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K3–Fibrations and Heterotic-Type II String Duality

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Abstract

We analyze the map between heterotic and type II N=2 supersymmetric string theories for certain two and three moduli examples found by Kachru and Vafa. The appearance of elliptic j-functions can be traced back to specializations of the Picard-Fuchs equations to systems for K_3 surfaces. For the three-moduli example we write the mirror maps and Yukawa couplings in the weak coupling limit in terms of j-functions; the expressions agree with those obtained in perturbative calculations in the heterotic string in an impressive way. We also discuss symmetries of the world-sheet instanton numbers in the type II theory, and interpret them in terms of S–duality of the non-perturbative heterotic string.

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1. Introduction

In a very interesting recent paper [1], Kachru and Vafa provided concrete evidence of the conjecture that the exact non-perturbative behavior of the heterotic string compactified on $K_3 \times T_2$ is governed by certain Calabi-Yau (CY) manifolds [2], and can effectively be described in terms of type II strings [3].

In particular, there are examples [1,4] of CY's that, when taken as background of type II theories, lead to prepotentials that reproduce certain perturbative corrections of the heterotic theory in the weak string coupling limit (for non-zero coupling, one expects new stringy non-perturbative phenomena [5] to become visible, analogous to rigid N=2 Yang-Mills theory [6]). It is known from explicit heterotic string computations [7,8] that these corrections are given in terms of elliptic j-functions in the T_2 moduli. That precisely such combinations of j-functions really do appear [9] in the moduli spaces of certain CY manifolds, is highly suggestive, and at first rather surprising.

One of the purposes of this letter is to gain insight in the origin of such modular functions in the moduli spaces of certain Calabi-Yau's. We will show that this can be very simply understood in terms of specializations of Picard-Fuchs equations, and more abstractly in terms of CY manifolds being elliptic or K_3 fibrations. This understanding opens up the possibility of a more systematic construction of CY manifolds that describe the exact quantum theory of $N=2$ supersymmetric heterotic strings. In particular, it also allows us to perform further non-trivial checks on the original examples of Kachru and Vafa, by explicitly writing certain "Yukawa couplings"† in terms of j-functions.

We will also briefly investigate the symmetry structure of certain models, linking the symmetries of the CY instanton expansion to the perturbative and non-perturbative T- and S-dualities of the quantum heterotic string. In particular, we find evidence that in some models there is a symmetry of exchanging the heterotic dilaton S with a target space moduli field, T.

† What we mean are the triple derivatives of the prepotential, which have in the present context, strictly speaking, the interpretation of anomalous magnetic moments.

2. Modular properties of certain Calabi-Yau moduli sub-spaces

The appearance of j-functions is the key for making the relationship of heterotic strings compactified on $K_3 \times T_2$ with type II compactifications evident [1]. Many of the examples of "dual" Calabi-Yau threefolds in [10,11] are actually elliptic fibrations over rational surfaces, or K_3 fibrations over rational curves. In this section we show how these properties lead to the crucial modular properties of the mirror map (and Yukawa couplings) in the weak string coupling domain.

In fact, K_3 fibrations are of natural interest for the conjectured duality between heterotic and type II compactifications, because they automatically give rise to prepotentials in the large complex structure limit of the form: $\mathcal{F} = sQ(t) + C(t)$ (where Q is quadratic polynomial and C is a cubic polynomial). That is, s is a good candidate for the heterotic dilaton.

Specifically, consider the model $X_{12}(1,1,2,2,6)_2^{-252}$,$^\diamond$ which is the first of the examples discussed in [1]. The defining polynomial is

$$p(x) \;=\; {x_1}^{12} + {x_2}^{12} + {x_3}^{6} + {x_4}^{6} + {x_5}^{2} - 12\psi\, x_1x_2x_3x_4x_5 - 2\phi\, {x_1}^{6}{x_2}^{6} \;, \qquad (1)$$

and the weak coupling limit was identified in [1] with $\overline{y} = \frac{1}{\phi^2} \to 0$ and $\overline{x} = -\frac{2\phi}{(12\psi^2)^3}$ finite. In terms of these variables, the Picard-Fuchs operators look ($\theta_x \equiv x\partial_x$, etc.):

$$\begin{aligned} \mathcal{D}_1 &= \theta_x^2\,(\theta_x - 2\,\theta_y) - 8\,x\,(6\,\theta_x + 5)(6\,\theta_x + 3)(6\,\theta_x + 1) \\ \mathcal{D}_2 &= \theta_y^2 - y\,(2\,\theta_y - \theta_x + 1)(2\,\theta_y - \theta_x)\;. \end{aligned} \qquad (2)$$

One way of understanding why a modular function appears in the $y \to 0$ limit is via the following two steps. First, the surface (1) is a K_3 fibration [9] in that there is a linear system $|L|$ generated by the polynomial of degree one, whose divisors are described after the substitution $x_1 = \lambda x_2$ and the single-valued variable change $\widetilde{x}_1 = x_1^2$ as following family of degree 12 K_3 hypersurfaces in $\mathbb{P}^{(1,1,1,3)}$:

$$\mathcal{K}\;:\quad (1 + \lambda^{12} - 2\phi\lambda^6)\widetilde{x}_1^6 + x_3^6 + x_4^6 + x_5^2 - 12\psi\lambda\widetilde{x}_1x_3x_4x_5 = 0 \qquad (3)$$

As divisors in $|L|$ are disjoint, $|L|\cdot|L| = 0$ holds and thus the cubic intersection form has indeed the desired property. Moreover, taking the above limit $\phi \to \infty$, $\psi \sim \phi^{1/6}\widetilde{\psi}$

$\diamond$ We use the notation of [10], e.g. $X_{d_1,\ldots,d_r}(w_1,\ldots,w_n)^{\chi}_{h_{1,1}}$ is a complete intersection (hypersurface) of multi-degree $d_1,\ldots,d_r$ in weighted projective space $\mathbb{P}^{n-1}(w_1,\ldots,w_n)$ (if all $w_i = 1$ we omit them) with Euler number χ and Betti number $h_{1,1}$.

and $\lambda \sim \phi^{-1/6}$ all terms in (3) stay finite, and $x = -\frac{2}{(12\widetilde{\psi}^2)^3}$ can be identified as the canonical one-parameter deformation of $\mathcal{K}$.

Second, there is strong evidence that one-modulus deformations of K_3 surfaces are intrinsically related to modular functions [12]. That is, it was observed in [13] that the W_3-invariant of any single-modulus Picard-Fuchs operator of K_3 vanishes, and since W_2 transforms under coordinate changes $z \to \zeta(z)$ as $W_2 \to W_2 + \{\zeta, z\}$, it is possible to rewrite the PF operator, after gauging away W_2, as $\mathcal{D} = \partial_t^3$. In order to implement this gauging, one needs to solve a Schwarzian differential equation of the form

$$\{x, t_x\} \;=\; 2Q(x)\,(\partial_{t_x} x(t_x))^2 \tag{4}$$

for some $Q(x)$. Its solution $t_x(x)$ is given by a triangle function $s(x)$, whose inverse yields a modular function that is automatically associated with some discrete subgroup of $PSL(2,\mathbb{R})$. (Equivalently, the mirror map $x(q_x)$, where $q_x \equiv e^{2\pi i t_x}$, is given by the ratio of two independent solutions of the associated PF-system.)

It was observed in [12] that in all examples investigated so far this subgroup is given by a subgroup of the modular group $SL(2,\mathbb{Z})$ (possibly together with some extra Atkin-Lehner involutions), and the authors conjectured this to be true for general one-modulus deformations of K_3 arising from orbifold constructions.[‡] In the present example, $Q(x) = \frac{1-1968x+2654208x^2}{4x^2(1728x-1)^2}$, $t_x = s(\frac{1}{2}, \frac{1}{3}, 0; j(q_x))$, and this leads indeed to $x = 1/j(q_x)$ (this feature of the mirror map for vanishing y was noticed first in [9]).

The occurrence of this kind of specialization to K_3 surfaces, with similar modular properties, is actually ubiquitous in the class of complete intersection (hypersurface) CY spaces. Consider, for example, the families

$$\begin{aligned} A:&\ X_8(1,1,2,2,2)_2^{-86} \\ B:&\ X_{6,4}(1,1,2,2,2,2)_2^{-132} \\ C:&\ X_{4,4,4}(1,1,2,2,2,2,2)_2^{-112} \end{aligned} \tag{5}$$

and their associated PF systems, whose relevant parts are

$$\begin{aligned} A:&\ \mathcal{D}_1 = \theta_x^2(2\theta_y - \theta_x) + 4x(4\theta_x+3)(4\theta_x+2)(4\theta_x+1) \\ B:&\ \mathcal{D}_1 = \theta_x^2(2\theta_y - \theta_x) + 6x(2\theta_x+1)(3\theta_x+2)(3\theta_x+1) \\ C:&\ \mathcal{D}_1 = \theta_x^2(2\theta_y - \theta_x) + 8x(2\theta_x+1)^3\ . \end{aligned} \tag{6}$$

‡ More precisely, they conjectured the mirror maps to be given by Thompson series, which have an intrinsic relationship to modular functions and to the representation theory of the Convay-Norton groups.

Together with the first example (1), these examples represent selected one-modulus K_3 fibrations, and the PF operators (2),(6) effectively reduce under $y \to 0$ to the following list of K_3 operators [12]: ♮

	K_3family	PF operator	mod. group
$\mathcal{K}$	$X_6(1,1,1,3)$	$\theta^3 - 8x(6\theta+5)(6\theta+3)(6\theta+1)$	$\Gamma_0(1) \equiv \Gamma$
$\mathcal{K}_A$	X_4	$\theta^3 - 4x(4\theta+3)(4\theta+2)(4\theta+1)$	$\Gamma_0(2)_+$
$\mathcal{K}_B$	$X_{3,2}$	$\theta^3 - 6x(2\theta+1)(3\theta+2)(3\theta+1)$	$\Gamma_0(3)_+$
$\mathcal{K}_C$	$X_{2,2,2}$	$\theta^3 - 8x(2\theta+1)^3$	$\Gamma_0(4)_+$

Model A was briefly discussed in [1], where it was conjectured that the relevant modular group should be given by an extension of some $\Gamma_0(2^k)$; from the table we can infer that this is indeed true, with $k=1$. The commensurability relations of the K_3 mirror maps $x(q_x)$ with the j-function were explicitly worked out in [12]:

$$\begin{aligned} \mathcal{K}:\ P(j,x) &= 1 - jx = 0\,, \\ \mathcal{K}_A:\ P(j,x) &= 1 + 432x - jx + 62208x^2 + 207jx^2 + 2985984x^3 \\ &\quad - 3456jx^3 + j^2x^3 = 0\ , \qquad \text{etc.} \end{aligned}$$

For the first model we immediately recover $x = 1/j(q_x)$. Similarly, the mirror maps for the models A,B,C, when written in the form $1/x(q_x) - c$ (with $c = 104, 42, 24$), specialize to the Hauptmodul of $\Gamma_0(N)$ for $N = 2,3,4$, while $y(q_x=0,q_y) = \frac{q_y}{(1+q_y)^2}$. They are given by certain Thompson series [14], which can be written in terms of modular functions as follows:

$$\begin{aligned} A:\ x(q_x,q_y=0) &= \frac{16(\eta(q_x)\eta(q_x^2))^8}{(\vartheta_3^4+\vartheta_0^4)^4} \\ B:\ x(q_x,q_y=0) &= \left(\frac{\eta^{12}(q_x)}{\eta^{12}({q_x}^3)} + 729\frac{\eta^{12}({q_x}^3)}{\eta^{12}(q_x)} + 54\right)^{-1} \\ C:\ x(q_x,q_y=0) &= \frac{\eta^{24}(q_x)\eta^{24}({q_x}^4)}{\eta^{48}({q_x}^2)} \end{aligned}$$

These expressions might be useful for further checks on the conjectured heterotic-type II string duality.

♮ $\Gamma_0(N)_+$ denotes a group that in general includes certain Atkin-Lehner involutions besides $\Gamma_0(N)$; see [14] for details.

A simple generalization of (1) and example A would be to take K_3 hypersurfaces in weighted $\mathbb{P}^3$ of the form $X_d(1,w_3,w_4,w_5)$, and consider as CY the K_3 fibration $X_{2d}(1,1,2w_3,2w_4,2w_5)^*$. From the examples we have checked, it appears that the discriminant naturally is of the form $\triangle = \triangle(K_3)^2 + \ \dots$, where the dots denote terms which vanish in an appropriate (weak coupling) limit. There are 95 transversal families of such K_3 surfaces [15]; 31 of them with $w_1 = 1$ give rise to transversal Calabi-Yau configurations and are listed in the Appendix. It would be very interesting to investigate whether these CY manifolds describe non-perturbative quantum heterotic strings.

In fact, the surface $X_{24}(1,1,2,8,12)_3^{-480}$, which was studied too in [1], is precisely of this type. The defining polynomial is given by

$$p = x_1^2 + x_2^3 + x_3^{12} + x_4^{24} + x_5^{24} - 12\psi_0 x_1 x_2 x_3 x_4 x_5 - 2\psi_1 (x_3 x_4 x_5)^6 - \psi_2 (x_4 x_5)^{12}\ , \quad (7)$$

and variables that are appropriate near the point of maximal unipotent monodromy in the large complex structure limit are: $x = -\frac{2\psi_1}{1728^2\psi_0^6}$, $y = \frac{1}{\psi_2^2}$, $z = -\frac{\psi_2}{4\psi_1^2}$. For $y \to 0$, the PF-system

$$\begin{aligned} \mathcal{D}_1 &= \theta_x\,(\theta_x - 2\,\theta_z) - 12\,x\,(6\,\theta_x + 5)\,(6\,\theta_x + 1) \\ \mathcal{D}_2 &= \theta_y^2 - y\,(2\,\theta_y - \theta_z + 1)\,(2\,\theta_y - \theta_z) \\ \mathcal{D}_3 &= \theta_z\,(\theta_z - 2\theta_y) - z(2\,\theta_z - \theta_x + 1)\,(2\,\theta_z - \theta_x) \end{aligned} \quad (8)$$

degenerates to the two moduli system of the generic fiber, given by a K_3 family of type $X_{12}(1,1,4,6)$. Actually, this K_3 is in itself a elliptic fibration over $\mathbb{P}^1$ with generic fiber $X_6(1,2,3)$, as can be seen in an analogous way.

It is quite clear that elliptic fibrations lead very directly to modular functions. Specifically, we present below a table of elliptic curves $\mathcal{E}$, noticing that the present example corresponds to the last entry.

$*$ For K_3 fibrations one can always choose a basis s.t. $\mathcal{F} = sQ(t) + C(t)$ and $\int c_2 \wedge s = 24$. Conversely given this topological data one has still to find the suitable projection map. For hypersurfaces in toric varieties there seem to be the combinatorical condition, that the K_3 polyhedron is embedded in the Calabi-Yau polyhedron.

	elliptic family	PF operator	$j(x)$	mod. subgroup
$\mathcal{E}_1$	$X_3(1,1,1)$	$\theta^2 - 3x(3\theta+2)(3\theta+1)$	$\frac{(1+216x)^3}{x(1-27x)^3}$	$\Gamma(3)$
$\mathcal{E}_2$	$X_4(1,1,2)$	$\theta^2 - 4x(4\theta+3)(4\theta+1)$	$\frac{(1+192x)^3}{x(1-64x)^2}$	$\Gamma(2)$
$\mathcal{E}_3$	$X_6(1,2,3)$	$\theta^2 - 12x(6\theta+5)(6\theta+1)$	$\frac{1}{x(1-432x)}$	Γ^*

Table 1: *Families of elliptic curves $\mathcal{E}$, their Picard-Fuchs operators, commensurability relations of the mirror-map $x(q)$ as defined in [10,11] with the j-function, and the relevant modular subgroup of $SL(2,\mathbb{Z})$. In the first two cases the Hauptmodul of $\Gamma(3)$ and $\Gamma(2)$ is related to the mirror map by removing from $1/x$ the constant term. In the third case the commensurability polynomial is of genus one, meaning that one needs two generators to define the function field on $\mathcal{E}_3$.*

The PF operator of $\mathcal{E}_3$ obviously represents the $y,z \to 0$ limit of (8). From the commensurability relation of the mirror map of $\mathcal{E}_3$ with $j(q_x)$ we can immediately see that the mirror map of (7) in the limit $z,y \to 0$ is given by

$$x(q_x) \;=\; \frac{2}{j(q_x) + \sqrt{j(q_x)(j(q_x)-1728)}}\,. \tag{9}$$

In addition, it follows from analyzing the corresponding degeneration limits of the PF-system that the mirror-maps $y(q_y)$ (and $z(q_z)$) specialize to rational functions on the boundary of the moduli space, $x = z = 0$ ($x = y = 0$, resp.): $y(q_y) = \frac{q_y}{(1+q_y)^2}$, $z(q_z) = \frac{q_z}{(1+q_z)^2}$. We will use the solution (9) below to provide further evidence in favor of the conjectured heterotic-type II string duality.

Moreover, we can infer from Table 1 that the mirror map $x(q_x)$ (and $y(q_y)$) of the hypersurface of bidegree $(3,3)$ in $\mathbb{P}^1 \times \mathbb{P}^1$, denoted $X_{3|3}(1,1,1|1,1,1)^{-162}_{2,83}$, of [10] for $y = 0$ ($x = 0$) is related to the Hauptmodul of $\Gamma(3)$. Finally, we find that the mirror map† $X_{12}(1,1,1,3,6)^{-324}_{3(1),165}$ is related to the Hauptmodul of $\Gamma(2)$ at the boundary $y = z = 0$.

† The problem of including the twisted sector in the analysis of the PF-system was recently solved in [16].

3. The three-moduli example $X_{24}(1,1,2,8,12)$ revisited

We have seen how the appearance of elliptic functions in the mirror maps of Calabi–Yau compactifications can naturally be understood in terms of their special structure as fibrations, at least in the case of one modulus, where we could use the results known in the mathematical literature. Unfortunately, an analogous treatment for more than one modulus does not seem to exist. We will now show that the mirror map and Yukawa couplings of the three-moduli example of [1], $X_{24}(1,1,2,8,12)$, can nevertheless be written in terms of elliptic functions in the weak coupling limit. This will provide a further impressive non–trivial check on the conjecture of equivalence of the corresponding $N=2$ heterotic and type II strings.

Following Kachru and Vafa [1], we identify $y \sim e^{-\alpha S} \to 0$ with the weak coupling limit of the heterotic string theory. This identification is motivated by the fact that the discriminant locus of the mirror CY becomes a perfect square, representing the splitting [6] of the classical $SU(2)$ singularity into two branches in the quantum theory. Specifically, the discriminant is

$$\triangle = [\ (1-\overline{z})^2 - \overline{y}\ \overline{z}^2\ ] \times [\ ((1-\overline{x})^2 - \overline{x}^2\overline{z})^2 - \overline{y}\ \overline{x}^4\ ] \times [\ 1-\overline{y}\ ] \equiv \triangle_1 \times \triangle_2 \times \triangle_3\ , \quad (10)$$

where $\overline{x} = 432x, \overline{y} = 4y, \overline{z} = 4z$. For $\overline{y} \to 0$, $\triangle$ degenerates into quadratic factors that have the following significance with respect to gauge symmetry enhancements in the heterotic theory:

$$\begin{aligned} \triangle_1 = 0 : T = U \qquad & SU(2) \\ \triangle_2 = 0 : T = U = i \quad & SU(2) \times SU(2) \\ T = U = \rho \quad & SU(3)\ . \end{aligned}$$

The discriminant factor $\triangle_3$ has the interpretation of a strong-coupling singularity in the heterotic theory. The conjectured duality between the type II theory and the heterotic theory implies that the perturbative $SO(2,2,\mathbb{Z})$ symmetry of the latter theory should be encoded in the former one. Indeed it turns out that in the limit $\overline{y} \to 0$ the mirror map for $\overline{x}$ and $\overline{z}$ can be written in terms of elliptic j-functions. More precisely, using (9) and the fact that T and U should enter symmetrically, we find:

$$\begin{aligned} \overline{x} &= q_1 + \sum_{m+n>1} a_{mn} q_1^m\ q_3^n = \frac{\mu}{2} \frac{j(T)+j(U)-\mu}{j(T)j(U) + \sqrt{j(T)(j(T)-\mu)}\sqrt{j(U)(j(U)-\mu)}} \\ \overline{z} &= q_3 + \sum_{m+n>1} b_{mn} q_1^m\ q_3^n = \frac{(j(T)j(U) + \sqrt{j(T)(j(T)-\mu)}\sqrt{j(U)(j(U)-\mu)})^2}{j(T)j(U)(j(T)+j(U)-\mu)^2}, \end{aligned} \quad (11)$$

where $\mu \equiv j(i) = 1728$ and where we have defined $q_1 \equiv q_T$, $q_3 \equiv q_U/q_T$ ◇ (we also have verified directly that this corresponds to solutions of the PF equations).

Although on the first glance complicated, eqs. (11) can be recognized as appropriate generalization of the various limits. That is, for the $SU(2)$ enhanced line, $T = U$, we find

$$\overline{x} = \frac{\mu}{2j} , \qquad \overline{z} = 1 ,$$

and consequently $\triangle_1 = 0$. For the points of further enhancement we get

$$T = U = i \ : \ \overline{x} = \frac{1}{2} , \ \text{ and } \qquad T = U = \rho \ : \ \overline{x} = \infty ,$$

such that in addition $\triangle_2 = 0$. Moreover, in the limit $U \to i\infty$ we recover (9):

$$\overline{z} = 0 , \qquad \overline{x} = \frac{\mu}{2} \frac{1}{j(T) + \sqrt{j(T)(j(T) - \mu)}} \tag{12}$$

(and similarly for $T \to i\infty$). This equation also reflects the invariance of the defining polynomial (1) under a subgroup of general automorphisms: $x_i \to x_i$, $i = 3,4,5$, $x_2 \to x_2 + a(x_3x_4x_5)^2$, $x_1 \to x_1 + b(x_3x_4x_5)^3 + cx_2x_3x_4x_5$ that acts non-trivially on the moduli space as follows:

$$\psi_0 \to i\psi_0 \ , \psi_1 \to \psi_1 + 2\mu\psi_0^6, \psi_2 \to \psi_2 \tag{13}$$

and hence:

$$\chi_1 \to \frac{\chi_1}{\chi_1 - 1} \quad (\mathbb{Z}_2)$$

on $\chi_1 \equiv 1/\overline{x}$. Identifying the invariant expression with $j(T)$ reproduces (12). Note also that the symmetry (13) exchanges the factors of the discriminant (10):

$$\triangle_1 \to \overline{x}^4 \triangle_2 \ , \quad \triangle_2 \to \frac{1}{(1-\overline{x})^4} \triangle_2 \ .$$

The identifications (11) provide a further, highly non–trivial check on the conjectured string duality. For this purpose we need the translation of the Yukawa couplings (that were determined in [10]) in terms of S, T and U, as well as the expression for the mirror map of the third Calabi–Yau modulus, $\overline{y}$. It has the general form

$$\overline{y} = q_2 \sum_{m+n \geq 1} q_1^m q_3^n + \mathcal{O}(q_2^2) \equiv q_s f_y(q_1, q_3) + \mathcal{O}(q_2^2) \ , \tag{14}$$

◇ For the definition of the special coordinates $t_i = 1/(2\pi i) \ln q_i$, see ref. [10].

with $q_s = e^{-8\pi^2 S}$, where S will be identified with the (tree level) dilaton of the heterotic string. Then, using

$$
\begin{aligned}
&\partial_T \overline{x} = -j_T(T) \frac{\sqrt{j(U)(j(U)-\mu)}}{\sqrt{j(T)(j(T)-\mu)}} \frac{1}{j(T)+j(U)-\mu} \overline{x}(1-\overline{x}) \\
&\partial_T \overline{y} = \overline{y}\ \partial_T \ln f_y(q_1, q_3) \\
&\partial_T \overline{z} = -j_T(T) \overline{z}\ \times \\
&\frac{\sqrt{j(T)(j(T)-\mu)}(j(T)+j(U)-\mu) - 2j(T)\sqrt{j(U)(j(U)-\mu)}(1-\overline{x})}{j(T)\sqrt{j(T)(j(T)-\mu)}(j(T)+j(U)-\mu)} \\
&\partial_S \overline{x} = \partial_S \overline{z} = 0\ , \quad \partial_S \overline{y} = -8\pi^2\ \overline{y}\ ,
\end{aligned}
$$

($j_T(T) \equiv \partial_T j(T)$) and the analogous relations obtained by exchanging T and U, the CY Yukawa couplings given in [10] when written in terms of S, T, U read:

$$
\begin{aligned}
\widetilde{K}_{SSS} &= \widetilde{K}_{SST} = \widetilde{K}_{SSU} = \widetilde{K}_{STT} = \widetilde{K}_{SUU} = 0 \\
\widetilde{K}_{STU} &= 1 \\
\widetilde{K}_{TTT} &= \frac{i}{2\pi} \frac{E_4(T)E_4(U)E_6(U)(E_4(T)^3 - E_6(T)^2)}{E_4(U)^3 E_6(T)^2 - E_4(T)^3 E_6(U)^2} \\
&= -\frac{1}{4\pi^2} \frac{j_T(T)^2 j(U)(j(U)-\mu)}{(j(T)-j(U)) j(T)(j(T)-\mu) j_U(U)} \\
\widetilde{K}_{TTU} &= -\frac{i}{2\pi} \frac{E_4(T)^2 E_6(T)(E_4(U)^3 - E_6(U)^2)}{E_4(U)^3 E_6(T)^2 - E_4(T)^3 E_6(U)^2} + \frac{i}{2\pi} \partial_T \ln f_y(q_1, q_3) \\
&= \frac{1}{4\pi^2} \frac{j_T(T)}{j(T)-j(U)} + \frac{i}{2\pi} \partial_T \ln f_y(q_1, q_3)\ .
\end{aligned}
\tag{15}
$$

Here $E_{4,6}$ are the normalized Eisenstein series, and $\widetilde{K}_{ABC} = 1/(2\pi^2\omega_0)^2 K_{ABC}$, where $\omega_0 = E_4(T)^{1/4} E_4(U)^{1/4}$ is the fundamental period and the transition from K_{ABC} to $\widetilde{K}_{ABC}$ corresponds to going to the canonical gauge. The expressions (15) must be compared with the results from perturbative string calculations performed in the heterotic theory [7,8]. We find indeed perfect agreement ! Note also that corrections to K_{TTT} arising from the T-dependence of $\overline{y}$ cancel in a very non–trivial way. This is just as expected: for the coupling K_{TTT} we do know the exact expression from the calculations in the heterotic theory, while the corrections to K_{TTU} proportional to $f_y(q_1, q_3)$ are not known (this coupling has been determined only to the leading order in $T - U$).

As was pointed out to us [17], the mirror map for $\overline{y}$ provides a further independent consistency check. In fact, since $\overline{y}$ is invariant under the CY monodromy group that contains the modular groups for T and U, its logarithm should to be identified with the modular invariant dilaton defined in [7]:

$$S_{inv} = S - \frac{1}{2}\partial_T \partial_U h^{(1)}(T,U) - \frac{1}{8\pi^2}\ln(j(T)-j(U)) + \text{const.}\ . \tag{16}$$

Here $h^{(1)}(T,U)$ is the moduli dependent one-loop contribution to the holomorphic prepotential. Since $h^{(1)}(T,U)$ transforms as modular form of weight $(-2,-2)$ under (T,U) duality transformations up to terms quadratic in T and U [7,8], $\partial_T^3\partial_U^3 h^{(1)}(T,U)$ is a modular form of weight $(4,4)$. Consistency of (16) and (14) then implies

$$\partial_T^2\partial_U^2 \ln f_y(q_1,q_3) = M(q_1,q_3) + \alpha\, \partial_T^2\partial_U^2 \ln(j(T)-j(U))\ , \tag{17}$$

where M is a modular form of weight $(4,4)$ that is regular in the fundamental domain, except for a fourth order pole at $T=U$. We indeed find that (17) is fulfilled, with $\alpha = 1$ and $M = 4\pi^2\partial_U^3 \widetilde{K}_{TTT}$.

4. Duality symmetries

Having successfully met further non-trivial checks on the heterotic–type II duality for the model $X_{24}(1,1,2,8,12)$, we now address some questions beyond leading perturbation theory. Adopting the duality hypothesis, we know that important information about the non–perturbative S- and T-duality of the heterotic string must be encoded in the symmetries of the CY moduli space. Its monodromy properties have been explicitly worked out [9] for the two-moduli example $X_8(1,1,2,2,2)$, and can be straightforwardly determined for other examples as well, using known results for the periods. Another type of symmetries arise from the defining polynomial, such as the automorphisms (13). Alternatively, we may directly look for symmetries in the instanton expansions, which reflect some of the monodromy properties.

More specifically, analyzing various instanton expansions, we are lead to consider the following symmetries acting on q_i :

$a)$	$X_{24}(1,1,2,8,12)$	$q_1 \to q_1 q_3$	$q_3 \to 1/q_3$	$(q_2 \equiv 0)$	$\overline{z}$	$T \leftrightarrow U$
$b)$	"	$q_3 \to q_2 q_3$	$q_2 \to 1/q_2$		$\overline{y}$	?
$c)$	$X_8(1,1,2,2,2)$	$q_1 \to q_1 q_2$	$q_2 \to 1/q_2$		$\overline{y}$	?
	$X_{12}(1,1,2,2,6)$	$q_1 \to q_1 q_2$	$q_2 \to 1/q_2$		$\overline{y}$	?

We indicated in the last two colomns the relevant modulus and the interpretation of the symmetry in terms of the dual heterotic string; evidence for what the question marks should stand for will be presented below.

Symmetry a) is easily identified as the mirror symmetry of the heterotic string that exchanges the Kähler and complex structure moduli of the compactification torus, T_2. We did not found a simple generalization to $q_2 \neq 0$, possibly indicating a non-perturbative breaking of this symmetry. Furthermore $\overline{z}|_{q_3=1} \neq 1$ for $q_2 \neq 0$, that there is a shift of the singularity $T = U$. However a clarification of these points is of course strongly connected to the precise relation between heterotic and CY moduli at all orders in q_2, an information which is beyond the present knowledge.

A consequence of the inversion symmetry $q_3 \to 1/q_3$ is that the mirror map has the property that powers of q_3 are accompagnied by a sufficient number of powers of q_1. Translated into the heterotic string language, this means that no negative powers of q_T, q_U appear in the expansions of physical quantities such as the Yukawa couplings.[†] In fact, this property can be traced back to the special form of the basis vectors of the Mori cone. Furthermore, note that the only association of t_3 with the moduli T, $U \in \mathbb{H}^+$ of the heterotic string that is consistent with the symmetry $q_3 \to 1/q_3$, is given by: $q_3 = q_T/q_U$. Finally, remember from section 2 that the mirror map for $\overline{z}$ reduces to $q_3/(1+q_3)^2$ in the limit $q_1, q_2 \to 0$.

A crucial observation is that these features are shared also by the other two symmetries, b) and c). However, these transformations are quite different from the point of view of the heterotic string, in that they involve q_2 that is related to the dilaton S.

If we allow q_2 to have additional dependence on T, and define for convenience $q_2 = q'_S q_T^\alpha$, the symmetry c) translates to $T \to (1+\alpha)T + S'$, $S' \to -(1+\alpha)S' - \alpha(2+\alpha)T$. Realizing the shift symmetries $t \to t+1$ for $t = t_1, t_2, S', T$, requires then α to be an integer. Imposing in addition the reasonable condition that the positivity of the imaginary parts of S' and T (which are $\sim g^{-2}$ and $\sim R^2$, respectively) is preserved, enforces $\alpha = -1$, and this implies: $q_2 = q_S/q_T$. That is, the natural interpretation of the symmetry c) is a symmetry under exchange of S and T (and similarly of S and U for b)) ! Note also that a linear change of variables, $S' \to S' + \alpha T$, does not alter the expressions for the Yukawa couplings, apart from a constant shift.

† More precisely, the mirror map for $\overline{z}$ has an additional overall factor of q_3 that is compatible with (15).

Finally, we point out a remarkable property of the mirror maps $\overline{z}, \overline{y}$ ($\overline{y}$) in the models X_{24} (X_8, X_{12}). That is,

$$\overline{z}_{X_{24}}|_{\substack{q_2=0\\q_3=1}} = 1 \ , \quad \overline{y}_{X_{24}}|_{q_2=1} = 1 \ , \quad \overline{y}_{X_8,X_{12}}|_{q_2=1} = 1 \ ,$$

are independent of q_1 (and of q_3 for the second equation). Comparing with the discriminants of these models, we see that the singularities $\overline{z} = 1$ ($cf., T = U$), $\overline{y} = 1$ (cf., strong coupling) are precisely at the fixed points of the symmetries a)–c). This means that the singularities correspond to $t_i = 0$ of the relevant moduli. Keeping in mind the relation $t_i = (\sum_i A_i^j \omega_j)/\omega_0$, where ω_i are the (generically independent) components of the CY period vector, we see that this implies the vanishing of certain linear combinations of the periods. Hence, according to the $N = 2$ mass formula [18], Bogomolnyi states with the appropriate quantum numbers should become massless at these points in the moduli space (provided they exist). The fact that these singularities are not of the simple conifold type indicates that the ideas described in [5] might apply in a more general context.

5. Conclusions

We have amassed further, and we think convincing, evidence in favor of the conjectured duality between heterotic strings compactified on $K_3 \times T_2$, and type II strings compactified on certain Calabi-Yau manifolds. We have also gained insight in the modular properties of certain CY theories in the weak coupling limit, and believe that by focusing on K_3 fibrations, many more examples can be systematically studied. We also analyzed the symmetry structure of some models, linking the symmetries of the CY instanton expansion to the perturbative and non-perturbative T- and S-dualities of the quantum heterotic string. We hope that the methods described in this paper can be developed further, allowing to make new, concrete and truly non-perturbative predictions for $N=2$ supersymmetric string theory.

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Appendix: Calabi-Yau manifolds that are K_3 fibrations

	$h_{1,1}$	$h_{2,1}$	χ	deg.	weights
1	2	86	-168	8	$(1,1,2,2,2)$
2	2	128	-252	12	$(1,1,2,2,6)$
3	3	99	-192	10	$(1,1,2,2,4)$
4	3	243	-480	24	$(1,1,2,8,12)$
5	4	148	-288	16	$(1,1,2,4,8)$
6	4	190	-372	20	$(1,1,2,6,10)$
7	5	101	-192	12	$(1,1,2,4,4)$
8	5	121	-232	14	$(1,1,2,4,6)$
9	5	161	-312	18	$(1,1,2,6,8)$
10	7	143	-272	20	$(1,1,4,4,10)$
11	7	271	-528	36	$(1,1,4,12,18)$
12	8	104	-192	16	$(1,1,4,4,6)$
13	8	164	-312	24	$(1,1,4,6,12)$
14	8	194	-372	28	$(1,1,4,8,14)$
15	9	111	-204	18	$(1,1,4,6,6)$
16	9	125	-232	20	$(1,1,4,6,8)$
17	9	153	-288	24	$(1,1,4,8,10)$
18	9	321	-624	48	$(1,1,6,16,24)$
19	10	194	-368	32	$(1,1,6,8,16)$
20	10	220	-420	36	$(1,1,6,10,18)$
21	10	376	-732	60	$(1,1,8,20,30)$
22	11	131	-240	24	$(1,1,6,8,8)$
23	11	143	-264	26	$(1,1,6,8,10)$
24	11	167	-312	30	$(1,1,6,10,12)$
25	11	227	-432	40	$(1,1,8,10,20)$
26	11	251	-480	44	$(1,1,8,12,22)$
27	11	485	-960	84	$(1,1,12,28,42)$
28	12	164	-304	32	$(1,1,8,10,12)$
29	12	186	-348	36	$(1,1,8,12,14)$
30	12	318	-612	60	$(1,1,12,16,30)$
31	13	229	-432	48	$(1,1,12,16,18)$

Table A.1: *Simple hypersurfaces in weighted* $\mathbb{P}^4$ *which are* K_3 *fibrations.*

	$h_{1,1}$	$h_{2,1}$	d_1	d_2	weights
1	2	68	6	4	$(1,1,2,2,2,2)$
2	3	69	6	6	$(1,1,2,2,2,4)$
3	4	84	8	8	$(1,1,2,2,4,6)$
4	4	76	8	6	$(1,1,2,2,4,4)$
5	6	138	16	12	$(1,1,2,6,8,10)$
6	6	102	12	10	$(1,1,2,4,6,8)$
7	6	98	12	8	$(1,1,2,4,6,6)$
8	6	82	10	8	$(1,1,2,4,4,6)$
9	6	70	8	8	$(1,1,2,4,4,4)$
10	8	76	12	8	$(1,1,4,4,4,6)$
11	9	81	12	12	$(1,1,4,4,6,8)$
12	9	75	12	10	$(1,1,4,4,6,6)$
13	10	122	20	16	$(1,1,4,8,10,12)$
14	10	98	16	14	$(1,1,4,6,8,10)$
15	10	94	16	12	$(1,1,4,6,8,8)$
16	10	84	14	12	$(1,1,4,6,6,8)$
17	10	76	12	12	$(1,1,4,6,6,6)$
18	12	128	24	20	$(1,1,6,10,12,14)$
19	12	108	20	18	$(1,1,6,8,10,12)$
20	12	104	20	16	$(1,1,6,8,10,10)$
21	12	96	18	16	$(1,1,6,8,8,10)$
22	13	139	28	24	$(1,1,8,12,14,16)$
23	13	121	24	22	$(1,1,8,10,12,14)$
24	13	117	24	20	$(1,1,8,10,12,12)$
25	14	166	36	32	$(1,1,12,16,18,20)$

Table A.2: *Simple complete intersections Calabi-Yau spaces in weighted* $\mathbb{P}^5$ *which are* K_3 *fibrations.*

References

[1] S. Kachru and C. Vafa, *Exact Results for $N = 2$ Compactifications of the Heterotic String*, preprint HUTP-95/A016, `hep-th/9505105`.

[2] A. Ceresole, R. D'Auria and S. Ferrara, Phys. Lett. 339 (1994) 71, `hep-th/9408036`; A. Ceresole, R. D'Auria, S. Ferrara and A. Van Proeyen, *On Electromagnetic Duality in Locally Supersymmetric N=2 Yang–Mills Theory*, preprint CERN-TH.7510/94, POLFIS-TH. 08/94, UCLA 94/TEP/45, KUL-TF-94/44, `hep-th/9412200`; *Duality Transformations in Supersymmetric Yang-Mills Theories coupled to Supergravity*, preprint CERN-TH 7547/94, POLFIS-TH. 01/95, UCLA 94/TEP/45, KUL-TF-95/4, `hep-th/9502072`; M. Billo', A. Ceresole, R. D'Auria, S. Ferrara, P. Fre', T. Regge, P. Soriani and A. Van Proeyen, *A Search for Non-Perturbative Dualities of Local N=2 Yang–Mills Theories from Calabi–Yau Threefolds*, preprint SISSA 64/95/EP, POLFIS-TH 07/95, CERN-TH 95/140, IFUM 508/FT, KUL-TF-95/18, UCLA/95/TEP/19, `hep-th/9506075`.

[3] A. Klemm, W. Lerche, S. Theisen and S. Yankielowicz, Phys. Lett. 344 (1995) 169, `hep-th/9411048`.

[4] S. Ferrara, J. A. Harvey, A. Strominger and C. Vafa, *Second-Quantized Mirror Symmetry*, preprint EFI-95-26, `hep-th/9505162`.

[5] A. Strominger, *Massless Black Holes and Conifolds in String Theory*, ITP St. Barbara preprint, `hep-th/9504090`.

[6] N. Seiberg and E. Witten, Nucl. Phys. *B426* (1994) 19, `hep-th/9407087`; Nucl. Phys. *B431* (1994) 484, `hep-th/9408099`.

[7] B. de Wit, V. Kaplunovsky, J. Louis and D. Lüst, *Perturbative Couplings of Vector Multiplets in $N = 2$ Heterotic String Vacua*, `hep-th/9504006`.

[8] I. Antoniadis, S. Ferrara, E. Gava, K.S. Narain and T.R. Taylor, *Perturbative Prepotential and Monodromies in N=2 Heterotic Superstring*, `hep-th/9504034`.

[9] P. Candelas, X. de la Ossa, A. Font, S. Katz and D. Morrison, Nucl. Phys. *B416* (1994) 481, `hep-th/9308083`.

[10] S. Hosono, A. Klemm, S. Theisen and S.-T. Yau, Comm. Math. Phys. 167 (1995) 301, `hep-th/9308122`.

[11] S. Hosono, A. Klemm, S. Theisen and S.-T. Yau, Nucl. Phys. B433 (1995) 501, hep-th/9406055.

[12] B. H. Lian and S.-T. Yau, *Arithmetic Properties of the Mirror Map and Quantum Coupling*, preprint hep-th/9411234

[13] W. Lerche, D. Smit and N. Warner, Nucl. Phys. B372 (1992) 87, hep-th/9108013.

[14] J. Conway and S. Norton, Bull. Math. Soc. 11 (1979) 308.

[15] T. Yonemura, Tohuku Math. J. 42 (1990) 351.

[16] P. Berglund, S. Katz and A. Klemm, *Mirror Symmetry and the Moduli Space for Generic Hypersurfaces in Toric Varieties*, CERN-TH-7528/95, IASSNS-HEP-94/106, hep-th/9506091.

[17] S. Theisen and C. Vafa, private communication.

[18] See the third reference in [2]; G. Cardoso, L. Ibañez, D. Lüst and T. Mohaupt, in preparation.