has been stated above, we develop quantization program for the system which includes not just pure gravitational field, but also fermionic and electromagnetic fields. It was noted by Ashtekar *et al* [8] that there exists a natural possibility of unification gravity with other gauge fields in the Hamiltonian framework; it is to enlarge the gauge group of pure gravity $SL(2,C)$ to a group which describes a unified gauge field. The first work along this line [6] concerned the loop representation for such a unified theory and its connection to the knot theory. We continue the developement of loop representation for the unified gauge field. It turnes out that the enlargening of gauge degrees of freedom, and therefore the enlargening of the symmetry group, leads to some appealing features of quantum theory in the loop representation. One of them is that the loop operators act even simpler than in the case of pure gravity: in the latter case momentum operators act with a result which includes both a loop and its inverse; this is connected to the fact that loop variables corresponding to a loop and to its inverse coincide. In the case of the unified field these two quantities become independent so the loop variables acquire *orientation*. The difference from the case of pure gravity is that the loop operators never change this orientation when they act in the "loop" space. The Poisson algebra of loop variables is described solely in terms of breaking, rearranging and rejoining loops.

The example of the loop technique for fermions coupled to gravitational field was given by Morales-Técolt and Rovelli [7]. Unlike these authors we consider the full-featured case when two independent fermionic fields are present. Fermionic fields are described by two complex Grasmann-valued spinor fields so the "loop" variables, which are mixed "gauge – fermionic" quantities, are even Grassmann algebra elements. We construct the loop representation in which the action of quantum analogs of these variables can again be described in a geometric way as operation of gluing curves.

The organization of this paper is as follows. In Sec. II we remind briefly the properties of our system in the Hamiltonian formulation and introduce the unified Einstein-Maxwell gauge field. In this section we also obtain the Hamiltonian formulation for the Grassmann-valued fermionic fields. In Sec. III we introduce the loop variables and compute the Poisson algebra structure. Sec. IV is devoted to constructing a representation of the Poisson algebra obtained in the previous section. We find a representation space, choose a basis in which the "loop" operators become simple and discuss their hermiticity with respect to a certain scalar product.

II. HAMILTONIAN FORMULATION

We begin with the action for gravity and matter fields. Fix a four-manifold $\mathcal{M}$, which is topologically a direct product $I\!R \times \Sigma$ for some three-manifold Σ. In the framework of Ashtekar variables the Lagrangian density for gravity $\mathcal{L}_E$ is the functional of an anti-Hermitian soldering form $\sigma^a_A{}^{A'}$ and a self-dual connection ${}^4A_{aA}{}^B$ on $\mathcal{M}$ [8]

$$\mathcal{L}_E(\sigma, A) = G^2(\sigma)\sigma^a_A{}^{A'}\sigma^b{}_{BA'}{}^4F_{ab}{}^{AB}, \tag{1}$$

where (σ) is the determinant of the soldering form (see eq. (24) below), ${}^4F_{ab}$ is the curvature tensor of the connection ${}^4A_a$. We take here that the self-dual connection has the dimension of 1/m, what is rather unusual; the convenience of such a choise will become clear when we define a unified Einstein-Maxwell field. The factor G is the fundamental constant; G is set to have a dimension of 1/m. The other fundamental constants we have set to be $\hbar = c = 1$. Thus, the action is dimensionless.

It is important to note that the action functional (1) is complex because of the complexity of ${}^4A_{aA}{}^B$ connection. Although a soldering form is restricted to be anti-Hermitian so that the space-time metric will always be real (with Lorentzian signature), the complexity of a self-dual connection demands the suitably chosen *reality conditions* on the system's phase space to be imposed in order to obtain real general relativity.

The connection $\mathcal{D}_a$ defined by ${}^4A_{aA}{}^B$ and by electromagnetic vector-potencial a_a via ${}^4\mathcal{D}_a\lambda_A = \partial_a\lambda_A + {}^4A_{aA}{}^B\lambda_B + a_a\lambda_A$ acts only on unprimed spinors. Thus, we shall take the Dirac Lagrangian density for fermionic fields ξ^A, $\eta^{A'}$ (Grassmann-valued) to be

$$\mathcal{L}_D(\xi,\eta,\sigma,A,a) = \sqrt{2}(\sigma)\Big[\sigma^a{}_{AA'}[\bar{\xi}^{A'4}\mathcal{D}_a\xi^A - ({}^4\mathcal{D}_a\bar{\eta}^A)\eta^{A'}] + \frac{im}{\sqrt{2}}[\bar{\eta}_A\xi^A - \bar{\xi}^{A'}\eta_{A'}]\Big]. \tag{2}$$

The Lagrangian density for electromagnetic field is

$$\mathcal{L}_{Em}(a,\sigma) = \tfrac{1}{2}\,(\sigma)g^{ac}g^{bd}\,{}^4f_{ab}{}^4f_{cd}, \tag{3}$$

where f_{ab} is the curvature tensor of a_a.

The total action of the theory is the sum

$$S = S_E + S_D + S_{Em}.$$

In order to develop the canonical quantization program we should pass on to the Hamiltonian framework, carrying out a space+time decomposition of the action (see [8] for details). Then the action takes the following form

$$S = \int dt \int_{\Sigma_t} d^3x\Big(Tr\,\tilde{E}^a\mathcal{L}_tA_a + \mathcal{L}_t\xi^A\tilde{\pi}_A + \mathcal{L}_t\bar{\eta}^A\tilde{\omega}_A + \tilde{e}^a\mathcal{L}_ta_a$$

$+\underset{\sim}{N}\tilde{\tilde{C}}(A,E,\xi,\tilde{\pi},\bar{\eta},\tilde{\omega},a,\tilde{e})$ - Hamiltonian constraint

$+N^a\tilde{C}_a(A,E,\xi,\tilde{\pi},\bar{\eta},\tilde{\omega},a,\tilde{e})$ - Diffeomorphism constraint

$+({}^4A\ t)_B{}^A\tilde{C}_A{}^B(A,E,\xi,\tilde{\pi},\bar{\eta},\tilde{\omega},a,\tilde{e})$ - gauge transformations constraint (spin basis rotations)

$+({}^4a\ t)\ \tilde{c}\ (A,E,\xi,\tilde{\pi},\bar{\eta},\tilde{\omega},a,\tilde{e})\Big).$ - gauge transformations constraint (phase rotations)

As usual in generally covariant field theories the Hamiltonian will be the sum of constraints. The following part of the Section is devoted to the analysis of this expression.

A. Einstein-Maxwell unified field

Let us for the moment restrict our consideration to the gauge part of the Hamiltonian. The last two terms in the Hamiltonian are the generators of local gauge transformations on the phase space. These transformations are: rotations of the complexified spin basis at each spatial point which form the group $SL(2,C)$ (see, for instance, [13] for the discussion of the underlying geometric structure) and phase rotations which form the group $U(1)$; the gauge fields lie in the corresponsing Lie algebras. Therefore, the full gauge group, which is formed by all internal space symmetry transformations, is $SL(2,C)\times U(1)$. From the Hamiltonian, i.e. geometric, point of view it is superfluous to distinguish the two gauge fields – the dynamical variables of the theory should be a connection on some bundle over the spatial manifold (which takes values in the Lie algebra of the gauge group) and its conjugate momentum. Thus, we should regard the two independent connection fields of initial Lagrange formulation as the two parts of one connection field – the unified Einstein-Maxwell field.

So we are to choose the new "coordinates" on the phase space of the system which will correspond to the unified gauge field. The expression for the new gauge variables is straightforward. Let us choose the new connection field to be

$$ {}^4\mathcal{A}_{aA}{}^{B} = {}^4A_{aA}{}^{B} + {}^4a_a\, \delta_A^B. \tag{4} $$

Then the initial Einstein and Maxwell connection fields can be expressed through ${}^4\mathcal{A}$ as follows

$$ {}^4A_A^B = {}^4\mathcal{A}_A^B - \tfrac{1}{2}(Tr\, {}^4\mathcal{A})\delta_A^B, \tag{5} $$

$$ {}^4a = \tfrac{1}{2}(Tr\, {}^4\mathcal{A})\delta_A^B. \tag{5a} $$

Having introduced the unified connection field $\mathcal{A}$ we are to define the corresponding momentum field $\mathcal{E}$. We shall take it in the form

$$ \tilde{\mathcal{E}}^a_A{}^B = \tilde{E}^a_A{}^B + \tfrac{1}{2}\tilde{e}^a\delta_A^B, \tag{6} $$

so that it is the canonically conjugate to $\mathcal{A}$

$$ \{\tilde{\mathcal{E}}^a_{CD}(x), \mathcal{A}_b^{AB}(y)\} = -\,\delta^3(x-y)\delta_b^a\delta_D^A\delta_C^B. \tag{7} $$

Here $\mathcal{A}$ is the pullback of ${}^4\mathcal{A}$ to the tree-manifold. The factor $\frac{1}{2}$ in (6) is important[1]; it provides the correct (canonical) commutational relations between the connection field and its momentum (7). The gravitational and electromagnrtic momentum fields can also be expressed through the unified field

$$ \tilde{E}^a_A{}^B = \tilde{\mathcal{E}}^a_A{}^B \;-\; \tfrac{1}{2}\, Tr(\tilde{\mathcal{E}}^a)\delta_A^B \tag{8} $$

$$ \tilde{e}^a = Tr(\tilde{\mathcal{E}}^a) \tag{8a} $$

Having these relations it is a simple exercise to rewrite the constraints in terms of the unified fields. The last two terms in the Hamiltonian are the Gauss law constraints for the gravitational and electromagnetic fields

$$ ({}^4A\,t)_A^B\; D_a\, \tilde{E}^a_B{}^A + \; ({}^4a\,t)\partial_a\tilde{e}^a, $$

where

$$ D_a = \partial_a + A_a. $$

We can express it in terms of the new Lagrange multiplier $({}^4\mathcal{A}\;t)$, so the Gauss law constraint for the unified field takes the form

[1] It has been overlooked in [6]

$$\frac{\delta(S_E + S_{Em})}{\delta({}^4\mathcal{A}_A^B t)} = \mathcal{D}_a \tilde{\mathcal{E}}_B^{a\ A} = 0, \tag{9}$$

where we introduced

$$\mathcal{D}_a = \partial_a + \mathcal{A}_a. \tag{10}$$

The gauge part of the diffeomorphism constraint

$$\frac{\delta(S_E + S_{Em})}{\delta N^a} = -\ Tr(\tilde{E}^b F_{ab}) -\ \tilde{e}^b f_{ab} \tag{11}$$

expressed through the unified variables takes the form

$$\frac{\delta(S_E + S_{Em})}{\delta N^a} = -\ Tr(\tilde{\mathcal{E}}^b \mathcal{F}_{ab}). \tag{12}$$

Again, the factor $\frac{1}{2}$ from (6) was necessary to cansel the factor 2 which appeared from the trace operation. Here we introduced the curvature field $\mathcal{F}_{ab}$ of the connection $\mathcal{A}_a$

$$\mathcal{F}_{ab} := 2\,\mathcal{D}_{[a}\mathcal{A}_{b]} = 2\,\partial_{[a}A_{b]} + 2\partial_{[a}a_{b]} + [A_a, A_b] = F_{ab} + f_{ab}.$$

B. Fermionic part

For the Dirac action the space-time decomposition leads to the following expression (see [8] for details)

$$S_D = \int dt \int_{\Sigma_t} d^3x\ \{-i(\sigma)[(\xi\dagger)_A \mathcal{L}_t \xi^A\ +\ (\bar{\eta}\dagger)_A \mathcal{L}_t \bar{\eta}^A]\ +\ \mathcal{H}(\xi, \xi\dagger, \bar{\eta}, \bar{\eta}\dagger)\}, \tag{13}$$

where $\mathcal{H}$ means the Hamiltonian functional. The $\dagger$-operation here descends from the complex conjugation on the Grassmann algebra of $SL(2,C)$ spinors and satisfies the following properties (a) $(a\alpha_A + b\beta_A)^\dagger = a^*\alpha_A^\dagger + b^*\beta_A^\dagger$; (b) $(\alpha_A^\dagger)^\dagger = -\alpha_A$; (c) $(\alpha^A)^\dagger\alpha_A \geq 0$; (d) $(\epsilon_{AB})^\dagger = \epsilon_{AB}$; (e) $(\alpha_A\beta_B)^\dagger = \alpha_A^\dagger\beta_B^\dagger$, for all Grassmann fields α_A and β_B and complex functions a, b. Being Grassmann-valued, the fermionic fields anti-commute. So having rearranged them in (13) we got the different from [8] sign in square brackets. We define the momentum fields by the left variational derivatives

$$\tilde{\pi} := \frac{\vec{\delta}\ S}{\delta\ \mathcal{L}_t \xi^A} = i(\sigma)(\xi\dagger)_A, \tag{14}$$

$$\tilde{\omega} := \frac{\vec{\delta}\ S}{\delta\ \mathcal{L}_t \bar{\eta}^A} = i(\sigma)(\bar{\eta}\dagger)_A.$$

Then the action takes the form

$$S_D = \int dt \int_{\Sigma_t} d^3x\ [\mathcal{L}_t \xi^A \tilde{\pi}_A\ +\ \mathcal{L}_t \bar{\eta}^A \tilde{\omega}_A\ +\ \mathcal{H}(\xi, \tilde{\pi}, \bar{\eta}, \tilde{\omega})]. \tag{15}$$

The momentum fields have appeared at the right side to the configurational fields because of the usage of the left derivatives in the momentum field definition. The full Hamiltonian density for the spinor fields is

$\mathcal{H}(\xi, \tilde{\pi}, \bar{\eta}, \tilde{\omega}) =$

$$\underset{\sim}{N} \left[G^{-2} \, \tilde{E}^{a\,B}_{A} [\mathcal{D}_a \xi^A \tilde{\pi}_B \, + \, \mathcal{D}_a \bar{\eta}^A \tilde{\omega}_B] \, + \, im[(\sigma)^2 \bar{\eta}_A \xi^A \, + \, \tilde{\pi}^A \tilde{\omega}_A] \right] \\ - \, ({}^4\!\mathcal{A}\, t)^A_B [\xi^B \tilde{\pi}_A \, + \, \bar{\eta}^B \tilde{\omega}_A] \\ - \, N^a [\mathcal{D}_a \xi^A \tilde{\pi}_A \, + \, \mathcal{D}_a \bar{\eta}^A \tilde{\omega}_A], \tag{16}$$

where we used the "unified" Lagrange multiplier $({}^4\!\mathcal{A}\, t)$ (see (4)).

The equations of motion are now straightforward from the variational principle. Using the left variation which turns into zero at the initial and final time points one finds the dynamics

$$\mathcal{L}_t \xi^A = - \, \frac{\delta \, H}{\delta \, \tilde{\pi}_A} \quad ; \quad \mathcal{L}_t \bar{\eta}^A = - \, \frac{\delta \, H}{\delta \, \tilde{\omega}_A}$$

$$\mathcal{L}_t \tilde{\pi}_A = - \, \frac{\delta \, H}{\delta \, \xi^A} \quad ; \quad \mathcal{L}_t \tilde{\omega}_A = - \, \frac{\delta \, H}{\delta \, \bar{\eta}^A}.$$

So the evolution of any functional on the system phase space is given by

$$\mathcal{L}_t f(\xi, \tilde{\pi}, \bar{\eta}, \tilde{\omega}) = \{ \, H \, , \, f \, \},$$

where the Poisson structure on the phase space is defined via

$$\{ \, f \, , \, g \, \} = - \, \int d^3x \Big[\frac{\delta \, f}{\delta \, \xi^A} \, \frac{\delta \, g}{\delta \, \tilde{\pi}_A} \, + \, \frac{\delta \, f}{\delta \, \tilde{\pi}_A} \, \frac{\delta \, g}{\delta \, \xi^A} \, + \, \frac{\delta \, f}{\delta \, \bar{\eta}^A} \, \frac{\delta \, g}{\delta \, \tilde{\omega}_A} \, + \, \frac{\delta \, f}{\delta \, \tilde{\omega}_A} \, \frac{\delta \, g}{\delta \, \bar{\eta}^A} \Big]. \tag{17}$$

All functional derivatives in this formula are left. Then one can obtain the Poisson brackets between the canonical variables

$$\{ \, \xi^A(x), \, \xi^B(y) \} = 0 \quad ; \quad \{ \, \tilde{\pi}_A(x), \, \tilde{\pi}_B(y) \} = 0; \tag{18}$$

$$\{ \, \tilde{\pi}_B(y), \, \xi^A(x) \} = - \, \delta^A_B \delta(x - y) \tag{19}$$

and analogously for $\bar{\eta}, \tilde{\omega}$ fields.

C. Hamiltonian constraint

The Hamiltonian constraint of the theory consists of the three parts

$$\frac{\delta S_E}{\delta \underset{\sim}{N}} = \tfrac{1}{2}\frac{1}{G^2}\, Tr(\tilde{E}^a \tilde{E}^b F_{ab}) \tag{20}$$

$$\frac{\delta S_{Em}}{\delta \underset{\sim}{N}} = \frac{1}{32}\,\frac{1}{(G^2)^4}(\sigma)^{-2}\, Tr(\tilde{E}^a \tilde{E}^c)\ Tr(\tilde{E}^b \tilde{E}^d)\,(e_{ab}e_{cd}\ + b_{ab}b_{cd}) \tag{20a}$$

$$\frac{\delta S_D}{\delta \underset{\sim}{N}} = G^{-2}\tilde{E}^{a\ B}_{A}(\mathcal{D}_a \xi^A \tilde{\pi}_B\ +\ \mathcal{D}_a \bar{\eta}^A \tilde{\omega}_B)\ +\ im\,((\sigma)^2 \bar{\eta}_A \xi^A\ +\ \tilde{\pi}^A \tilde{\omega}_A), \tag{20b}$$

where $b_{ab} = 2\ f_{ab}$. Having introduced the unified connection field and the corresponding conjugate momentum, we shall express the Hamiltonian constraint in terms of these fields. This gives for the Einstein part of the Hamiltonian

$$\frac{1}{c}\,\frac{\delta S_E}{\delta \underset{\sim}{N}} = \tfrac{1}{2}\frac{1}{G^2}\ \underset{\sim}{\eta}_{abc}\, Tr(\tilde{\mathcal{E}}^a \tilde{\mathcal{E}}^b \tilde{\mathcal{B}}^c). \tag{21}$$

Here we introduced the magnetic field $\tilde{\mathcal{B}}^a$ as the dual of the curvature of the unified field

$$\mathcal{F}_{ab} = \underset{\sim}{\eta}_{abc}\tilde{\mathcal{B}}^c,$$

so that it has the dimension and the weight of $\tilde{\mathcal{E}}^a$. The tensor $\underset{\sim}{\eta}_{abc}$ is the totally antisymmetric tensor of weight -1.

The other two parts of the Hamiltonian become

$$\begin{aligned}\frac{\delta S_{Em}}{\delta \underset{\sim}{N}} = \frac{1}{32}\,\frac{1}{(G^2)^4}(\sigma)^{-2}\underset{\sim}{\eta}_{abe}\underset{\sim}{\eta}_{cdf}\ Tr(\tilde{\mathcal{E}}^a \tilde{\mathcal{E}}^c)\Big(&Tr(\tilde{\mathcal{E}}^b \tilde{\mathcal{E}}^d)\, Tr(\tilde{\mathcal{B}}^e)\, Tr(\tilde{\mathcal{B}}^f) \\ -\ Tr(\tilde{\mathcal{E}}^b \tilde{\mathcal{E}}^d)\, Tr(\tilde{\mathcal{E}}^e)\, Tr(\tilde{\mathcal{E}}^f)\ -\ &Tr(\tilde{\mathcal{E}}^b)\, Tr(\tilde{\mathcal{E}}^d)\, Tr(\tilde{\mathcal{B}}^e)\, Tr(\tilde{\mathcal{B}}^f)\Big)\end{aligned} \tag{21a}$$

$$\frac{\delta S_D}{\delta \underset{\sim}{N}} = G^{-2}(\tilde{\mathcal{E}}^{a\ B}_{A}\ -\ \tfrac{1}{2}\, Tr(\tilde{\mathcal{E}}^a)\delta^B_A)(\mathcal{D}_a \xi^A \tilde{\pi}_B\ +\ \mathcal{D}_a \bar{\eta}^A \tilde{\omega}_B)\ +\ im\,((\sigma)^2 \bar{\eta}_A \xi^A\ +\ \tilde{\pi}^A \tilde{\omega}_A). \tag{21b}$$

Let us also give here the complete (including fermionic degrees of freedom) expression for the Gauss law and diffeomorphism constraints in terms of the Einsein-Maxwell field

$$\frac{\delta S}{\delta N^a} = -\ Tr(\tilde{\mathcal{E}}^b \mathcal{F}_{ab})\ -\ (\mathcal{D}_a \xi^A \tilde{\pi}_A\ +\ \mathcal{D}_a \bar{\eta}^A \tilde{\omega}_A) \tag{22}$$

$$\frac{\delta S}{\delta ({}^4\!\mathcal{A}\ t)_A^{\ B}} = -\ (\xi^A \tilde{\pi}_B\ +\ \bar{\eta}^A \tilde{\omega}_B)\ +\ \mathcal{D}_a \tilde{\mathcal{E}}^{a\ A}_{B}. \tag{23}$$

As we have seen, the Hamiltonian constraint contains the determinant of the soldering form

$$(\sigma)^2 = -\ \frac{1}{3\sqrt{2}}\ \underset{\sim}{\eta}_{abc}\, Tr(\tilde{\sigma}^a \tilde{\sigma}^b \tilde{\sigma}^c). \tag{24}$$

So we need its expression through the unified variables. It is given by

$$(\sigma)^2 = \frac{i}{12\, G^6}\, \underset{\sim}{\eta}_{abc}\, Tr(\tilde{\mathcal{E}}^a \,-\, \tfrac{1}{2}\, Tr(\tilde{\mathcal{E}}^a))(\tilde{\mathcal{E}}^b \,-\, \tfrac{1}{2}\, Tr(\tilde{\mathcal{E}}^b))(\tilde{\mathcal{E}}^c \,-\, \tfrac{1}{2}\, Tr(\tilde{\mathcal{E}}^c)). \tag{25}$$

This accomplishes the aim of this Section, which was to obtain all the constraints of the Hamiltonian framework expressed in terms of the unified gauge and the fermionic fields. We will conclude by pointing out that in the form (21) the Hamiltonian is not polynomial in $\tilde{\mathcal{E}}^a$ variables because of the presence of the factor $(\sigma)^{-2}$ in the electromagnetic part; this might cause problems in constructing the corresponding quantum operator. Possible solution of the problem was proposed by Ashtekar *et al* [8]. Multiplying the Hamiltonian constraint by $(\sigma)^2$ one may restore its polynomial character; the Hamiltonian constraint becomes a density of weight four (therefore the corresponding Lagrange multiplier - lapse function - becomes a density of weight minus three). The other possible way to tackle this problem is discussed in [16].

III. ALGEBRA OF LOOP VARIABLES

In this Section we construct the loop variables and discuss the Poisson braket algebra which they generate. Loop variables are gauge invariant non-local quantities built from the dynamical variables and associated with curves and ribbons. Open curves are necessary to embrace the fermionic degrees of freedom.

A. Configurational loop variables

The set of variables which we call configurational loop variables will play a crucial role in quantization procedure. Let us denote the space of unified connection fields (the space of connections on a certain $SL(2,C) \times U(1)$ bundle over Σ) by $\mathcal{A}$ and consider the Wilson loop functional on $\mathcal{A}$

$$(\gamma) \equiv T_\gamma[\mathcal{A}] := Tr\, \mathcal{P}\, \exp \oint_\gamma \mathcal{A}; \tag{26}$$

or, using the matrix U of parallel transport (with the connection $\mathcal{A}$) of spinors along a curve $\gamma : [0,1] \to \Sigma$

$$U[\gamma]_A^{\ B} = \mathcal{P} \exp \int_\gamma d\,\tau\, \dot{\gamma}^a\, \mathcal{A}_{aA}{}^B,$$

$$(\gamma) = Tr\; U[\gamma].$$

The main difference from the case of pure gravity is that

$$U[\gamma^{-1}] \neq U[\gamma] \tag{27}$$

because of the presence of the additional electromagnetic part in the connection field. The loop quantities form a set of *complex* coordinates on the configurational space $\mathcal{A}/\mathcal{G}$ (we denoted by $\mathcal{A}/\mathcal{G}$ the quotient space of $\mathcal{A}$ with respect to the action of gauge transformations) in the sense that for a pair of gauge not equivalent fields $\mathcal{A}_1$ and $\mathcal{A}_2$ there exist such a pair of loops γ_1 and γ_2 that $(\gamma_1)[\mathcal{A}_1] \neq (\gamma_2)[\mathcal{A}_2]$. Then (27) means that the loop variables (γ) and (γ^{-1}) are independent coordinate variables. As we will see this independance is closely connected with the complexity of our loop variables.

Loop quantities form an over-complete set of coordinates in the sense that they satisfy the following identities [6]

1. They are invariant under reparametrization of loops. If γ' is a reparametrized loop $\gamma'(s) = \gamma\,(\,f(s)\,)$ then

$$(\gamma') = (\gamma).$$

2. The Mandelstam identity. For any three loops α, β, and γ intersecting at a point one has

$$(\alpha)(\beta)(\gamma) = (\alpha \circ \beta)(\gamma) + (\alpha)(\beta \circ \gamma) + (\alpha \circ \gamma)(\beta) - (\alpha \circ \beta \circ \gamma) - (\alpha \circ \gamma \circ \beta).$$

   This, on the first sight cumbersome relation has replaced the analogous one for the case of pure gravity (because of, as we have mentioned, the independence of (γ) and (γ^{-1}) variables).

Unlike the case of pure gravity these loop variables are not invariant under retracing operation

$$(\gamma \circ \eta \circ \eta^{-1}) \neq (\gamma),$$

where $\circ$ means the composition of loops which intersect at a point, η^{-1} is the inverse of a curve η. This makes the loop algebra more complicated; we will see in Sec. IV how this difficulty can be avoided.

As configurational loop variables involving the fermionic degrees of freedom we will take certain (see below) even Grassmann algebra elements. The infinite-dimensional Grassmann algebra is generated by the anticommuting complex objects – that is by the infinite set of our dynamical field variables $\xi(x), \bar{\eta}(x), \tilde{\pi}(x), \tilde{\omega}(x)$. A basis of the Grassmann algebra is formed by the powers of the algebra generators. Having these said let us consider the following even elements associated with open curves

$$(\xi|\gamma|\bar{\eta}) := Tr\{\xi\, U[\gamma]\, \bar{\eta}\} = \xi^A\, U[\gamma]_A^{\ B}\, \bar{\eta}_B, \tag{28}$$

which we will regard as the fermionic configurational variables[2]

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[2]This is the point where our approach differs from that of Morales-Técolt and Rovelli [7]. As the quantities involving fermionic fields they considered $(\psi|\gamma|\psi)$ (in our notations). Because of the anticommutative character of Grassmann variables this quantity turns into zero when the corresponding curve shrink to a point. However, it is one of the reasons for which we introduce the loop variables that the quantum Hamiltonian operator can be defined as the certain loop limit of the operator constructed from the basic loop operators. In this sense the loop quantities quadratic on a Grassmann field can not serve as the basic dynamical variables.

We propose the usefull notation in which any loop quantity is denoted by a Greek letter in parenthesis. Since ends of the curve correspond to the fermionic degrees of freedom, it is convinient to include the symbols of fermionic fields in paranthesis on both sides of a loop symbol to get a symbol which describes the mixed quantity. Thus, γ in the above expression is the open curve with ends marked by $\xi, \bar{\eta}$; we will always put ξ at the *final* point of a curve and η will mark the *initial* (recall that any curve (loop) has an orientation)

FIG. 1. Fermionic variables are associated with open curves.

The introduced quantities satisfy the relations analogous to these associated with closed loops

1. Reparametrization invariance

$$(\xi|\gamma'|\bar{\eta}) = (\xi|\gamma|\bar{\eta}).$$

2. The Mandelstam identity. Consider a curve α and a point s on it. This point divides α into two parts for which we use the special notation

$$\alpha = \alpha_{/s} \circ \alpha_s.$$

Thus, α_s is the part of the α from the begining to the point s and $\alpha_{/s}$ is the remaining part. Then for any three curves α, β, and γ intersecting at a point the following identity holds

$$(\xi|\alpha|\bar{\eta})(\xi|\beta|\bar{\eta})(\xi|\gamma|\bar{\eta}) =$$

$$\begin{aligned} &= (\xi|\alpha_{/i} \circ \beta_i|\bar{\eta})(\xi|\beta_{/i} \circ \alpha_i|\bar{\eta})(\xi|\gamma|\bar{\eta}) + (\xi|\alpha_{/i} \circ \gamma_i|\bar{\eta})(\xi|\gamma_{/i} \circ \alpha_i|\bar{\eta})(\xi|\beta|\bar{\eta}) \\ &+ (\xi|\beta_{/i} \circ \gamma_i|\bar{\eta})(\xi|\gamma_{/i} \circ \beta_i|\bar{\eta})(\xi|\alpha|\bar{\eta}) - (\xi|\gamma_{/i} \circ \alpha_i|\bar{\eta})(\xi|\beta_{/i} \circ \gamma_i|\bar{\eta})(\xi|\alpha_{/i} \circ \beta_i|\bar{\eta}) \\ &- (\xi|\gamma_{/i} \circ \beta_i|\bar{\eta})(\xi|\beta_{/i} \circ \alpha_i|\bar{\eta})(\xi|\alpha_{/i} \circ \gamma_i|\bar{\eta}), \end{aligned} \tag{29}$$

or using graphical notation for $(\xi|\alpha|\bar{\eta})$

FIG. 2. The Mandelstam identity for open curves

Again, there is no retracing identity

$$(\xi|\gamma_{/s} \circ \alpha \circ \alpha^{-1} \circ \gamma_s|\bar{\eta}) \neq (\xi|\gamma|\bar{\eta}).$$

So we have introduced the quantities (γ) and $(\xi|\gamma|\bar{\eta})$. The set of these variables form the Abelian algebra under the Poisson brackets and will play the role of "coordinates" in the loop representation.

B. "Momentum" loop variables.

Let us first construct the pure gauge quantities. We will associate such momentum variables with piecewise analitic strips, i.e. piecewise analitic imbeddings $S : [0,1] \times (0,1) \to \Sigma$. Inserting the momentum field $\tilde{\mathcal{E}}^a$ at points of loop, one can construct the following gauge invariant loop quantities linear in the field $\tilde{\mathcal{E}}$

$$(\gamma)^a(s) := Tr\{U[\gamma_{/s}]\, \tilde{\mathcal{E}}^a(s)\, U[\gamma_s]\}; \tag{30}$$

The higher orders in $\tilde{\mathcal{E}}$ are constructed in a similar way

$$(\gamma)^{a_1,\ldots,a_n}(s_1,\ldots,s_n) := Tr\{U[\gamma_{/s_n}]\, \tilde{\mathcal{E}}^{a_n}(s_n) U[\gamma_{s_n/s_{n-1}}] \cdots U[\gamma_{s_2/s_1}]\, \tilde{\mathcal{E}}^{a_1}(s_1)\, U[\gamma_{s_1}]\}. \tag{31}$$

These quantities are almost what we need as the momentum variables. As we have stated, they are gauge invariant but, because of their vector character, they transform under the action of diffeomorphisms somewhat complicately. We shall construct the other quantities which we associate with piecewise strips and which transform under diffeomorphisms as geometrical objects (i.e. the transformed quantity is of the same type but associated to another strip – a transformed one).

Let us first construct the basic, linear in momentum field variables. These are

$$(S) := \int_S d\, s^{ab}(p)\, \eta_{abc}(\, \gamma(p)\,)^c(p). \tag{32}$$

Here $\gamma(p)$ is a loop which goes through a point p on S, and η_{abc} denotes the Levi-Civita tensor density on Σ. The loop family $\gamma(p)$ is supposed to cover all the strip surface (the loops $\gamma(p)$ and $\gamma(p')$ for different p, p' may coincide). The quantity defined is a gauge invariant functional on the phase space associated with a strip S

FIG. 3. Linear in $\tilde{\mathcal{E}}$ momentum variables are associated with strips.

Next the variables of higher orders in $\tilde{\mathcal{E}}$ are to be constructed. Again, we will associate them with piecewise strips through the certain averaging procedure but, since in this case we have more than one momentum field to be averaged, we split a strip into pieces and average each $\tilde{\mathcal{E}}$ field over its own piece of strip. This procedure makes sure that the points where momentum fields are taken are not coincide, what is of much importance for the regularization program.

Let S be a strip which consists of two parts S_1 and S_2

FIG. 4. A piecewise strip assotiated with the higher order momentum variable.

Taken two points p_1, p_2 (each on a different part of S) one can draw a loop $\gamma(p_1, p_2)$ through these points. Suppose the loop family $\gamma(p_1, p_2)$ covers the strip (moreover, it is supposed to cover each part of the strip when a point on the other part is fixed.) Then we can define the second order in $\tilde{\mathcal{E}}$ momentum variable as the gauge invariant functional associated with the strip S

$$(S_1 \circ S_2) := \int_{S_1} d\,s^{ab}(p_1)\,\eta_{abc} \int_{S_2} d\,s^{a'b'}(p_2)\,\eta_{a'b'c'}\,(\,\gamma(p_1;p_2)\,)^{cc'}(p_1;p_2). \tag{33}$$

In a similar way one can construct the variables of higher orders in $\tilde{\mathcal{E}}$.

The loop quantities involving fermionic momentum fields are

$$(\xi|\gamma|\tilde{\pi}) := Tr\{\xi\, U[\gamma]\, \tilde{\pi}\} \tag{34}$$

$$(\tilde{\omega}|\gamma|\bar{\eta}) := Tr\{\tilde{\omega}\, U[\gamma]\, \bar{\eta}\} \tag{34a}$$

$$(\tilde{\pi}|\gamma|\tilde{\omega}) := Tr\{\tilde{\pi}\, U[\gamma]\, \tilde{\omega}\}. \tag{34b}$$

Again γ is a curve with ends marked by the corresponding fermionic variables. These quantities are represented by

FIG. 5. Fermionic momentum variables.

C. Loop variables algebra

The introduced loop variables are functionals on the phase space and the Poisson algebra they generate can be computed. It is induced by the Poisson structure on the space of gauge and fermionic fields ((7) and (19) respectively). Our aim is to describe the resulting algebra of loop variables in a graphical form.

The brackets of loop quantities with those including gauge momentum fields can be obtained by using the following useful expression for the matrix U

$$U[\gamma]_A^{\,B} = \int ds^a \, U[\gamma_{/s}]_A^{\,C} \, \mathcal{A}_a(s)_C^{\,D} \, U[\gamma_s]_D^{\,B}.$$

So one gets

$$\{\,(S),\,(\beta)\,\} = (\gamma^{(S)} \circ \beta), \tag{35}$$

where $\gamma^{(S)}$ is the loop from the loop family covering S which intersect with the loop β. One can represent these brackets by

FIG. 6. The Poisson brackets between a strip and a loop variables.

The brackets of (γ) with the higher orders variables can also be computed but in order to describe the result in a graphical form we need the objects like half-strip–half-curve. Instead of introducing such strange objects we computed the other Poisson brackets

$$\{\{\,(S_1 \circ S_2),\,(\alpha)\,\},\,(\beta)\,\} = (\gamma^{(S_1 \circ S_2)} \circ \alpha \circ \beta), \tag{36}$$

where $\gamma^{(S_1 \circ S_2)}$ is that loop from the family covering $S_1 \circ S_2$ which goes through the points of intersection of $S = S_1 \circ S_2$ with α and β.

It is tempting to represent it in the following graphical form

FIG. 7. The brackets with the higher order variable.

The brackets of (34) with coordinate loop quantities are given by

$$\int d^3x\,\{\,(\xi|\beta(x)|\tilde{\pi}(x)),\,(\xi|\gamma|\bar{\eta})\,\} = (\xi|\beta\circ\gamma|\bar{\eta}); \tag{37}$$

in the right side of this expression β is the curve from the family $\beta(x)$ whose initial point coincide with the final point of γ. There is the graphical representation for these brackets

FIG. 8.

Computing the other brackets one gets

$$\int d^3x\,\{\,(\tilde{\omega}(x)|\beta(x)|\bar{\eta}),\,(\xi|\gamma|\bar{\eta})\,\} = (\xi|\gamma\circ\beta|\bar{\eta}); \tag{37a}$$

FIG. 9.

$$\int d^3x\, d^3y \,\{\, (\tilde{\pi}(x)|\beta(x;y)|\tilde{\omega}(y)),\, (\xi|\gamma|\bar{\eta})\,\} = (\xi|\gamma\circ\beta|\tilde{\pi}) \,+\, (\tilde{\omega}|\beta\circ\gamma|\bar{\eta}); \tag{37b}$$

FIG. 10.

So, as we have seen, the algebra of introduced "loop" variables can be expressed solely in terms of geometrical objects: loops, curves and strips. Because of the natural action of the diffeomorphism transformations on the introduced variables (namely as on geometrical objects), the elements of the quotient algebra of these variables with respect to the giffeomorphisms' action have a clear geometrical meaning. They are represented by classes of diffeomorphism equivalent curves, loops and strips. The algebraic (induced Poisson) structure on this quotient algebra is given by *the same relations* (35)-(37) but understanding as the relations for classes of equivalence. This fact helps one to solve the problem of finding the solutions of the diffeomorphism constraint in the loop representation.

IV. THE LOOP REPRESENTATION

Constructing the quantum representation for our system we will mostly follow the programm of Asthekar group (for recent developments see [14]); however, the approach described below is more "physical", albeit naive. It is why we attach such importance to the visualization of all relations that we define the representation graphically: the corresponding operators, their action on states and states themselves will be described solely in terms of geometrical objects and operations with them. We run into not a new in theoretical physics situation that the objects it operate with are simpler to draw than to express mathematically. However, the recent progress in the program declared by Ashtekar group allows provides us with the framework for rigorous discussion. Although the program has not yet been accomplished (at least its results are not generally known) the whole picture is getting clear and we will try to outline it.

The program of quantization of generally covariant field theories proposed in the number of publications (see [14] and references therein) uses the idea to realize the quantum representation space as the representation space of Abelian sub-algebra of dynamical variables – the C^*-algebra of configurational variables. The construction is the infinite-gimensional generalization of the standart coorsinate representation in quantum mechanic: the space of coordinate representation is the represenation space of Abelian algebra of $\hat{x}$ operators, i.e. $Span\{|\,x\,\rangle\}$ where $\hat{x}|\,x\,\rangle = x|\,x\,\rangle$. The canonical realization of this space is the space $L^2(I\!R, dx)$ of functions $\varphi(x)$ over the spectrum of $\hat{x}$ operator, i.e. over $I\!R$. Other variables are represented by derivative operators on L^2. The infinite-dimensional case repeats all these points: the representation space is the space of all continuous functionals on the spectrum of the configuration variables algebra. The "momentum" variables require some work to be rigorously gefined (the projective limit technique in [14]), but naively they are represented by variational derivative operators.

In order to take advantage of the standard representaion theory of C^*-algebras we have to define the C^*-algebra of configurational variables.

A. C^*-algebra of configurational variables.

The most natural candidate for this algebra is the algebra of our configurational loop variables over complex numbers. Its elements would be complex even Grassmann numbers and it is tempting to define the $*$-operator as the $\dagger$-operator from the Grassmann algebra. However we run into problems on this way. First, because of the complexity of Ashtekar connection, the parallel transport matrix is *not* unitary (it belongs to $SL(2,C) \times U(1)$ group); therefore, there is no retraicing identity for our algebra elements. Next, our loop variables behave somewhat complicately under the complex conjugation operation; for example, given any loop γ there may not exists such a loop γ' that $(\gamma)^* = (\gamma')$; . But it is still not the worst. Because of the non-unitarity of $U(\gamma)$ the elements (γ) are *not* bounded so the natural sup-norm $\| \cdot \|$

$$\| (\gamma) \| := \sup_{\mathcal{A}} |(\gamma)| \tag{38}$$

does *not* exists on this algebra. Owing to these facts the case of Lorentzian general relativity remained a problem by the last time. The situation has changed considerably after the coherent state transform had appeared [15].

This transform incorporates in a natural way the reality conditions which one should impose on the complex phase space of general relativity. Being complex the $SL(2,C)$ connection field A bears some unphysical information and one can expect that its "real" $SU(2)$ part will play a role when the reallity conditions are imposed. Indeed, the representation space of Lorentzian general relativity can be realized as the space of holomorphic functionals of complex (generalized) Ashekar connection. Then, as it has been shown in [15], there exists the isomorphism (given by the coherent state transform) between this space and the space of functionals of (generalized) $SU(2)$ connection. This isomorphism provides us with the representation of real general relativity we look for. It is the representation in the space of holomorphic functionals of complex Ashtekar connection which is isomorphic to the representation of $SU(2)$ variables algebra in the space of functionals of $SU(2)$ connection.

So we have got to construct the representation of $SU(2)$ variables algebra regarding it as the representation isomorphic in a certain way to the required one. For the case of pure gravity this actually has been done [14] and our aim is to show that the construction allows the natural enlargening to the case when the gauge field and the fermionic matter present.

Having this said let us describe the C^*-algebra of $SU(2)$ configurational variables. It is formed by the same "loop" quantities $(\gamma), (\xi|\gamma|\bar{\eta})$ (with the multiplication, additive and $\dagger$-operations from the Grassmann algebra); the only difference is that the parallel transport matrix $U[\gamma]$ becomes now unitary so the algebra generators satisfy the following properties:

- The complex loop quantities (γ) are connected with thier complex conjugate as

$$(\gamma)^* = (\gamma^{-1}).$$

- Because of the key relation $U[\gamma^{-1}] = U^\dagger[\gamma]$ for the $U(2)$ connection holonomy matrix loop variables satisfy the retracing identity

$$\begin{aligned} (\gamma \circ \eta \circ \eta^{-1}) &= (\gamma), \\ (\xi|\gamma_{/s} \circ \alpha \circ \alpha^{-1} \circ \gamma_s|\bar{\eta}) &= (\xi|\gamma|\bar{\eta}). \end{aligned} \tag{39}$$

  This means that the elements of our C^*-algebra are associated with classes of equivalence of loops and curves. These classes for the case of pure gravity are called hoops and it seems reasonable to keep this name and for equivalence classes of curves. Two loops (or curves) belong to the same equivalence class (or hoop) if they define the same loop quantities for all fields $\mathcal{A}, \xi, \bar{\eta}$.

- Being unitary $U[\gamma]$ has a bounded trace. So there exists the sup-norm (38) on the introduced algebra which of generators is

$$\| (\gamma) \| = 2.$$

- The action of $\dagger$-operation on the fermionic "loop" variables is a consequence of our momentum fields definition (14) and is given by

$$\begin{aligned} (\sigma(x))(\sigma(y))\,(\xi(x)|\gamma|\bar{\eta}(y))^\dagger &= (\tilde{\omega}(y)|\gamma^{-1}|\tilde{\pi}(x)), \\ (\sigma(x))\,(\xi(x)|\gamma|\tilde{\pi}(y))^\dagger &= (\sigma(y))\,(\xi(y)|\gamma^{-1}|\tilde{\pi}(x)), \\ (\sigma(y))\,(\tilde{\omega}(x)|\gamma|\bar{\eta}(y))^\dagger &= (\sigma(x))\,(\tilde{\omega}(y)|\gamma^{-1}|\bar{\eta}(x)). \end{aligned} \tag{40}$$

  These relations are, in fact, the reality conditions which one should impose on the fermionic phase space in order to single out its real part. It is worthwile to note that we have chosen the form in which they are non-polynomial in $\tilde{\mathcal{E}}$ variable (because of presence of (σ)).

- Fermionic "loop" algebra generators have the following norm (see the Appendix A. for the definition of the norm on Grassmann algebra)

$$\| (\xi|\gamma|\bar{\eta}) \| = \sqrt{|\, Tr\, U[\gamma]\, U[\gamma^{-1}]|} = \sqrt{2}.$$

So the algebra of these variables becomes an Abelian *-algebra with norm $\| \cdot \|$ (which satisfies the relation $\| A\, A^\dagger \| = \| A \|^2$) and we can take a completion to obtain a C^*-algebra of configurational "loop" variables.

B. Representation space.

Having the C^*-algebra of configurational variables we are at the point to implement the standard representation theory. According to Gelfand an Abelian C^*-algebra is isomorphic to the algebra of all continuous functions on its spectrum. Let us give the description of the spectrum of our loop variables algebra. Denote by $\mathcal{F}$ the space of all (satisfying a certain boundary conditions) fields $\mathcal{A}_{aA}^{B}(x), f^A(x), g^A(x)$ where $\mathcal{A}_a \in u(2)$ and f, g are complex spinor fields (*non*-Grassmann-valued) which take values in fibres F of $\mathcal{G}$-bundle over Σ. The corresponding space quotient by the gauge transformations will be $\mathcal{F}/\mathcal{G}$ where $\mathcal{G} = SU(2) \times U(1) = U(2)$. Then each point of $\mathcal{F}/\mathcal{G}$ defines a liniar homomorphism ω from the loop variables algebra to C (a character) as follows:

$$\omega_{\mathcal{A},f,g}((\gamma)) := Tr(\exp \oint_\gamma \mathcal{A}),$$

$$\omega_{\mathcal{A},f,g}((\xi|\gamma|\bar{\eta})) := Tr(f \ \exp \int_\gamma \mathcal{A} \ g).$$

The spectrum is the set of all characters so we have that the points of $\mathcal{F}/\mathcal{G}$ distinguish the elements of our algebra spectrum; it is easy to show that $\mathcal{F}/\mathcal{G}$ is dence (in Gelfand topology) in the spectrum, so we will denote the later by $\overline{\mathcal{F}/\mathcal{G}}$. This space becomes a quantum configurational space of our theory. As in the case of pure gravity its limit points are distributions which we shall regard as generalized fields (in the sense of Dirac's δ-function). We will denote the generalized fields by the same symbols $\mathcal{A}, f, g$; so $\overline{\mathcal{F}/\mathcal{G}} = \{\mathcal{A}, f, g\}$.

So the space of loop algebra representation is the space $C^0(\overline{\mathcal{F}/\mathcal{G}})$ of all continuous functions over $\overline{\mathcal{F}/\mathcal{G}}$. This space, however, is too large to define integral and differential calculus on it. The construction of a smoler space, measure and differential calculus on this smoler space has been proposed by Ashtekar *et al* [14]. They proposed to regard the quantum configuration space of infinite-dimensional case as *the projective limit* of finite-dimensional configurational spaces of gravity on floating lattices. Then the representation space becomes the space $Cyl(\overline{\mathcal{F}/\mathcal{G}})$ of cylindrical functionals over the algebra spectrum. By a cylindrical functional on $\mathcal{F}/\mathcal{G}$ we understand a map Ψ

$$\Psi = \Phi(f^1, \ldots, f^k, \ g^1, \ldots, g^m, \ \mathcal{P}\exp \int_{\gamma_1} \mathcal{A}, \ldots, \mathcal{P}\exp \int_{\gamma_n} \mathcal{A})$$

$$\Psi : (F^k \times F^m \times \mathcal{G}^n) \to C.$$

There is a natural measure μ on $Cyl(\overline{\mathcal{F}/\mathcal{G}})$ and the differential calculus which are defined by the projective limit from the finite-dimensional configurational spaces [14]; this gives rise to the Hilbert representation space $L^2(\overline{\mathcal{F}/\mathcal{G}}, \mu)$.

In order to construct the *loop representation* we choose a certain basis in the representation space $Cyl(\overline{\mathcal{F}/\mathcal{G}})$ so that the loop variables become in this basis simple operators which can be interpreted in terms of creation and annihilation operators. The idea is very similar to one which is used to define, for example, the momentum representation in quantum mechanics. One chooses the basis formed by all proper states $|\,p\,\rangle$ of the momentum operator (the corresponding waive-functions are $\sim \ \exp i\,px$) and defines all the operators by their action on states from this basis. Our idea is to introduce the basis of “loop” states which in some sense are the proper states of the momentum loop operators. The loop operators become the operators of creation and annihilation of loops and curves in this basis.

It is convinient to use Dirac's notation and denote the following functionals in our space by Dirac's kets

$$|\,\alpha\,\rangle = \begin{cases} Tr(f\ \mathcal{P}\exp\int_\alpha \mathcal{A}\, g) \quad or \\ \\ Tr(\mathcal{P}\exp\oint_\alpha \mathcal{A}) \end{cases}$$

depending on whether α has ends or not; and in a similar way a “multiloop” state is

$$|\,\alpha,\beta\,\rangle = \begin{cases} Tr(f\ \mathcal{P}\exp\int_\alpha \mathcal{A}\, g)\, Tr(f\ \mathcal{P}\exp\int_\beta \mathcal{A}\, g) \\ \qquad \ldots \end{cases} \qquad , etc.$$

The order in which loops are taken to compose a multiloop state is not important.

These states form the basis in the representation space and we will call them n-loop states. Configurational loop variables become operators of multiplication and Dirac's notations allow us to express their action simply by

$$(\hat{\gamma})\ |\,\alpha\,\rangle = |\,\alpha,\gamma\,\rangle$$

$$(\xi|\hat{\gamma}|\eta)\ |\,\alpha\,\rangle = |\,\alpha,\gamma\,\rangle.$$

In a similar manner they act on the states containing more ”loops”. We see, therefore, that if one thinks about state $|\,\alpha_1,\alpha_2,\cdots,\alpha_n\,\rangle$ as about state containing n ”loops” then the action of the coordinate operators consists in simply the adding of one more ”loop” to the state. It is tempting to regard these operators as “creation” operators. The basis can be obtained acting on a cyclic vector $|\,\alpha\,\rangle$ by these creation “loop” operators.

Let us exemine the basis introduced more thoroughly. Due to the identities satisfied by Wilson functionals this basis is overcomplete. Its elements are linear dependent, so some of them may be rewritten as linear combinations of others. For example, any three-loop state may be first realized as the state with three loops intersecting at a point and then reduced to the sum of states containing two and one loop

FIG. 11. The reduction of a three loop state.

Thus, as it has been found by Gambini and Pullin [6], any n-loop state may be reduced to a linear combination of two- and one-loop states.

Consider then a state containing two loops and one open curve. Repeating the above procedure, we may reduce this state to a linear combination of states containing one loop, one open curve and merely open curve states

FIG. 12. An open curve state reduction.

Thus, any state containing n curves and m loops may be reduced to a linear combination of states containing n curves and one loop or n-curve states (the number of ends in a state cannot be reduced). The set of irreducible elements of our basis consists of states

- n curves no loops, n $= 0, 1, \cdots$,
- n curves one loop, n $= 0, 1, \cdots$,
- two loops.

Having this "loop" basis in the representation space we are ready to define the action of other operators by defining it on the basis elements.

C. The "momentum" operators.

In the last part of this Section we construct the representation of the classical "momentum" loop variables in the space considered, i.e. we build operators

$$(\hat{S}_1 \circ \ldots \circ \hat{S}_n)$$

$$(\xi|\hat{\gamma}|\tilde{\pi})$$

$$(\tilde{\omega}|\hat{\gamma}|\bar{\eta})$$

$$(\tilde{\pi}|\hat{\gamma}|\tilde{\omega}),$$

so that their commutational relations coincide to the first degres in $\hbar$ with the Poisson brackets of their classical analogs. Note that we represent the Poisson brackets by *commutational* relations even though the variables involve Grassmann fields.

The advantage of the built representation is that we have simultaneously two equivalent descriptions of operator's action. The first, visual one is based on graphical representation of states and operators, i.e. on dealing with Dirac's kets $|\,\alpha\,\rangle$. The second description is based on representing states as functionals of generalized fields and operators act in the space of functionals. In this later one there is a naive way to define the "momentum" operators; one shold just use the corresponding classical expressions and replace all momentum fields by the functional derivative operators. The resulting operators will act in the space of functionals of generalized fields $\mathcal{A}, f, g$. No problems arise there with operator ordering because functional derivative operators commute when they act at different space points. These construction leads to operators whose action on the introduced basis vectors can be described graphically. Moreover, this graphical description can be used as the alternative definition of "momentum" operators. We shall give the result in both descriptions.

First, let us define the "strip" operators, which involve only gauge degrees of freedom. The quantity of the first order in $\tilde{\mathcal{E}}$ is represented by the following operator

$$(\hat{S}) \circ |\,\alpha\,\rangle := i\,|\,\gamma^S \circ \alpha\,\rangle, \tag{41}$$

where γ^S is the loop from the loop family covering S which intersect with loop α (when there is no intersection of α with the strip the result is zero). The graphical representation of this operator is

FIG. 13.

i.e. the operator adds the loop to one loop state and glues these loops in the only way compatible with their orientation. As we have stated above the "loop" states are "proper" states of our momentum operators in the sense that the result of their action on a n-loop state is also a n-loop state (when the strip intersect more than one loop from the state the result will be the sum of n-loop states). Next, we have to define the operators which correspond to higher order "momentum" variables. For the quantity introduced as the two strip variable we propose the following definition

$$(\hat{S}_1 \circ \hat{S}_2) \circ |\, \alpha, \beta \,\rangle := (i)^2 \,|\, \alpha \circ \gamma^{S_1 \circ S_2} \circ \beta \,\rangle. \tag{42}$$

Here $\gamma^{S_1 \circ S_2}$ is the loop from the family $\gamma(p_1; p_2)$ which covers $S_1 \circ S_2$ such that it intersect with the both loops α, β. If the state it acts on does not contain two loops which intersect each with its own part of the strip the action is set to be zero. The graphical representation of this operator is

FIG. 14.

These operators serve as annihilation operators in our representation because they reduce the number of loops in the state they act on.

Let us define the fermionic "momentum" operators. The construction is straightforward from the form of commutational relations with the configurational variables. We shall define

$$\int d^3x \,(\xi|\hat{\gamma}(x)|\tilde{\pi}(x)) \circ |\, \alpha \,\rangle := i \,|\, \gamma \circ \alpha \,\rangle. \tag{43}$$

Here we integrated over the first point of the loop in order to have the same density at the right and left sides of the expression and the curve γ at the right side is that one from the family $\gamma(x)$ whose final point coincide with the initial point of α. The graphical description of this operator is

FIG. 15.

i.e. it prolonges α by adding the corresponding curve to the final point. This, linear in fermionic momentum field operator does not change the number of open ends in a state. The other linear in momentum field operator is defined in a similar way

$$\int d^3x\,(\tilde{\omega}(x)|\hat{\gamma}(x)|\bar{\eta}) \circ |\,\alpha\,\rangle := i\,|\,\alpha \circ \gamma\,\rangle; \tag{44}$$

FIG. 16.

the only difference with the previous operator is that this one adds the corresponding curve to the initial point of α.

$$\int d^3x\,d^3y\,(\tilde{\omega}(x)|\hat{\gamma}(x;y)|\tilde{\pi}(y)) \circ |\,\alpha,\beta\,\rangle := (i)^2\,|\,\beta \circ \gamma \circ \alpha\rangle \,+\, (i)^2\,|\,\alpha \circ \gamma \circ \beta\,\rangle; \tag{45}$$

FIG. 17.

This operator glues two different curves so it reduces the number of open ends in a state; it requires at least two curves to be in the state in order that its action is non-trivial. When there is more than one loop in the state the operators act as derivative operators by the Leibniz rule on each loop.

Restoring the $\hbar$ factor in all relations one can easy check that the above defenitions really give a representation of the classical algebra, i.e. that the commutators among defined operators (scalled by the factor $i\hbar$) turn into their classical analogs when $\hbar$ goes to zero.

D. Measure and the hermiticity problem.

We write the i factor defining the quantum operators hoping that it will provide the correct properties under hermician conjugation operation. Let us discuss this problem describing the measure in the representation space $Cyl(\overline{\mathcal{F}/\mathcal{G}})$. This space can be thought of as a projective limit (see [14]) of finite-dimensional spaces so there exists a natural diffeomorphism-invariant measure μ on it. Thus, one can define a scalar product of two "loop" states by

$$\langle \gamma' \,|\, \gamma \rangle := \int_{\overline{\mathcal{F}/\mathcal{G}}} \overline{T_{\gamma'}[\mathcal{A}]} T_{\gamma}[\mathcal{A}]\, d\mu \tag{46}$$

and

$$\langle \gamma' \,|\, \gamma \rangle := \int_{\overline{\mathcal{F}/\mathcal{G}}} \overline{Tr(f\; \mathcal{P} \exp \int_{\gamma'} \mathcal{A}\; g)}\, Tr(f\; \mathcal{P} \exp \int_{\gamma} \mathcal{A}\; g)\, d\mu \tag{46a}$$

for open "loop" states. This scalar product is diffeomorphism-invariant so for any diffeomorphism transformation φ

$$\langle \gamma' \,|\, \varphi \circ \gamma \rangle = \langle \gamma' \,|\, \gamma \rangle.$$

Note also that from the properties of measure μ follows that a scalar product of two loop states does not turn to zero only if the corresponding loops intersect. It is easy to check that the configurational operators we defined has the correct hermitian conjugate

$$\langle \gamma' \,|\, (\hat{\alpha}) \circ \gamma \rangle = \langle (\hat{\alpha}^{-1}) \circ \gamma' \,|\, \gamma \rangle.$$

Let us consider the momentum operators. Classicaly, the quantity (S) associated with a strip behaves under the complex conjugation operation as $\overline{(S)} = -(S^{-1})$ where the strip (S^{-1}) is the same strip but foliated by loops of opposit direction. One should have a similar relation on quantum level. This relation is easy to prove due to a simple character of the operator's action. Indeed, consider the quantity $\langle \gamma' \,|\, (S) \circ \gamma \rangle$. It is equal to the product of two loop states (multiplied by i) $i \langle \gamma' \,|\, \alpha^S \circ \gamma \rangle$ which does not turn to zero only if the both loops γ', γ intersect with the strip. Due to the diffeomorphism-invariance of the scalar product this quantity is an invariant of two loops; this means that it has the same value for any pair of two intersecting loops. It follows from the definition of the operator $(\hat{S})$ that

$$\langle (\hat{S}^{-1}) \circ \gamma' \,|\, \gamma \rangle = -\, i \langle (\alpha^S)^{-1} \circ \gamma' \,|\, \gamma \rangle.$$

Because we assume that these two loops intersect with the strip the last scalar product in this formula is the same invariant of two intersecting loops

$$\langle (\alpha^S)^{-1} \circ \gamma' \,|\, \gamma \rangle = \langle \gamma' \,|\, \alpha^S \circ \gamma \rangle.$$

Thus we have the relation

$$(\hat{S})^{\dagger} = -\, (\hat{S}^{-1})$$

which coincides with the relation on the classical level. In summary let us say that we have obtained the quantization of pure gauge part of our system such that all operators are well defined operators in the loop space with correct behaviour under hermitian conjugation (with respect to the scalar product defined). As to fermionic operators, on the quantum level one should impose more complicated conditions (40) which include ($\hat{\sigma}$) operator. We have seen in Sec.II that this quantity is not polynomial in $\tilde{\mathcal{E}}$ field so in order to define the corresponding operator one needs a certain regularization procedure (which should include an integration over some sub-space of Σ in order to get a nonlocal quantity). For the representation described it has been done in [16]. One can consider the reality conditions (40) as conditions which allow one to choose a regularization method and construct operator ($\hat{\sigma}$) with required properties.

It is left a few words to be said about the solving of the diffeomorphism constraint in our approach. This is particularly simple while describing the states and operators graphically for, as we have stated, the diffeomorphism constraint generates a flow on the quantum operator algebra which preserves the structure of its commutational relations. In other words, the quantum algebra is described in terms of geometrical objects and diffeomorphism constraint is represented in quantum case by a generator of transformations of these objects; commutational relations are written in terms of geometrical objects and the same relations hold for all representatives of the diffeomorphic equivalent object classes. In order to pick up the physical states, i.e. to solve the constraint, one should find a representation of the "physical" operators which lie in the corresonding quotient algebra. In the approach described this will be the representaion in the space of equivalence classes of loops and curves and all "physical" operators will act on classes of diffeomorphic equivalent objects. The other approach for solving the diffeomorphism constraint is the averaging procedure [14].

V. DISCUSSION

We have constructed the representation for our quantum system in which classical loop variables became operators in the "loop" space. The representation we built differs from the loop representation of pure gravity in the following important points:

1. Unlike loops describing pure gravitational exitations loops and curves of the unified theory are oriented.
2. Momentum loop operators (pure gauge as well as fermionic ones) act on "loop" states merely prolonging these "loops" in the only compatible with their orientation way.

We have proved that the operators describing pure gauge degrees of freedom have the correct properties under the hermitian conjugation operation with respect to the scalar product (46). As to fermionic operators, in order to prove their hermitian properties one has first to construct the ($\hat{\sigma}$) operator which corresponds to the square root of a determinant of the metric; these problems are discussed in our following work [16].

One can propose an interesting classification of the constructed "loop" operators in terms of creating and annihilating operators. We have seen that the operators corresponding to configurational loop variables act by adding a "loop" to a state so it is natural to regard them as creating operators. This terminology is especially good for an operator represented by a curve because the open ends of a curve in our formalism correspond to fermions [3]. It can easily be shown that this operator actually creates a pair of "fermions" of different charge sign. Indeed, one can define the charge in a state as an eighenvalue of the charge operator for this state. Classical charge is the generator of gauge transformations; it is the quantity

$$i\,Q = -\int_\Sigma d^3x\,C(x)\,(\xi^A(x)\tilde{\pi}_A(x) + \bar{\eta}^A(x)\tilde{\omega}_A(x)),$$

where $C(x)$ is an arbitrary (real, integrable) function. One can define the quantum charge operator with the regularization procedure of point splitting. The result is a well defined operator which act only on the ends of curves in a state. Each final point on a curve gives -1 while initial points give $+1$. Thus, the result of this operator's action on *any* state in our representation is zero[4]. This means that our "fermions" are born only in pairs with their "anti-particles"and that all the states in our representation are electrically neutral.

The operators corresponding to the quantities linear in momentum fields do not change neither number of loop nor number of ends in a state; in this sense "loop" states are "eighenstates" of these operators. And finaly there are momentum operators of higher orders which reduce the number of "loops" in a "loop" state; it is natural to call them annihilating operators. The result of such operator's action is not zero only when there are two (or more) "loops" in the state it act on; moreover, these "loops" should intersect with the object associated with the operator. Note then that there are no operators which can "kill" a state containing only one curve (or only one loop): the result of action of any sequence of our operators on a state with one curve (loop) will still contain at least one curve (loop). Therefore, the states containing pure gravitational exitations are inaccessible from the states with fermionic exitations. This fact immediately follows from the formalism; it means that the representation constructed is reducible one because the representation space contains an invariant sub-space – it is the sub-space of all states which contain open ends (these states may contain or may not contain loops). Thus, the irreducible representation of our system is realized on the sub-space of all possible states with fermionic exitations.

Let us conclude by speculating on a possible physical meaning of the formalism obtained. First, it describes the unified theory; this means that the gravitational and electromagnetic fields enter the formalism only in a certain combination. On the quantum level exitations of these fields are described by loops and curves; the formalism predicts that there do not exist pure gravitational or pure electromagnetic quantum exitations and these fields appear in the theory only *together*. Second, we have seen that the space of irreducible representation is the space of all states with open ends. Let us denote an arbitrary element from the

[3]Or rather to fermionic degrees of freedom because of the lack of interpritation in terms of particles when no background structure present.

[4]This is what one whould expect from the demand of gauge invariance.

irreducible representation space by $|\,\rangle$. The letter O means that this state contains at least two open ends. Physical states $\langle\, Ph \,|$ (i.e. annihilated by the Hamiltonian operator) are to be constructed by the averaging procedure from the states $|\, O\,\rangle$. As it is discussed in [14] the physical states belong to the dual of the initial representation space; the physical state $\langle\, Ph_O \,|$ corresponding to some $|\, O\,\rangle$ is defined via

$$\langle\, Ph_O \,|\, \alpha \,\rangle := \int_0^\infty dt \,\langle\, O \,|\, \exp i\, t \hat{H} \,|\, \alpha \,\rangle.$$

Thus, physical states are represented by diffeomorphism-invariant functionals of loops and curves. This definition allows one to regard a physical state as a complicated combination (continual sum) of "usual" states; in this sense *any* physical state is a combination of states containing open ends. We have stated that there do not exist pure gravitational or pure electromagnetic exitations in the theory. The above consideration shows that there do not even exist pure gauge exitations: any physical state is constructed in a certain way from the states which contain fermionic exitations. Thus, any physical state seems to contain quantum exitations of *all* the fields of the theory. If it is possible to construct a vacuum state with such a procedure it will contain exitations of all fields as well. One can draw an analogy with a convenctional field theory where the vacuum state also contains exitations of all the fields and regard the exitations in a vacuum state as quantum vacuum field "fluctuations". Sure, these facts are too formal to be regarded as physical predictions of quantum theory. However, in the situation when we lack any consistent interpritation of the formalism the formalism itself might serve a guide to find the physical meaning of our quantum theory.

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APPENDIX

Let us define a norm on Grassmann algebra so that the fermionic variables acquire a certain norm. Arbitrary Grassmann algebra element can be written as

$$f = f^0 + \int_\Sigma d^3x\,(\sigma(x)) f^1_{1\,A}(x)\xi^A(x) + \int_\Sigma d^3x\, f^1_{2\,A}(x)\tilde{\pi}^A(x) + \int_\Sigma d^3x\,(\sigma(x)) f^1_{3\,A}(x)\bar{\eta}^A(x) + \int_\Sigma d^3x\, f^1_{4\,A}(x)\tilde{\omega}^A(x) + \cdots$$

plus terms of higher order in algebra generators. We define its norm as the following sum

$$\| f \| := |f^0| + \int_\Sigma d^3x\,(\sigma(x))|f^1_{1\,A}| + \int_\Sigma d^3x\,(\sigma(x))|f^1_{2\,A}| + \int_\Sigma d^3x\,(\sigma(x))|f^1_{3\,A}| + \int_\Sigma d^3x\,(\sigma(x))|f^1_{4\,A}| + \cdots, \tag{A1}$$

where the norm of "coordinate" spinors is defined via $|f_A| = \sqrt{|f^A \bar{f}_A|}$. The norm on higher order Grassmann algebra sub-spaces (the higher terms in (A1)) is defined from the requirement that $\| f\, f^\dagger \| = \| f \| \, \| f^\dagger \|$, so, for example, the element

$$\int_\Sigma d^3x \int_\Sigma d^3y\,(\sigma(x))\,(\sigma(y))\,\xi^A(x)\, f_A^B(x,y)\,\bar{\eta}_B(y)$$

has the norm

$$\int_\Sigma d^3x \int_\Sigma d^3y\,(\sigma(x))\,(\sigma(y))\sqrt{|\, Tr\, f\, \tilde{f}|},$$

where $\tilde{f}^B_A = \bar{f}^B_A$ is the matrix Hermitian conjugate to f. Let us note that the definition leads to the unity norm of the algebra generators. It can also be checked that $\| f \| = \| f^\dagger \|$ so the norm satisfies all the requirements for a norm on C^*-algebra.

REFERENCES

[1] C. Rovelli, L. Smolin, Phys. Rev. Lett. **61**, 1155 (1988); Nuc. Phys. **B331**, 80 (1990).

[2] B. Brügmann, J. Pullin, Nuc. Phys. **B363**, 221 (1991).

[3] Abhay Ashtekar, Phys.Rev. **D36**, 1587 (1987).

[4] A. Ashtekar, C.J. Isham, Representations of the holonomy algebras of gravity and non-Abelian gauge theories, Class. Quant. Grav. **9**, 1433-67 (1992)

[5] B. Brügmann, R. Gambini, J. Pullin, Phys. Rev. Lett. **68**, 431 (1992); R.Gambini, J.Pullin, The Gauss linking number in quantum gravity. gr-qc/9310025, In "Knots and Quantum Gravity", ed. J.Baez, Oxford U. Press, (1994).

[6] R.Gambini, J.Pullin, Phys. Rev. **D47**, R5214, (1993).

[7] H.A.Morales-Tecolt, C.Rovelli, Phys.Rev.Lett **72**, 3642, (1994).

[8] A.Ashtekar, J.Romano, R.S.Tate, Phys. Rev. **D40**, 2572, (1989).

[9] A.Ashtekar, Overview and Outlook, gr-qc/9403038.

[10] A.Ashtekar, C.Rovelli, and L.Smolin, Phys.Rev.Lett., **69**, 237, (1992)

[11] C.Rovelli, L.Smolin, Phys.Rev.Lett., **72**, 446, (1994)

[12] C.Rovelli, L.Smolin. Discreteness of area and volume in quantum gravity, gr-qc/9411005.

[13] John C. Baez, Spin Networks in Nonperturbative Quantum Gravity, To appear in the proceedings of the AMS Short Course on Knots and Physics, San Francisco, Jan. 2-3, 1995, gr-qc/9501016.

[14] A.Ashtekar, J.Lewandowski, D.Marolf, J.Mourão, T.Thiemann, Quantization of diffeomorphism invariant theories of connections with local degrees of freedom, gr-qc/9504018.

[15] A.Ashtekar, J.Lewandowski, D.Marolf, J.Mourão, T.Thiemann, Coherent State Transforms for Spaces of Connections, gr-qc/9412014.

[16] K.Krasnov, Volume operator in quantum Einsein-Maxwell theory, in preparation.