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Aspects of Duality in $N = 2$ String Vacua \star

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ABSTRACT

We collect further evidence for the proposed duality between $N = 2$ heterotic and type II string vacua in a specific model suggested by Kachru and Vafa. In the gauge sector the previous analysis is extended; it is further shown that the duality also holds for the one-loop gravitational couplings to the vector multiplets.

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1. The recent advances in understanding non-perturbative aspects of $N = 2$ supersymmetric Yang-Mills theories [1] have raised the question whether similar techniques are applicable in string theory. One of the key elements in the work of Seiberg and Witten is the relation between a gauge theory at strong coupling and a ‘dual’ theory at weak coupling with magnetic monopoles (dyons) as elementary excitations. This dual theory can be analyzed in perturbation theory and, as a consequence, the exact non-perturbative low energy effective action for all values of the coupling constant is determined.

In order to apply such techniques to string theory, a similar strong – weak coupling duality has to be established. For $N = 2$ string theories it has been conjectured that such a duality exists between heterotic vacua compactified on the six-dimensional manifold $K_3 \times T^2$ and type II vacua compactified on a Calabi-Yau threefold [2–5]. The coupling constant in string theory is a dynamical variable determined by the vacuum expectation value of the dilaton S . In $N = 2$ heterotic vacua S resides in an abelian vector multiplet while in type II vacua it is a member of a hypermultiplet [6]. Combining the facts that there are no gauge neutral couplings between vector and hypermultiplets [7] and that S organizes the string perturbation theory implies a non-renormalization theorem for both, heterotic and type II vacua. For the heterotic vacua the tree level couplings of the hypermultiplets are exact whereas the tree level couplings of the vector multiplets are corrected at one-loop and non-perturbatively. In type II vacua the situation is reversed and the tree level couplings of the vector multiplets are exact while the hypermultiplets suffer perturbative and non-perturbative corrections. Thus, if a string vacuum has a dual representation as both heterotic and type II the exact effective Lagrangian can be obtained by computing the couplings of the vector multiplets in the type II theory and the couplings of the hypermultiplets in the heterotic theory.

Concrete examples of ‘dual pairs’ of $N = 2$ string vacua have been suggested

in refs. [4,5] and non-trivial evidence for the proposed duality was found. One of the models considered in ref. [4] is a specific compactification of the heterotic string on $K_3 \times T^2$ with gauge group $U(1)^3$. The corresponding gauge bosons are the graviphoton, the vector partner of the dilaton and the vector partner of the toroidal modulus T . The second toroidal modulus U is locked at $U = T$. Thus the model has two $U(1)$ vector multiplets ($n_V = 2$) while the number of hypermultiplets turns out to be $n_H = 129$. Kachru and Vafa observe that there is an unique Calabi–Yau threefold $X_{12}(1, 1, 2, 2, 6)$, — the degree 12 hypersurface in $\mathbf{P}^4(1, 1, 2, 2, 6)$ — with $n_V = b_{(1,1)} = 2$ and $n_H = b_{(1,2)} + 1 = 129$ ($b_{(1,1)}$ and $b_{(1,2)}$ denote the number of $(1, 1)$ and $(1, 2)$ forms and the ‘+1’ counts the dilaton). Therefore this Calabi–Yau space is a good candidate for the dual type IIA string vacuum. The tree level couplings of the two vector multiplets are known exactly for the type II vacuum [8,9] while for the dual heterotic vacuum they have only been studied in perturbation theory [2,10,11]. Kachru and Vafa have given evidence that they agree at weak coupling. In this letter we extend their analysis of the gauge sector and further show that also for the gravitational coupling to vector multiplets the duality between the vacua holds. Our results in section 2 overlap with recent work of K. Narain and collaborators.

2. The couplings of the vector multiplets are encoded in a holomorphic prepotential F [7]. In the heterotic vacuum F^{het} has the weak coupling expansion

$$F^{\text{het}} = \frac{1}{2}ST^2 + h(T) + h^{\text{np}}(e^{-8\pi^2 S}, T), \quad (1)$$

where $\frac{1}{2}ST^2$ is the tree level contribution, $h(T)$ is the dilaton independent one-loop correction and h^{np} is generated non-perturbatively[* In ref. [10,11] it was shown that $h(T)$ is strongly constrained by its transformation properties under any exact quantum symmetry and by its singular behaviour at special points in

* The standard $N = 2$ non-renormalization theorem states that beyond one-loop there are no further perturbative corrections [12].

the moduli space where additional massless states appear. The model at hand has an exact $SL(2, \mathbf{Z})$ quantum symmetry (T -duality) which acts on the modulus T according to

$$T \rightarrow \frac{aT - ib}{icT + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{Z}), \quad (2)$$

while the dilaton S is invariant at the tree level. Using the formalism developed in ref. [10] it is straightforward to determine the transformations law of h under this $SL(2, \mathbf{Z})$

$$h(T) \rightarrow \frac{h(T)}{(icT + d)^4} + \frac{\Xi(T)}{(icT + d)^4}, \quad (3)$$

where Ξ is at most a quartic polynomial in T arising from the multivaluedness of $h(T)$. (In the absence of logarithmic singularities h would be a modular form of weight -4 .) The 5th derivative $\partial_T^5 h(T)$ does not suffer from any ambiguity and is a modular form of weight $+6$ [[†] The singularities of h are at $T = 1, \infty$; at $T = 1$ the gauge group $U(1)^3$ is enlarged to $SU(2) \times U(1)^2$ (with no additional massless hypermultiplets) and, as a consequence, $\partial_T^2 h$ develops a logarithmic singularity $\partial_T^2 h \sim -\frac{b_{SU(2)}}{8\pi^2} \ln(T-1)$][[‡] Hence, modular invariance (together with $b_{SU(2)} = -4$) dictates

$$\partial_T^2 h = \frac{1}{4\pi^2} \ln[j(iT) - j(i)] + \text{finite terms}, \quad (4)$$

where $j(iT)$ is the modular invariant j -function.

At $T = \infty$ the coupling $\partial_T^2 h$ should diverge at most like T^2 which implies that $\partial_T^5 h(T)$ is regular everywhere except at $T \sim 1$ where $\partial_T^5 h \sim \pi^{-2}(T-1)^{-3}$.

[†] Note that the n -th ordinary derivative $\partial_T^n f_{1-n}$ of a weight $n-1$ modular form f_{1-n} is a modular form of weight $n+1$.

[‡] A more extensive discussion of the singularities of h and its precise relation to the gauge coupling is given in refs. [13,10,11].

This determines $\partial_T^5 h(T)$ up to one arbitrary coefficient A

$$\partial_T^5 h = \pi \left[\frac{E_4^6}{E_6^3} + A \frac{E_4^3}{E_6} - (A+1) E_6 \right] , \quad (5)$$

where E_4, E_6 are the normalized Eisenstein functions of weight 4, 6, respectively. To determine the coefficient A we use the results of toroidal compactifications where, in addition to T , the modulus U is also unconstrained. For this case the third derivatives $\partial_T^3 h(T, U), \partial_U^3 h(T, U)$ have been determined in refs. [10,11]. In terms of the coordinates $\phi^\pm \equiv T \pm U$, it is possible to compute $\partial_{\phi^+}^5 h(\phi^+, \phi^-)$ at $\phi^- = 0$ from the knowledge of the third derivatives $\partial_T^3 h(T, U), \partial_U^3 h(T, U)$. $\partial_{\phi^+}^5 h(\phi^+, 0)$ is singular at $T = 1$ (where the enhanced gauge symmetry is $SU(2)^2$) and at $T = e^{i\pi/6}$ (where the enhanced gauge symmetry is $SU(3)$); nevertheless the singularity at $T = 1$ has to agree with eq. (5) since at that point the additional massless states which contribute to the β -function are identical in both theories. Matching the coefficients of the singular terms at $T = 1$ yields $A = -23/18$.

The analysis of refs. [10,11] also showed that at one-loop the dilaton S is no longer invariant under T -duality. Instead it transforms according to

$$S \rightarrow S - \frac{1}{3} \partial_T^2 \Xi + 2ic \frac{\partial_T(h + \Xi)}{(icT + d)} + 4c^2 \frac{h + \Xi}{(icT + d)^2} + i \text{const.} . \quad (6)$$

It is however possible to define a modular invariant coordinate by

$$S^{\text{inv}} := S + \frac{1}{3} [\partial_T^2 h(T) + L(T)] , \quad (7)$$

where the holomorphic $L(T)$ is modular invariant up to a shift by an imaginary constant. The difference $S^{\text{inv}} - S$ has to be finite for finite T and should not grow faster than a polynomial at $T \rightarrow \infty$. Therefore, eq. (4) determines

$$L = -\frac{1}{4\pi^2} \ln[j(iT) - j(i)] + \text{const.} . \quad (8)$$

It is important to note that S is a $N = 2$ special coordinate but S^{inv} is *not*.

Let us now turn to the type II string compactified on the Calabi–Yau threefold $X_{12}(1, 1, 2, 2, 6)$. The defining polynomial of the mirror manifold (which has $b_{(1,2)} = 2$) is given by [8,9]

$$p = z_1^{12} + z_2^{12} + z_3^6 + z_4^6 + z_5^2 + a_0 z_1 z_2 z_3 z_4 z_5 + a_1 z_1^6 z_2^6, \quad (9)$$

where a_0 and a_1 are the two complex structure deformations. The uniformizing variables at large complex structure are $x = \frac{a_1}{a_0}$ and $y = \frac{1}{a_1}$ and the manifold (9) has a conifold singularity at

$$(1 - 12^3 x)^2 - 4 \cdot 12^3 x^2 y = 0. \quad (10)$$

For generic values of y this is satisfied by two values of x , which coalesce for $y = 0$. This observation, which is reminiscent of the work of Seiberg and Witten [1], led Kachru and Vafa to identify $y = 0$ with the weak coupling limit of the dual heterotic vacuum.

In order to make this proposal more precise one has to find a map between the special coordinates S and T in the heterotic vacuum and the $(1, 1)$ deformations of the Calabi-Yau manifold. The special coordinates t_1 and t_2 on $X_{12}(1, 1, 2, 2, 6)$ are determined by the mirror map in terms of x and y [8,9]. The mirror map can be inverted, leading at weak coupling to

$$x = \frac{1}{j(q_1)} + O(q_2), \quad y = q_2 g(q_1) + O(q_2^2), \quad q_j = e^{2\pi i t_j}, \quad (11)$$

where $g(q_1)$ is a power series in q_1 , normalized to $g(0) = 1$. The first few coefficients are recorded in [8] or can be computed using the computer program of [14]. One is now led to the following identification of the special coordinates in the two vacua: $t_1 = iT$, $t_2 = 4\pi i S$ [★

★ Note that eq. (6) implies that S is ambiguous up to a quadratic polynomial in T . We acknowledge usefull discussions with P. Mayr on that point.

Once the coordinates have been identified one has to check the identity of the two prepotentials $F^{\text{het}} = F^{\text{II}}$. For the Calabi-Yau manifold the Yukawa couplings Y_{ijk} are computed in refs. [8,9] and when expressed in terms of special coordinates they are determined by the third derivative of the prepotential $Y_{ijk} = \partial_{t_i} \partial_{t_j} \partial_{t_k} F^{\text{II}}$. Using the formulæ of [8,9] it is straightforward to compute in the weak coupling limit ($y \rightarrow 0$)

$$\partial_T^3 F^{\text{II}} = \frac{1}{4\pi^2} \frac{E_4}{\omega_0^2} \partial_T (\ln[j(iT) - j(i)] - \frac{3}{2} \ln g(iT)) , \quad \partial_T^2 \partial_S F^{\text{II}} = \frac{E_4}{\omega_0^2} , \quad (12)$$

while $\partial_S^2 \partial_T F^{\text{II}}$ and $\partial_S^3 F^{\text{II}}$ vanish in this limit. Here we have chosen a convenient overall normalization of F^{II} . ω_0 is the fundamental period of the Calabi-Yau manifold and it appears in the transformation of the Yukawa couplings to special coordinates [15]; it plays the role of the homogeneous $N = 2$ coordinate X_0 . The two equations in (12) are consistent if

$$\omega_0^2(x, y = 0) \equiv \left(\sum_{n=0}^{\infty} \frac{(6n)!}{(3n)!(n!)^3} j(iT)^{-n} \right)^2 = E_4(iT) , \quad (13)$$

where the first equation follows from the explicit series representation of ω_0 [8,9]. We have checked this identity perturbatively in q_1 ; an analytic proof has been given by O. Ogievetsky [16]. Inserting eq. (13) into (12) we arrive at

$$\partial_T^2 F^{\text{II}} = S + \frac{1}{4\pi^2} \ln[j(iT) - j(i)] - \frac{3}{8\pi^2} \ln g(iT) . \quad (14)$$

Now we are prepared to compare the two prepotentials. Equating (1) with (14) ($F^{\text{het}} = F^{\text{II}}$) implies

$$\partial_T^2 h = \frac{1}{4\pi^2} \ln[j(iT) - j(i)] - \frac{3}{8\pi^2} \ln g(iT) , \quad (15)$$

where $g(iT)$ has been defined in eq. (11). Inserting (15) into (5) now facilitates a non-trivial check of the consistency of the proposed duality. We were able to

verify the consistency of eqs. (15) and (5) up to order 10 in q_1 . Furthermore, inserting eqs. (15), (8) into (7) we find

$$S^{\text{inv}} = S - \frac{1}{8\pi^2} \ln g(iT) , \quad \Rightarrow \quad e^{-8\pi^2 S^{\text{inv}}} = e^{-8\pi^2 S} g(iT) \equiv y . \quad (16)$$

Hence, at leading order the Calabi-Yau coordinate y precisely corresponds to the invariant dilaton defined by eq. (16) and therefore is modular invariant.

3. So far we have concentrated on the duality in the gauge couplings of the two vacua. It is possible to extend the analysis and show the duality also between the gravitational couplings of the vector multiplets. In $N = 2$ supergravity a particular combination of higher derivative curvature terms (including $R\tilde{R}$) reside in the square of the (chiral) Weyl superfield [17]. Its coupling to the (abelian) vector multiplets is governed by a holomorphic function F_1 . In type II vacua F_1^{II} is only generated at the one-string loop level and in ref. [18] a general prescription for its computation in terms of topological amplitudes was given. For the particular Calabi-Yau threefold $X_{12}(1, 1, 2, 2, 6)$ F_1^{II} has the expansion

$$F_1^{\text{II}} = -\frac{2\pi i}{12}(52t_1 + 24t_2) + \sum_{jk} \left[2d_{jk} \ln \eta_0(q_1^j q_2^k) + \frac{1}{6} n_{jk} \ln(1 - q_1^j q_2^k) \right] , \quad (17)$$

where $\eta_0(q) = \prod_1^\infty (1 - q^n) = q^{-1/24} \eta(q)$. The first few coefficients d_{jk}, n_{jk} have been explicitly computed in ref. [8] .

If the proposed duality is to hold it should be possible to identify the same coupling also in the heterotic vacuum. In complete analogy with eq. (1) F_1^{het} has a weak coupling expansion

$$F_1^{\text{het}} = 24 S + h_1(T) + \text{non-perturbative} , \quad (18)$$

where the factor of 24 is the standard normalization of the curvature couplings. As before $h_1(T)$ is strongly constrained by its modular properties and its singularities on the moduli space. Near $T \sim 1$ the singular contribution to $h_1(T)$

coincides with the correction for $\partial_T^2 h$ since no additional gauge singlets become massless and we have

$$h_1 = \frac{1}{4\pi^2} \ln[j(iT) - j(i)] + \text{finite terms} . \quad (19)$$

On the other hand the modular transformation properties of $h_1(T)$ are determined from the holomorphic or modular anomaly of this coupling [19,20,18]. Repeating the analysis for the gravitational couplings of the vector multiplets we find that the non-holomorphic coupling

$$g_{\text{grav}}^{-2} = \text{Re} F_1^{\text{het}}(S, T) + \frac{b_{\text{grav}}}{16\pi^2} \left(\log \frac{M_{\text{Pl}}^2}{p^2} + K(S, T) \right) \quad (20)$$

has to be modular invariant. Here $b_{\text{grav}} = 2[n_H - (n_V - 1) + 22] = 300$ is the one-loop coefficient of the ‘gravitational β -function’ [20] and K is the Kähler potential given by

$$K = -\ln(S + \bar{S} - V_{\text{GS}}) - 2 \ln(T + \bar{T}) , \quad (21)$$

where $V_{\text{GS}} = 4(T + \bar{T})^{-2}(h + \bar{h}) - 2(T + \bar{T})^{-1}(\partial_T h + \partial_{\bar{T}} \bar{h}) .$

The requirement of keeping g_{grav}^{-2} modular invariant uniquely determines

$$F_1^{\text{het}} = 24 S^{\text{inv}} + \frac{1}{4\pi^2} \ln[j(iT) - j(i)] - \frac{300}{4\pi^2} \ln \eta^2(iT) . \quad (22)$$

We compared eqs. (22) and (17) (using an appropriate normalization) as a power series in q_1 and found agreement up to fourth order. This is a further check of the duality between the two string vacua.

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