A Generalization of Haldane state-counting procedure and π-deformations of statistics.

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Abstract

We consider the generalization of Haldane's state-counting procedure to describe all possible types of exclusion statistics which are linear in the deformation parameter g. The statistics are parametrized by elements of the symmetric group of the particles in question. For several specific cases we determine the form of the distribution functions which generalizes results obtained by Wu. Using them we analyze the low-temperature behavior and thermodynamic properties of these systems and compare our results with previous studies of the thermodynamics of a gas of g-ons. Various possible physical applications of these constructions are discussed.

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1 Introduction

The importance of the notion of statistics for the formulation of the quantum many-body problem and investigation of its macroscopic properties is very well established. Moreover in the last decade it was realized that there are dynamical models where interparticle forces can be regarded as purely statistical interactions. The most famous of these is a system of particles with Aharonov- Bohm type interactions [1, 2] (anyons in 2+1 dimensions) and models solved by the Thermodynamic Bethe Anzatz [3, 4, 9]. These two examples demonstrate the possibility of constructing intermediate statistics which are neither Bose-Einstein nor Fermi-Dirac statistics.

It is obvious that there are at least two ways to deform statistics to interpolate between Fermi and Bose statistics. The first is to deform the exchange factor which appears when particles are permuted (exchange statistics) and the second is to change the allowed occupation numbers for each quantum state (exclusion statistics). This latter possibility was initially explored by Haldane [5].

Here we will summarize the definition of exclusion statistics for one species of particles, for simplicity. Following Ref.[5] let us consider the system confined to a finite region for which the number K of independent single particle states is finite and extensive, i.e. proportional to the size of the region where the particle resides. Then the statistics are determined by the consequence of adding a second particle, keeping all coordinates of the existing particles and external properties (size etc.) of the system fixed. In general an N-particle wave function with fixed coordinates of $N-1$ particles can be expanded in a basis of wave functions of the Nth particle. It is important that in the presence of $N-1$ particles the number of allowed single particle states $d(N)$ is not in general equal to K but can depend on the number $N-1$. If we impose a state homogeneity condition (i.e. independence of $d(N)$ on the particular choice of state for the $N-1$ particles) and particle homogeneity condition (i.e. independence $\frac{d(N+m)-d(N)}{m}$ on m) then Haldane's exclusion statistical parameter g is defined by

$$g = -\frac{d(N+m)-d(N)}{m} \tag{1}$$

for any choice of N and m. Using the homogeneity properties it is possible use the definition in the form $g = d(1) - d(2)$ from which it immediately follows that such statistical interactions make their first contribution at the level of the second virial coefficient.

Applying (1) to the usual Bose and Fermi ideal gases gives $g = 0$ for the Bose case (i.e. the number $d(N)$ does not depend on N) and $g = 1$ for the Fermi case (i.e. after the inclusion of $N-1$ particles the next particle can occupy only $K-N+1$ states and hence $d(N) = K-N+1$). These straightforward examples make the definition of the statistical deformation (1) very attractive.

However one complication arises: the definition of a fractional dimension for the Hilbert space associated with both single and many particle states. This question is also closely related to the construction of quantum statistical mechanics for a system with such 'g-particles' – state-counting which is needed to calculate the entropy and other thermodynamical quantities of the system. In his original paper [5] Haldane suggested calculating the dimension of the full Hilbert space for N-particle states W using (we will call this expression the Haldane-Wu state-counting procedure)

$$W = \frac{(d(N)+N-1)}{(d(N)-1)!(N)!} \qquad d(N) = K - g(N-1), \tag{2}$$

which was subsequently used by many authors [6, 4, 7] to describe the thermodynamic properties of g-ons. In our previous paper [10] we argued that this state-counting procedure is inconsistent with a step by step application the original definition of Haldane's statistics (1). In that paper we showed that expression (2) is actually one of the two simplest state-counting deformations of the exclusion statistics and so it is perhaps not surprising that this possibility was realized in many models. However the procedure for state-counting is closely connected with the construction of a fractional dimensional Hilbert space and could not be performed self-consistently without it.

In reference [10] we tried to resolve at least some of the difficulties connected with the fractionality of the Hilbert space dimension. Our main suggestion was to treat the notion of Haldane's dimension, and the corresponding statistics, in a probabilistic spirit. Motivated by the experience of dimensional regularization we defined the dimension of a space as a trace of diagonal 'unit operator' where the diagonal matrix elements are not unity in general but are the probabilities to find a system in a given state. These probabilities are then uniquely defined by the statistics of the particles with homogeneity properties or, equivalently, by a statistical interaction.

In this paper we want to consider and classify the generalizations of the Haldane-Wu state-counting procedure. They describe all the possible types of exclusion statistics which depend linearly on the deformation parameter g. As we will show, these statistics (which we term π-statistics) are parametrized by elements of the symmetric group assorted with the particles in question. For several particular cases of deformations we derive the equations for distribution functions which generalize the equation obtained by Wu. These are then used to analyze the low-temperature behavior and thermodynamic properties of the systems and compare our results with the thermodynamics of a gas of g-ons studied earlier.

The paper is organized as follows. In the next section we describe in more details state-counting procedures for bosons, fermions and the Haldane-Wu ideal gas. These are used to motivate the introduction of a general construction of exclusion statistical deformations defined in section 3. In section 4 we briefly summarize the main points of the derivation of the thermodynamics for a gas obeying the Haldane-Wu state-counting procedure. Then in section 5 we investigate the simplest example of π-statistics associated with the identical permutation. More sophisticated permutations will be explored in section 6.

2 State-counting procedure. Simple examples

Let us consider the familiar counting procedure for many-body states for fermions and bosons in some detail. It is well-known that the number of quantum states for N identical fermions occupying a set of single-particle K states is given by the following expression:

$$W_f = \frac{K!}{N!(K-N)!} = K \cdot (K-1) \cdot (K-2) \ldots (K-N+1) \cdot \frac{1}{N!} \tag{3}$$

To understand the procedure of the construction we can consider the factor K as the number of ways to place a first particle in the system, $K-1$ as the corresponding number for a second particle and so on. In this procedure we consider the particles as distinguishable and at the end take into account indistinguishability by the factor $1/N!$.

In a similar manner we interpret the bosonic expression for the number of possible N-particle states, which is

$$W_b = \frac{(K+N-1)!}{N!(K-1)!} = K \cdot (K+1) \cdot (K+2) \ldots (K+N-1) \cdot \frac{1}{N!} \tag{4}$$

as a product of the number of ways to add the i-th particle to the system. It is easy to see that in contrast to the fermionic case, where the addition of each particle reduces the number of accessible states, in the bosonic case the number of states it increases by unity when a particle is added. A picture which provides an explanation of this strange fact appeals to quantum mechanical arguments.

Initially we assume that the particles are distinguishable and so we may associate different creation operators with them. These do not commute in general and this is the main origin of the distinction with particles obeying Boltzmann statistics. For the first particle noncommutativity does not contribute and we may place it in K possible states. To see why the number of states for the second particle increases we presume that the Hilbert space of the system is factorized into a tensor product of the Hilbert spaces for each single particle state. Then we obtain $K-1$ ways to place the particle in the empty states and *two* possibilities for the state occupied by the first particle: due to the noncommutativity of the creation operators for the first (a_1^+) and second (a_2^+) particles the state vectors $a_1^+ a_2^+ |0>$ and $a_2^+ a_1^+ |0>$ are in principle different (in the case of Boltzmann particles these states are identical). As a result we obtain $K-1+2$ accessible states for the second particle. By close analogy with the previous step, at the third step we have $K-2$ possibilities to add to empty states and $2 \cdot 2$ possibilities to add to the two single-occupied states. And so on. As a consequence we obtain the product of factors in equation (4). The additional factor of $1/N!$ makes our particles indistinguishable. It is not difficult to see that for the same reasons for Boltzmann particles we obtain $\frac{K^N}{N!}$ which also can be considered from state-counting procedure point of view.

So for both fermions and bosons the expression for W contains a product of N brackets which are interpreted as the number of the accessible states for the corresponding particle. We will now construct in the same manner intermediate statistics which reduce to fermionic or bosonic statistics for particular values of the deformational parameter.

We can now state the simplest deformation of the state-counting procedure, which is natural and interpolates between Bose- and Fermi- cases. We suppose that the increase in the number of available single particle states due to the addition of one particle is not 1, as it was for bosons, nor -1, as it was for fermions, but is α. Then the number for states of N particles occupying a set of K single particle states is given by

$$W = \frac{1}{N!} \cdot K \cdot (K+\alpha) \cdot (K+2\alpha) \ldots (K+\alpha(N-1)) \tag{5}$$

This expression reduces to W_b when $\alpha = 1$ and to W_f when $\alpha = -1$ in such a manner that the lth bracket of Eq.(4) is transformed into the lth one of Eq.(3). In the next section we will consider this simple example in detail.

The Haldane-Wu state-counting procedure can also be viewed in this manner. It gives the following dimension for the N-particle space:

$$W = \frac{(K + (N-1)(1-g))!}{N!(K - gN - (1-g))!} \tag{6}$$

which also interpolates between the Bose- and Fermi- expressions. Originally the expression was introduced by Haldane [5] to describe a state-counting procedure for particles with exclusion statistics and subsequently was used by many authors [6, 4, 7] to investigate the thermodynamic properties of a gas of such particles (g-ons). One can rewrite (6) as follows

$$\begin{aligned} W &= \frac{1}{N!} \cdot (K + (N-1)(1-g))! \cdot (K + (N-1)(1-g) - 1)! \ldots (K - g(N-1))! \\ &= \frac{1}{N!} \cdot \prod_{l=0}^{N-1} (K - g(N-1) + l)! \; . \end{aligned} \tag{7}$$

In this form we can see again that each of the brackets in the expression interpolates from one of brackets in Eq.(4) (by $g = 0$) to another one in Eq.(3) (by $g = 1$). Moreover the lth bracket of Eq.(4) is transformed into $(n - l + 1)$th one of Eq.(3), i.e. brackets of Eq.(4) correspond to brackets of Eq.(3) in reverse order by replacement of $g = 0$ by $g = 1$ in Eq.(8).

Motivated by these two examples, in the next section we give the general definition of deformed state-counting procedures which are based on the linearity of the deformation of single-particle dimensions and on the symmetric group of the system of particles.

3 State-counting procedure: general construction

In this section we will describe the general case of state-counting deformations of statistics which are linear in g. These deformations differ from Haldane's original exclusion statistics but are still of an exclusion type as they are concerned with occupation number considerations (as in eqs.(8) and (5)). Moreover we will show that the expressions (8) and (5) represent the two simplest examples of such deformations.

The definition of the deformed statistics (which depend linearly on g) consists in stating a one to one correspondence between the brackets of Eq.(4) and Eq.(3). It is clear that there are $N!$ possibilities of such a correspondence and they are parametrized by the general element of the group of permutations of the set of N distinguishable objects. The discussion in the previous section implies that the general expression for the number of states is:

$$W = \frac{1}{N!} \cdot \prod_{l=0}^{N-1} (K - gl + (1-g)\pi_l) \; , \tag{8}$$

where π_l is the lth member of a permutation of $1, \ldots, N$, denoted by π. The cases $g = 0$ and $g = 1$ correspond respectively to bosons and fermions independently of the choice of π. For the identical permutation ($\pi = id : \pi_l = l$) expression (8) reduces to Eq.(5) with $\alpha = 1 - 2g$ while for the inverse case ($\pi_l = N - l - 1$) we find Eq.(8). It is obvious that expression (8) represents all possible linear (in term of g) exclusion statistics. Henceforth we will term the statistics parametrized by the permutation π as π-*statistics*.

Let us now note that not all permutations π lead to distinct thermodynamic behavior in the thermodynamic limit $K, N \to \infty$, $N/K = const$. Indeed, permutations differing only by a finite number of pairwise transpositions give the same statistics in the thermodynamic limit. Moreover only the permutations which differ by an infinite number, $M = O(N)$, of pairwise transpositions which in their order change the positions of the particles on infinite number $M' = O(N)$ give distinguishable thermodynamic results. So if we are interested in thermodynamic quantities we should consider not the original group of permutations S but the quotient group $S' = S/H$ where H is the subgroup of permutations for which the numbers M and M' are $o(N)$ (we imply that an arbitrary but finite number of elements in the products are allowed). The simplest example of such transposition $\pi \in H$ can be easy found: $\pi_l = l + 1$ for $l = 0, \ldots, N-2$ and $\pi_{N-1} = 0$. In this case M' is equal N but $M = 1$ and is negligible if $N \to \infty$. In the following we will be interested specifically in π-statistics with $\pi \in S'$ for $N \to \infty$.

Following [6] we can consider systems containing particles of different species. We introduce the notation: K_i is the number of independent states of a single particle of species i, N_i is the number of particles of species i. Then the number of many-body states at fixed $\{N_i\}$ in the framework of π-statistics is given by the expression:

$$W = \prod_i \frac{1}{(N_i)!} \cdot \prod_{l=0}^{N_i - 1} (K_i - g_i l_i + (1 - g_i)\pi^{(i)}_{l_i} - g_{ik}\pi^{ik}_{l_k}) \ , \tag{9}$$

where, as above, π^i is a permutation of the particles of i-th species and the mapping π^{ik} is a mapping of the k-species into the i-th one. If the parameters g_{ik} are not zero or the mappings π^{ik} are not trivial ($\pi^{ik}_{l_k} = 0$) we will call such π-statistics as *mutual π-statistics*. This construction exactly generalizes the Haldane-Wu mutual statistics and models the dependence of the number of states for particles of species i on the particle number N_j of other species. We will not deal with mutual statistics in detail but the generalizations are straightforward.

4 Thermodynamical properties of Haldane-Wu gas

This section is included to make the picture more complete and the references more convenient. It is devoted to the brief description of the thermodynamic properties of a gas using the Haldane-Wu state-counting procedure (6):

$$W = \prod_i \frac{(K_i + (N_i - 1)(1 - g))!}{N_i!(K_i - gN_i - (1 - g))!} \ .$$

Starting with this counting procedure it is possible to construct thermodynamics in the standard manner. In the thermodynamic limit, the number of particles N_i, as well as the number of single-particle states K_i, becomes infinite. But the occupation numbers $n_i = N_i/K_i$ still remain finite. The entropy of the system $\mathrm{S} = \ln W$ (we set Boltzmann's constant equal to unity). By definition, the ideal gas with $\{N_i\}$ particles has a total energy of the following form:

$$E = \sum_i N_i \epsilon_i$$

with constant ϵ_i. For such gases, the thermodynamic potential Ω can be evaluated by the minimizing

$$\Omega = E - TS - \mu \sum_i N_i$$

with respect to the variation of the densities n_i (here T is the temperature and μ is the chemical potential).

The resulting thermodynamics may be summarized as follows [6]:

$$\Omega = -T \sum_i K_i \ln \frac{1 + w_i}{w_i} \ ,$$

where the function w_i is defined by the following equation:

$$w_i^g (1 + w_i)^{1-g} = e^{(\epsilon_i - \mu)/T} \ . \tag{10}$$

Using the same notation the distribution functions n_i may be expressed as

$$n_i = 1/(w_i + g) \ . \tag{11}$$

This leads to fermionic-like behavior for cases with the value of the deformation parameter $0 < g \leq 1$ (i.e. except the case of real bosons) at low temperatures. In particular, at zero temperature the distribution function contains a 'Fermi-step':

$$n_i = \begin{cases} 0 & \text{if } \epsilon_i > E_F \\ -1/\alpha & \text{if } \epsilon_i < E_F \end{cases} \ . \tag{12}$$

We will refer to the formulae (10,11,12) when other deformed state-counting procedures are considered in the sections below.

5 id-Statistics and its thermodynamics

In this section we derive the occupation number distribution for gas of particles obeying *id*-statistics and compare it with the result obtained in [6] for the permutation $\pi_X = \pi_l = N - l - 1$, i.e. with Haldane-Wu state-counting procedure.

Let us consider the identical permutation in Eq.(9): $\pi_l^{(i)} = l$ for all i. In this case we regain expression (5) for the number of states of particles of species i with $\alpha = 1 - 2g$. One can rewrite Eq.(9) using Γ-functions (due to the well-known property of Γ-function: $\Gamma(z+1) = z\Gamma(z)$) as follows

$$W = \prod_i \frac{\Gamma\left(\frac{K_i + \alpha N_i}{\alpha}\right) \alpha^{N_i}}{\Gamma\left(\frac{K_i}{\alpha}\right) (N_i)!} \ .$$

We consider an ideal gas of such particles, where the total energy is a direct sum

$$E = \sum_i N_i \epsilon_i \ ,$$

where ϵ_i is the energy of a particle of species i. Following the standard procedure [11], one can consider a grand canonical ensemble at temperature T and with chemical potential μ. Then the partition function is

$$Z = \sum_{\{N_i\}} W(\{N_i\}) \exp\left\{\sum_i N_i(\mu - \epsilon_i)/kT\right\} .$$

For very large K_i and N_i the summand has a very sharp peak around the set of most probable particle numbers $\{N_i\}$. Using the asymptotic approximation for Γ-functions ($\ln\Gamma(z) = z\ln z$) and introducing the average occupation number $n_i = N_i/K_i$, one can express $\ln W$ as follows

$$\ln W = \sum_i K_i \left\{ \left(\frac{1}{\alpha} + n_i\right) \ln(1 + \alpha n_i) - n_i \ln n_i \right\} .$$

The most probable distribution of n_i is determined by

$$\frac{\partial}{\partial n_i} \sum_i K_i \left\{ \left(\frac{1}{\alpha} + n_i\right) \ln(1 + \alpha n_i) - n_i \ln n_i + n_i \frac{\mu - \epsilon_i}{kT} \right\} = 0 ,$$

that leads to the expression for the average occupation number:

$$n_i = \frac{1}{\exp\left(\frac{\epsilon_i - \mu}{kT}\right) - \alpha} \tag{13}$$

This expression recovers the Bose, Boltzmann and Fermi distributions with $\alpha = 1$, $\alpha = 0$ and $\alpha = -1$ respectively.

Now we consider in detail the thermodynamics of a gas with $0 < \alpha \leq 1$ and demonstrate Bose-condensation at low temperature and values of the deformation parameter $\alpha \geq 0$ (which is equivalent to the inequality $g \leq 1/2$). The energy distribution of particles with the occupation number (13) is [11]

$$dN_\epsilon = \frac{SVm^{3/2}}{\sqrt{2}\pi^2 h^3} \cdot \frac{\sqrt{\epsilon}\, d\epsilon}{\exp\frac{(\epsilon - \mu)}{T} - \alpha} \tag{14}$$

where we use units such that Boltzmann's constant $k = 1$, spin degeneracy factor $S = 2s + 1$ (s being the spin of the particle), m is the mass of the particle, V is the total volume of the gas. Integrating with respect to ϵ we obtain the number of particles with energy $\epsilon > 0$ in the gas:

$$\frac{N}{V} = \frac{S(mT)^{3/2}}{\sqrt{2}\pi^2 h^3} \int_0^\infty \frac{\sqrt{z}\, dz}{e^{z - \mu/T} - \alpha} . \tag{15}$$

This formula determines the chemical potential μ of the gas as a function of its temperature T and density N/V.

From the condition $n_i \geq 0$ one can derive a restriction on μ:

$$\mu \leq -\ln\alpha \cdot T \ , \text{ where } \ \ln\alpha < 0 . \tag{16}$$

If the temperature of the gas is lowered at constant density N/V, the chemical potential μ given by (15) increases. It reaches the limiting value determined by the relation (16) at a temperature T_0, which can be determined from the equation

$$\frac{N}{V} = \frac{S(mT_0)^{3/2}}{\sqrt{2}\pi^2 h^3 \alpha} \int_0^\infty \frac{\sqrt{z}\ dz}{e^z - 1}$$

which gives the following value for T_0:

$$T_0 = \frac{3.31 h^2}{m S^{2/3}} \left(\frac{N}{V}\right)^{\frac{2}{3}} \cdot \alpha^{\frac{2}{3}} = T_0^{(b)} \cdot \alpha^{\frac{2}{3}} \tag{17}$$

with the temperature of Bose-Einstein condensation for bosons ($\alpha = 1$) $T_0^{(b)}$.

For $T < T_0$ Eq. (15) has no solution obeying the relation (16). This contradiction arises because we have actually neglected in (14) the particles with $\epsilon = 0$ by multiplying the expression by $\sqrt{\epsilon}$. In reality the situation for $T < T_0$ is as follows. Particles with energy $\epsilon > 0$ are distributed according to formula (14) with $\mu = -\ln\alpha \cdot T$:

$$dN_\epsilon = \frac{1}{\alpha} \cdot \frac{SVm^{3/2}}{\sqrt{2}\pi^2 h^3} \cdot \frac{\sqrt{\epsilon}\ d\epsilon}{e^{\epsilon/T} - 1} \ .$$

The total number of particles with energies $\epsilon > 0$ is

$$N_{\epsilon>0} = N(T/T_0)^{\frac{3}{2}}$$

and the number of particles in the lowest state with $\epsilon = 0$ is

$$N_{\epsilon=0} = N\left(1 - \left(\frac{T}{T_0}\right)^{\frac{3}{2}}\right) \tag{18}$$

Thus we obtain the Bose-Einstein condensation for particles obeying (13) for $0 < \alpha \leq 1$. Moreover the form of the expression (18) does not depend on α; for different values α we just obtain different values of Bose-Einstein condensation temperatures (17).

For $-1 \leq \alpha < 0$, it follows from (13) that

$$n_i \leq -\frac{1}{\alpha}$$

In this case at $T = 0$ the average occupation number for single particle states with a continuous energy spectrum has a 'fermionic' step-like distribution:

$$n_i = \begin{cases} 0 & \text{if } \epsilon_i > E_F \\ -1/\alpha & \text{if } \epsilon_i < E_F \end{cases} \ .$$

Now we can compare the results of this section with those derived for the inverse permutation π_X in [6]. Let us remember that for the inverse permutation the average occupation number at $T = 0$ has a fermi-like distribution for any value of the parameter except that which corresponds to bosons. In the present case for the identical permutation we obtained a fermion-like distribution for parameter values interpolating between Fermi and Boltzmann distributions ($0 < \alpha \leq 1$). For parameter values between Boltzmann and Bose distributions

($-1 \leq \alpha < 0$) we obtained a boson-like behavior at low temperatures (18). Furthermore, for the identical permutation we have the Boltzmann distribution at $\alpha = 1/2$ while for the inverse one the distribution tends to the Boltzmann distribution (at the same value of the parameter) only in the high temperature limit.

Now we are going to consider several more complicated examples of permutations and compare them with those discussed above.

6 Other permutations

As it is easy to see from the Eqn. (9), we have a lot of possibilities to construct statistics using different permutations. A more natural and simple way to do this is to divide the set of brackets into two equal parts (we assume the number of brackets is even, which is not important in the thermodynamic limit) and then consider the two simplest permutations (id- and inverse permutations) for these parts (see Figs.2-5). These constructions seem to be simple, however they display a great variety of thermodynamic properties.

6.1 The permutation π_{XX}

At first we will deal with the permutation where the first $N/2$ factors of the number of states for fermions correspond to those for bosons crosswise and the same correspondence takes place for the second $N/2$ brackets (Fig.2). It is natural to call this permutation π_{XX}.

We can formalize our rule representing the permutation in terms of the following expressions:

$$\pi_{XX}{}_l^{(1)} = \frac{N}{2} - l - 1 \ , \quad \text{for} \quad 0 \leq l \leq \frac{N}{2} - 1$$
$$\pi_{XX}{}_l^{(2)} = \frac{3}{2}N - l - 1 \ , \quad \text{for} \quad \frac{N}{2} \leq l \leq N - 1 \ .$$

Then using Eqn. (9) the number of many-body states for very large K_i and N_i is given by the equality:

$$W_{XX} = \prod_i \frac{1}{(N_i)!} \cdot \frac{\left(K_i + \frac{1-g}{2}N_i\right)!}{\left(K_i - \frac{g}{2}N_i\right)!} \cdot \frac{\left(K_i + \left(1 - \frac{3}{2}g\right)N_i\right)!}{\left(K_i + \frac{1-3g}{2}N_i\right)!}$$

Following the same procedure as in the previous section one can derive the expression for the average occupation number

$$n_{XX\,i} = \frac{1}{w_{XX}(\xi) + g/2}$$

where $\xi = \exp\left(\frac{\epsilon_i - \mu}{kT}\right)$ and $w_{XX}(\xi)$ satisfies the following equation:

$$w_{XX}^{\frac{g}{2}}(\xi) \cdot \left(w_{XX}(\xi) + \frac{1}{2}\right)^{\frac{1-g}{2}} \cdot \left(w_{XX}(\xi) + 1 - g\right)^{1 - \frac{3}{2}g} \cdot \left(w_{XX}(\xi) + \frac{1}{2} - g\right)^{\frac{-1+3g}{2}} = \xi \qquad (19)$$

This equation may be considered as an analogue of Wu's equation (10). It is interesting to note that not only the general form of this equation is similar to the form of Eqn.(10): the sum of the powers in the expression (19) is equal to unity as in (10). We will see that this property is common to the other permutations.

The average occupation number at $T = 0$ has a fermionic step-like distribution for all values of the parameter g except $g = 0$ (bosons):

$$n_{XX_i} = \begin{cases} 0 & \text{if } \epsilon_i > E_F \\ \frac{2}{g} \quad \text{for } g \leq 1/2 & \\ \frac{2}{3g-1} \quad \text{for } g > 1/2 & \text{if } \epsilon_i < E_F \end{cases} \tag{20}$$

where the Fermi level is a continuous function of g for $0 < g \leq 1$ but not a smooth function as it was for the inverse permutation π_X ($E_F = 1/g$). For the case $g = 1/2$ it is possible to solve the Eqn.(19) and obtain the following expression for the average occupation number:

$$\left(n_{XX} = \frac{1}{\sqrt{\frac{1}{16} + \exp(2\frac{\epsilon-\mu}{kT})}} \right)_{g=\frac{1}{2}} \tag{21}$$

Comparing this with previous cases we recall that for the inverse permutation π_X the same expression occurs with the replacement of 1/16 by 1/4 [6] and in the case of the identical permutation for $g = 1/2$ we have just the Boltzmann distribution. So in some sense the function n_{XX} is closer to the Boltzmann distribution at $g = 1/2$ than to Wu's distribution.

6.2 The permutation π_{XI}

The second example of this section is generated by the permutation where the first $N/2$ brackets of the number of states for fermions correspond to those for bosons crosswise and for the second $N/2$ brackets the identical permutation takes place (Fig.3). One can write the following expressions for it:

$$\pi_{XI_l}^{(1)} = \frac{N}{2} - l - 1 \ , \quad \text{for} \quad 0 \leq l \leq \frac{N}{2} - 1$$

$$\pi_{XI_l}^{(2)} = l \ , \quad \text{for} \quad \frac{N}{2} \leq l \leq N - 1 \ .$$

Then the number of many-body states can be expressed as:

$$W_{XI} = \prod_i \frac{1}{(N_i)!} \cdot \frac{\left(K_i + \left(\frac{N_i}{2} - 1\right)(1-g)\right)!}{\left(K_i - g\left(\frac{N_i}{2} - 1\right) - 1\right)!} \cdot \frac{\Gamma\left(\frac{K_i + (1-2g)N_i}{1-2g}\right)}{\Gamma\left(\frac{K_i + (1-2g)N_i/2}{1-2g}\right)} \cdot (1-2g)^{\frac{N_i}{2}} \ .$$

Following the same procedure as in previous sections (i.e.minimizing the thermodynamic potential Ω), we obtain an expression for the average occupation number:

$$n_{XI_i} = \frac{1}{w_{XI}(\xi) + g/2}$$

where the function $w_{XI}(\xi)$ as usual is defined by the equation:

$$w_{XI}^{\frac{g}{2}}(\xi) \cdot \left(w_{XI}(\xi) + \frac{1}{2}\right)^{\frac{1-g}{2}} \cdot \left(w_{XI}(\xi) + 1 - \frac{3}{2}g\right) \cdot \left(w_{XI}(\xi) + \frac{1-g}{2}\right)^{-\frac{1}{2}} = \xi \ .$$

For this permutation at $T = 0$ we also obtain a step-like distribution for any value of the parameter g except $g = 0$ (bosons):

$$n_{XI\,i} = \begin{cases} 0 & & \text{if } \epsilon_i > E_F \\ \frac{2}{g} & \text{for } g \le 2/3 & \\ \frac{1}{2g-1} & \text{for } g > 2/3 & \text{if } \epsilon_i < E_F \end{cases} \tag{22}$$

As for the permutation π_{XX} the Fermi level is a continuous, but not smooth, function of g for $0 < g \le 1$.

6.3 The permutation π_{IX}

The last example of such a type of permutation can be denoted by π_{IX}, illustrated by Fig.4 and described by the formulae:

$$\pi_{IX}{}_l^{(1)} = l \ , \quad \text{for} \quad 0 \le l \le \frac{N}{2} - 1$$

$$\pi_{IX}{}_l^{(2)} = \frac{3}{2}N - l - 1 \ , \quad \text{for} \quad \frac{N}{2} \le l \le N - 1$$

To escape repetition of standard arguments, we just state the main results for this case without comment:

$$W_{IX} = \prod_i \frac{1}{(N_i)!} \cdot \frac{\Gamma\left(\frac{K_i + (1-2g)N_i/2}{1-2g}\right)}{\Gamma\left(\frac{K_i}{1-2g}\right)} \cdot (1-2g)^{\frac{N_i}{2}} \cdot \frac{\left(K_i + \left(1 - \frac{3}{2}g\right) N_i\right)!}{\left(K_i + \frac{1-3g}{2}N_i\right)!} \ .$$

$$n_{IX\,i} = \frac{1}{w_{IX}(\xi) + \frac{3g-1}{2}}$$

$$w_{IX}^{\frac{3g-1}{2}}(\xi) \cdot \left(w_{IX}(\xi) + \frac{1}{2}\right)^{1-\frac{3}{2}g} \cdot \left(w_{IX}(\xi) + \frac{g}{2}\right)^{\frac{1}{2}} \cdot = \xi \tag{23}$$

In contrast with the previous examples, for this permutation at $T = 0$ the fermion distribution takes place only for values of the parameter $g > 1/3$ and the Fermi level is a continuous and smooth function of g:

$$n_{IX\,i} = \begin{cases} 0 & & \text{if } \epsilon_i > E_F \\ \frac{2}{3g-1} & \text{for } g > 1/3 & \text{if } \epsilon_i < E_F \end{cases} \ .$$

To investigate the behavior of particles with such statistics for $g \le 1/3$ we will consider the particular example $g = 1/3$. In this case the equation (23) can be solved and the average occupation number is given by

$$\left(n_{IX} = \frac{3}{\sqrt{9\xi^2 + \frac{1}{4}} - 1}\right)_{g=\frac{1}{3}} \ .$$

The condition $n \ge 0$ implies the following inequality for μ

$$\mu \le \ln(2\sqrt{3})kT \ .$$

It is similar to relation (16) which was obtained for the statistics generated by the identical permutation. Following the discussion of the previous section one can derive that Bose-Einstein condensation takes place in this case and the temperature for condensation is determined by the following equation

$$\frac{N}{V} = \frac{S(mT_0)^{3/2}}{\sqrt{2}\pi^2 h^3 \alpha} \int_0^\infty \frac{3\sqrt{z}\ dz}{\frac{1}{2}\sqrt{3e^z+1}-1} \ .$$

We obtain the Bose-Einstein condensation at $g = 1/3$ and it takes place at $g = 0$ (bosons). So one can conclude that the statistics generated by the permutation π_{IX} obey Bose-like behavior at low temperature for the values of the parameter $0 \leq g \leq 1/3$.

For the permutations π_{IX} and π_{XI} one can obtain the equivalent expressions for the average occupation number by $g = 1/2$:

$$\left(n_{XI} = \frac{1}{\sqrt{2}} \exp\left(-2\frac{\epsilon-\mu}{kT}\right) \left(\sqrt{1+4^5 \exp(4\frac{\epsilon-\mu}{kT})} - 1 \right)^{\frac{1}{2}} \right)_{g=\frac{1}{2}} \ .$$

In some sense this expression is closer to the Boltzmann distribution than (21). It can be connected with the presence of partially identical pieces in the basic permutation. Indeed, in the case of pure identical permutation in this limit we had a Boltzmann distribution. In other words we can try to characterize the thermodynamic properties in terms of permutation characteristics. Let us stress, however, that distribution functions of all the above examples have the Boltzmann limit at enough high temperature.

6.4 The permutation $\pi_{\bowtie}$

Let us now turn to the last example, which is a little more complicated. Consider the permutation that correlates the first $N/2$ brackets of the number of states for fermions with the second $N/2$ for bosons identically and the same correspondence takes place for the remaining brackets. It is similar to the identical permutation but the sets of the first and second $N/2$ brackets are related in a crosswise fashion (Fig.5). We will denote this permutation by the index $\bowtie$ and describe it by:

$$\pi_{\bowtie l}^{(1)} = \frac{N}{2} + l \ , \quad \text{for} \quad 0 \leq l \leq \frac{N}{2} - 1$$

$$\pi_{\bowtie l}^{(2)} = l - \frac{N}{2} \ , \quad \text{for} \quad \frac{N}{2} \leq l \leq N - 1 \ .$$

The main results are

$$W_{\bowtie} = \prod_i \frac{1}{(N_i)!} \cdot \frac{\Gamma\left(\frac{K_i+N_i(1-3/2\cdot g)}{1-2g}\right)}{\Gamma\left(\frac{K_i+N_i(1-g)/2}{1-2g}\right)} \frac{\Gamma\left(\frac{K_i+N_i(1-3g)/2}{1-2g}\right)}{\Gamma\left(\frac{K_i-N_i g/2}{1-2g}\right)} \cdot (1-2g)^{N_i} \ ,$$

$$n_{\bowtie i} = \frac{1}{w_{IX}(\xi) + \frac{g}{2}} \ ,$$

$$w_{\bowtie}^{\frac{g}{2}}(\xi)\cdot\left(w_{\bowtie}(\xi)+\frac{1}{2}\right)^{\frac{g-1}{2}}\cdot\left(w_{\bowtie}(\xi)+\frac{1}{2}-g\right)^{\frac{1-3g}{2}}\cdot\left(w_{\bowtie}(\xi)+1-g\right)^{1-\frac{3}{2}g}=\xi^{1-2g}\ . \qquad (24)$$

It is interesting to note that in this case at $T=0$ we obtain exactly the same results as for the permutation π_{XX}: $n_{\bowtie}=n_{XX}$ (20), i.e. we obtain the Fermi-like distribution at any value of parameter except that which corresponds to bosons and the Fermi level is a continuous, but not a smooth, function of the parameter.

The equation (24) obeys the identity at $g=1/2$. So for this case we have to consider the original expression for the number of many-body states for this permutation with $g=1/2$. Following the standard procedure one can obtain the equation for the average occupation number:

$$\left(\sqrt{1-\frac{n_{\bowtie}^2}{16}}=n_{\bowtie}\exp\left(\frac{1}{1-\frac{n_{\bowtie}^2}{16}}\right)\cdot\exp\left(\frac{\epsilon-\mu}{kT}\right)\right)_{g=\frac{1}{2}}\ .$$

6.5 Summary

We have considered four non-trivial examples of permutations obeying four different statistics. Summarizing our discussion, we can note that in only one case, π_{IX}, we obtained a Bose-like behavior at low temperature for some particular choices of parameters. In other three cases such behavior only takes place at the extreme parameter value corresponding to bosons and all other parameter values yield a Fermi-like distribution at $T=0$ with continuous but non-smooth Fermi level as a function of g. Moreover the results for the permutation π_{XX} and $\pi_{\bowtie}$ are identical at $T=0$. Comparing these examples with the two simplest permutations π_{id} and π_X, one can observe that the permutations π_{XX}, π_{XI} and $\pi_{\bowtie}$ are similar to the inverse one except for the fact that for these permutations the Fermi levels are non-smooth functions of the parameter g (20, 22). Finally the permutation π_{IX} is close to the identical one but the transition between the Bose- and Fermi-like behaviors occurs at a different value of g ($g=1/2$ for the identical permutation and $g=1/3$ for π_{IX}).

7 Conclusion

Summarizing our discussion, let us mention some possible physical applications of the constructions in this paper. The simplest example of π-statistics, obeyed by the identical permutation, has been recently considered in [12] as an alternative to the state-counting procedure for exclusion statistics. However one can note that similar statistics (up to constant), with integer positive values of the statistical parameter α, appeared long ago in the theory of statistics of donor and acceptor levels in semiconductors [13]. In the same manner, the Hubbard model with an infinite value of Coulomb interaction on a site can be considered as a gas of g-ons obeying $g=2$ statistics (a site with two single electron levels can be occupied by only one electron). Apparently, there are many other systems where such statistics arise naturally.

Another example of the π-statistics with $\pi\neq id$ can be represented by a system of anyons on a torus in a strong magnetic field when only the lowest Landau level is occupied. As was shown in reference [14] the number of many-body states in this case is given by

$$D = s\Phi \cdot \frac{(\Phi + N(1-\alpha) - 1)!}{(\Phi - \alpha N)!N!} \ ,$$

where Φ is the magnetic flux seen by the particle and α is the statistical parameter which is presented as $\alpha = k/s$ with positive coprime integers k and s. One can see that this expression corresponds to the number of many-body states for π-statistics with the permutation

$$\pi_0 = 0 \ , \qquad \pi_l = N - l \text{ for } 1 \leq l \leq N-1 \ ,$$

which can be termed π_{1X} and is illustrated in Fig.6. Let us remember that for the same system on a sphere the number of many-body states is described by Haldane-Wu statistics with the permutation π_X. So it is natural to expect the appearance of more complicated permutations on higher genus surfaces or taking into account higher Landau levels. Moreover, one can imagine a lot of possible physical speculations based on the statistics with π_{XX} or π_{XI}, π_{IX}. We will return to this subject in our forthcoming paper.

In conclusion, we considered the generalization of Haldane's state-counting procedure to describe all possible types of exclusion statistics which are linear in the deformation parameter g. The statistics are parametrized by elements of the symmetric group of the particles. For several particular cases we derived the equations for distribution functions which generalize results obtained by Wu. Using them we analyzed the low-temperature behavior and thermodynamic properties of these systems and compared our results with previous studies of the thermodynamics of a gas of g-ons. We speculated on the correlation between statistical properties of gas obeying π-statistics and the properties of the permutation π. Physical examples where these constructions are realized were discussed.

Figures

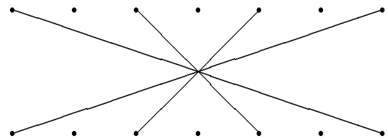

Fig.1 Illustration for the permutation π_X(Haldane-Wu state-counting procedure).

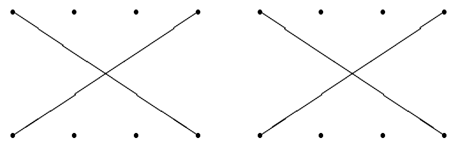

Fig.2 Illustration for the permutation π_{XX}.

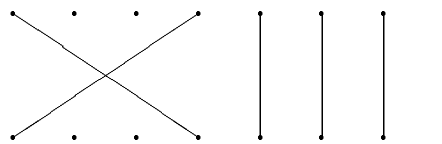

Fig.3 Illustration for the permutation π_{XI}.

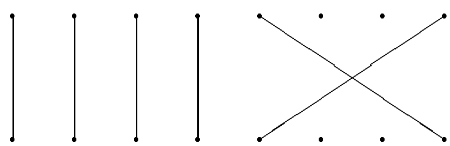

Fig.4 Illustration for the permutation π_{IX}.

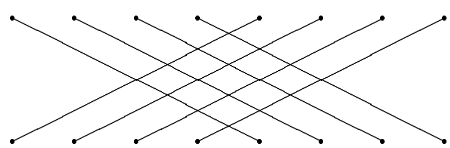

Fig.5 Illustration for the permutation $\pi_{\bowtie}$.

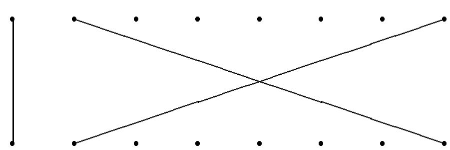

Fig.6 Illustration for the permutation π_{1X}.

References

[1] F.Wilczek, *Phys.Rev.Lett.* **48** 1144 (1982); **49** 957 (1982);

[2] A.Lerda, *'Anyons'*, Springer-Verlag (1992);

[3] F.D.M.Haldane, Z.N.C.Ha, J.C.Talstra, D.Bernard and V.Pasquier *Phys.Rev.Lett.* **69**, 2021 (1992);

[4] D.Bernard and Y.S.Wu, Preprint SPhT-94-043, UU-HEP /94-03;

[5] F.D.M.Haldane, *Phys.Rev.Lett.* **67**, 937 (1991);

[6] Y.S.Wu, *Phys.Rev.Lett.* **73**, 922 (1994);

[7] C.Nayak and F.Wilczek, *Phys.Rev.Lett.* **73** 2740 (1994);

[8] A.K. Rajagopal, *Phys.Rev.Lett.* **74**, 1048 (1995); M.V.N.Murthy and R.Shankar, *Phys.Rev.Lett.* **72**, 3629 (1994); S.B.Isakov, *Phys.Rev.Lett.* **73**, 2150 (1994); S.Ouvry, preprint IPNO/TH 94-76 (1994); A.Dasniéres de Veigy and S. Ouvry, *Phys.Rev.Lett.* **72**, 600 (1994); K.Byszuk, J.Spałek, *Phys.Rev.* **B 50**, N 16, 11403 (1994);

[9] Z.N.Ha, *Phys.Rev.Lett.* **73**, 1574 (1994); Preprint IASSNS-HEP-94/90 (1994);

[10] K.N.Ilinski, J.M.F.Gunn, Preprint hep-th/9403233;

[11] L.D.Landau and E.M.Lifshits, *Statistical Physics. Part 1*, Pergamon Press (1982);

[12] A.P.Polychronakos, UUITP-03/95 and hep-th/9503077;

[13] N.W.Aschcroft, N.D.Mermin, *Solid State Physics*, Saunders College, Philadelphia, Holt-Saunders Japan LTD, Tokyo, 1981;

[14] A.Fayyazuddin, D.Li, hep-th/9506032.