

Fusion of Dilute A_L Lattice Models

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Abstract

The fusion procedure is implemented for the dilute A_L lattice models and a fusion hierarchy of functional equations with an $su(3)$ structure is derived for the fused transfer matrices. We also present the Bethe ansatz equations for the dilute A_L lattice models and discuss their connection with the fusion hierarchy. The solution of the fusion hierarchy for the eigenvalue spectra of the dilute A_L lattice models will be presented in a subsequent paper.

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1 Introduction

The dilute $A-D-E$ lattice models [1, 2] are exactly solvable [3] restricted solid-on-solid (RSOS) models on the square lattice. These models resemble the $A-D-E$ lattice models of Pasquier [4] in that the spins or heights take their values on a Dynkin diagram of a classical $A-D-E$ Lie algebra. The properties of these two families of $A-D-E$ models,

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however, are quite distinct and recent studies have shown that the dilute A - D - E models exhibit some new and very interesting aspects.

First and foremost, in contrast to the A_L model of Andrews, Baxter and Forrester [5], the dilute A_L models, with L odd, can be solved [6] off-criticality in the presence of a symmetry-breaking field. In particular, in an appropriate regime, the dilute A_3 model lies in the universality class of the Ising model in a magnetic field and gives the magnetic exponent $\delta = 15$ [6]. In addition, Zamolodchikov [7] has argued that the magnetic Ising model in the scaling region is described by an E_8 scattering theory. Accordingly, related E_8 structures have recently been uncovered [8, 9] in the dilute A_3 model. Lastly, the dilute A - D - E lattice models give [10] lattice realizations of the complete unitary minimal series of conformal field theories [11]. This again is not the case for the A - D - E models of Pasquier.

An important step in the study of the A_L models of Andrews, Baxter and Forrester (ABF) was the fusion [12] of the elementary weights to form new solutions of the Yang-Baxter equations. Subsequently, it was shown [13] that the transfer matrices of the fused ABF models satisfy special functional relations which can be solved for the eigenvalue spectra of these models. Moreover, at criticality, these equations can be solved for the central charges [13] and conformal weights [14].

Following the developments for the A_L models of Andrews, Baxter and Forrester, we carry out in this paper the fusion procedure for the dilute A_L lattice models and derive a fusion hierarchy of functional relations satisfied by the fused transfer matrices. The solution of this fusion hierarchy for the eigenvalue spectra, central charges and conformal weights will be given in a subsequent paper [15].

Historically, the fusion procedure was first introduced in [16]. It has since been successfully applied [17, 12, 18, 19, 20, 26] to many solvable models in two-dimensional statistical mechanics.

The layout of this paper is as follows. In section 1.1 we define the dilute lattice models. In section 1.2 we present the Bethe ansatz for the commuting transfer matrices. Then in section 1.3 we discuss the fusion hierarchy and its connection to the Bethe ansatz. In section 2 we construct the fused face weights for elementary fusion. In section 3 we give in detail the procedure for constructing the completely symmetrically, $(n, 0)$ and antisymmetrically $(0, n)$ fused face weights. This is generalized in section 4 to construct the fused face weights for arbitrary fusion of mixed type (n, m) . Finally, in section 5, we present the fusion hierarchy of functional relations and Bethe ansatz for the general fusion of mixed type (n, m) . We summarize our results in section 6.

1.1 Dilute A_L Lattice Models

The dilute A_L lattice models [1] are restricted solid-on-solid (RSOS) models with L heights built on the A_L Dynkin diagram as shown in Fig 1(a). The elements $A_{a,b}$ of the adjacency matrix for this diagram are given by

$$A_{a,b} = A_{b,a} = \begin{cases} 1, & |a-b| = 1 \\ 0, & \text{otherwise.} \end{cases} \quad (1.1)$$

The face weights of the dilute A_L models are

$$\begin{aligned} W\left(\begin{array}{cc|c} a & a & u \\ a & a & \end{array}\right) &= \frac{\vartheta_1(6\lambda - u)\vartheta_1(3\lambda + u)}{\vartheta_1(6\lambda)\vartheta_1(3\lambda)} \\ &\quad - \left(\frac{S_{a+1}}{S_a} \frac{\vartheta_4(2a\lambda - 5\lambda)}{\vartheta_4(2a\lambda + \lambda)} + \frac{S_{a-1}}{S_a} \frac{\vartheta_4(2a\lambda + 5\lambda)}{\vartheta_4(2a\lambda - \lambda)}\right) \frac{\vartheta_1(u)\vartheta_1(3\lambda - u)}{\vartheta_1(6\lambda)\vartheta_1(3\lambda)} \\ W\left(\begin{array}{cc|c} a \pm 1 & a & u \\ a & a \pm 1 & \end{array}\right) &= W\left(\begin{array}{cc|c} a & a & u \\ a & a \pm 1 & \end{array}\right) = \frac{\vartheta_1(3\lambda - u)\vartheta_4(\pm 2a\lambda + \lambda - u)}{\vartheta_1(3\lambda)\vartheta_4(\pm 2a\lambda + \lambda)} \\ W\left(\begin{array}{cc|c} a & a & u \\ a \pm 1 & a & \end{array}\right) &= W\left(\begin{array}{cc|c} a & a \pm 1 & u \\ a & a & \end{array}\right) = \left(\frac{S_{a\pm 1}}{S_a}\right)^{1/2} \frac{\vartheta_1(u)\vartheta_4(\pm 2a\lambda - 2\lambda + u)}{\vartheta_1(3\lambda)\vartheta_4(\pm 2a\lambda + \lambda)} \\ W\left(\begin{array}{cc|c} a & a \pm 1 & u \\ a & a \pm 1 & \end{array}\right) &= W\left(\begin{array}{cc|c} a \pm 1 & a \pm 1 & u \\ a & a & \end{array}\right) \\ &= \left(\frac{\vartheta_4(\pm 2a\lambda + 3\lambda)\vartheta_4(\pm 2a\lambda - \lambda)}{\vartheta_4^2(\pm 2a\lambda + \lambda)}\right)^{1/2} \frac{\vartheta_1(u)\vartheta_1(3\lambda - u)}{\vartheta_1(2\lambda)\vartheta_1(3\lambda)} \quad (1.2) \\ W\left(\begin{array}{cc|c} a \pm 1 & a & u \\ a & a \mp 1 & \end{array}\right) &= \frac{\vartheta_1(2\lambda - u)\vartheta_1(3\lambda - u)}{\vartheta_1(2\lambda)\vartheta_1(3\lambda)} \\ W\left(\begin{array}{cc|c} a & a \mp 1 & u \\ a \pm 1 & a & \end{array}\right) &= -\left(\frac{S_{a-1}S_{a+1}}{S_a^2}\right)^{1/2} \frac{\vartheta_1(u)\vartheta_1(\lambda - u)}{\vartheta_1(2\lambda)\vartheta_1(3\lambda)} \\ W\left(\begin{array}{cc|c} a & a \pm 1 & u \\ a \pm 1 & a & \end{array}\right) &= \frac{\vartheta_1(3\lambda - u)\vartheta_1(\pm 4a\lambda + 2\lambda + u)}{\vartheta_1(3\lambda)\vartheta_1(\pm 4a\lambda + 2\lambda)} + \frac{S_{a\pm 1}}{S_a} \frac{\vartheta_1(u)\vartheta_1(\pm 4a\lambda - \lambda + u)}{\vartheta_1(3\lambda)\vartheta_1(\pm 4a\lambda + 2\lambda)} \\ &= \frac{\vartheta_1(3\lambda + u)\vartheta_1(\pm 4a\lambda - 4\lambda + u)}{\vartheta_1(3\lambda)\vartheta_1(\pm 4a\lambda - 4\lambda)} \\ &\quad + \left(\frac{S_{a\mp 1}}{S_a} \frac{\vartheta_1(4\lambda)}{\vartheta_1(2\lambda)} - \frac{\vartheta_4(\pm 2a\lambda - 5\lambda)}{\vartheta_4(\pm 2a\lambda + \lambda)}\right) \frac{\vartheta_1(u)\vartheta_1(\pm 4a\lambda - \lambda + u)}{\vartheta_1(3\lambda)\vartheta_1(\pm 4a\lambda - 4\lambda)} \end{aligned}$$

Here the crossing factors are

$$S_a = (-1)^a \frac{\vartheta_1(4a\lambda)}{\vartheta_4(2a\lambda)} \quad (1.3)$$

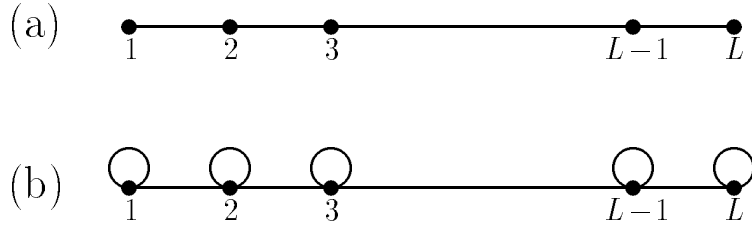


Figure 1: The A_L Dynkin diagram (a) and the effective adjacency diagram of the dilute A_L models (b).

and $\vartheta_1(u)$, $\vartheta_4(u)$ are standard theta functions of nome p with $|p| < 1$. Note that the effective adjacency matrix in the face weights is $I + A$, see Fig 1(b). We denote by $\text{val}(a)$ the number of allowed neighbours of height a ,

$$\text{val}(a) = \sum_b (I + A)_{a,b}. \quad (1.4)$$

Obviously, we have $\text{val}(a) \leq 3$ for the dilute A_L models. The weights (1.2) are normalized such that

$$W\left(\begin{array}{cc|c} d & c & 0 \\ a & b & \end{array}\right) = (I + A)_{a,b} (I + A)_{a,d} \delta_{a,c}, \quad (1.5)$$

are invariant under reflections along the diagonals

$$W\left(\begin{array}{cc|c} d & c & u \\ a & b & \end{array}\right) = W\left(\begin{array}{cc|c} d & a & u \\ c & b & \end{array}\right) = W\left(\begin{array}{cc|c} b & c & u \\ a & d & \end{array}\right) = W\left(\begin{array}{cc|c} b & a & u \\ c & d & \end{array}\right) \quad (1.6)$$

and satisfy the crossing symmetry

$$W\left(\begin{array}{cc|c} d & c & 3\lambda - u \\ a & b & \end{array}\right) = \left(\frac{S_a S_c}{S_b S_d}\right)^{1/2} W\left(\begin{array}{cc|c} c & b & u \\ d & a & \end{array}\right). \quad (1.7)$$

The dilute models admit four different physical branches. The spectral parameter u and the crossing parameter λ in the four branches take values

$$\begin{array}{lll} \text{branch 1:} & 0 < u < 3\lambda & \lambda = \frac{\pi}{4} \frac{L}{L+1} \\ \text{branch 2:} & 0 < u < 3\lambda & \lambda = \frac{\pi}{4} \frac{L+2}{L+1} \\ \text{branch 3:} & -\pi + 3\lambda < u < 0 & \lambda = \frac{\pi}{4} \frac{L+2}{L+1} \\ \text{branch 4:} & -\pi + 3\lambda < u < 0 & \lambda = \frac{\pi}{4} \frac{L}{L+1} \end{array} \quad (1.8)$$

At criticality, the face weights simplify to [1, 2]

$$\begin{aligned}
W\left(\begin{array}{cc|c} d & c & \\ a & b & u \end{array}\right) &= \begin{array}{c} d \\ \square \\ a \end{array} \begin{array}{c} c \\ \\ b \end{array} = \rho_1(u)\delta_{a,b,c,d} + \rho_2(u)\delta_{a,b,c}A_{a,d} + \rho_3(u)\delta_{a,c,d}A_{a,b} \\
&+ \sqrt{\frac{S_a}{S_b}}\rho_4(u)\delta_{b,c,d}A_{a,b} + \sqrt{\frac{S_c}{S_a}}\rho_5(u)\delta_{a,b,d}A_{a,c} + \rho_6(u)\delta_{a,b}\delta_{c,d}A_{a,c} \\
&+ \rho_7(u)\delta_{a,d}\delta_{c,b}A_{a,b} + \rho_8(u)\delta_{a,c}A_{a,b}A_{a,d} + \sqrt{\frac{S_a S_c}{S_b S_d}}\rho_9(u)\delta_{b,d}A_{a,b}A_{b,c}
\end{aligned} \tag{1.9}$$

with

$$\begin{aligned}
\rho_1(u) &= \frac{\sin 2\lambda \sin 3\lambda + \sin u \sin(3\lambda - u)}{\sin 2\lambda \sin 3\lambda} \\
\rho_2(u) &= \rho_3(u) = \frac{\sin(3\lambda - u)}{\sin 3\lambda} \\
\rho_4(u) &= \rho_5(u) = \frac{\sin u}{\sin 3\lambda} \\
\rho_6(u) &= \rho_7(u) = \frac{\sin u \sin(3\lambda - u)}{\sin 2\lambda \sin 3\lambda} \\
\rho_8(u) &= \frac{\sin(2\lambda - u) \sin(3\lambda - u)}{\sin 2\lambda \sin 3\lambda} \\
\rho_9(u) &= -\frac{\sin u \sin(\lambda - u)}{\sin 2\lambda \sin 3\lambda}.
\end{aligned} \tag{1.10}$$

Moreover, the crossing factors reduce to the nonnegative elements of the Perron-Frobenius eigenvector of the adjacency matrix given by

$$\sum_b A_{a,b} S_b = 2 \cos\left(\frac{\pi}{L+1}\right) S_a. \tag{1.11}$$

1.2 Commuting Transfer Matrices and Bethe Ansatz Equations

The dilute A_L models are exactly solvable because their face weights satisfy the Yang-Baxter equations

$$\begin{aligned}
&\sum_{g=1}^L W\left(\begin{array}{cc|c} f & g & \\ a & b & u \end{array}\right) W\left(\begin{array}{cc|c} e & d & \\ f & g & v \end{array}\right) W\left(\begin{array}{cc|c} d & c & \\ g & b & v-u \end{array}\right) \\
&= \sum_{g=1}^L W\left(\begin{array}{cc|c} e & g & \\ f & a & v-u \end{array}\right) W\left(\begin{array}{cc|c} g & c & \\ a & b & v \end{array}\right) W\left(\begin{array}{cc|c} e & d & \\ g & c & u \end{array}\right)
\end{aligned} \tag{1.12}$$

Diagrammatically, this equation is represented as follows

$$(1.13)$$

where the solid circle denotes summation over heights. This implies that the row transfer matrices $\mathbf{T}(u)$ commute. Here the elements of $\mathbf{T}(u)$ are given by

$$\langle \sigma | \mathbf{T}(u) | \sigma' \rangle = \prod_{j=1}^N W \left(\begin{array}{cc|c} \sigma'_j & \sigma'_{j+1} & u \\ \sigma_j & \sigma_{j+1} & \end{array} \right) \quad (1.14)$$

where the paths $\sigma = \{\sigma_1, \sigma_2, \dots, \sigma_N\}$ and $\sigma' = \{\sigma'_1, \sigma'_2, \dots, \sigma'_N\}$ are allowed configurations of heights along a row with periodic boundary conditions $\sigma_{N+1} = \sigma_1$ and $\sigma'_{N+1} = \sigma'_1$.

The eigenvalues $T(u)$ of the row transfer matrices $\mathbf{T}(u)$ can be calculated using a Bethe ansatz. Explicitly, the eigenvalues are given by [8]

$$T(u) = \omega \frac{s(u-2\lambda)s(u-3\lambda)}{s(2\lambda)s(3\lambda)} \frac{Q(u+\lambda)}{Q(u-\lambda)} + \frac{s(u)s(3\lambda-u)}{s(2\lambda)s(3\lambda)} \frac{Q(u)Q(u-3\lambda)}{Q(u-\lambda)Q(u-2\lambda)} + \omega^{-1} \frac{s(u)s(u-\lambda)}{s(2\lambda)s(3\lambda)} \frac{Q(u-4\lambda)}{Q(u-2\lambda)} \quad (1.15)$$

where

$$s(u) = \vartheta_1(u)^N, \quad Q(u) = \prod_{j=1}^N \vartheta_1(u - u_j) \quad (1.16)$$

and the zeros $\{u_j\}$ satisfy the Bethe ansatz equations

$$\omega^{-1} \left[\frac{\vartheta_1(u_j + \lambda)}{\vartheta_1(u_j - \lambda)} \right]^N = - \prod_{k=1}^N \frac{\vartheta_1(u_j - u_k + 2\lambda)\vartheta_1(u_j - u_k - \lambda)}{\vartheta_1(u_j - u_k - 2\lambda)\vartheta_1(u_j - u_k + \lambda)} \quad (1.17)$$

with $j = 1, \dots, N$ and $\omega = \exp(i\pi\ell/(L+1))$, $\ell = 1, \dots, L$. At criticality, these equations reduce, apart from the phase factors, to the Bethe ansatz equations of the Izergin-Korepin model [21, 22]. The Bethe ansatz equations ensure that the eigenvalues $T(u)$ are entire functions of u .

1.3 Fusion Hierarchy

Before discussing the fusion hierarchy, we recall some basic facts concerning $su(3)$. Let (n, m) , where n and m are nonnegative integers, denote the highest weight irreducible

representations of $su(3)$. Then the decomposition of the basic tensor product representations into irreducible representations is given by

$$\begin{aligned}(n, m) \otimes (1, 0) &= (n + 1, m) \oplus (n - 1, m + 1) \oplus (n, m - 1) \\ (n, m) \otimes (0, 1) &= (n, m + 1) \oplus (n + 1, m - 1) \oplus (n - 1, m).\end{aligned}\tag{1.18}$$

The irreducible representations can be represented by Young tableaux

$$(n, m) = \begin{array}{|c|c|c|c|c|} \hline & \cdots & & \cdots & \\ \hline & \cdots & & \cdots & \\ \hline \end{array} \tag{1.19}$$

$\underbrace{\hspace{1.5cm}}_m \quad \underbrace{\hspace{1.5cm}}_n$

so that, for example,

$$\begin{array}{c} \begin{array}{|c|c|c|c|c|} \hline & \cdots & & \cdots & \\ \hline & \cdots & & \cdots & \\ \hline \end{array} \otimes \begin{array}{|c|} \hline \\ \hline \end{array} = \begin{array}{|c|c|c|c|c|} \hline & \cdots & & \cdots & \\ \hline & \cdots & & \cdots & \\ \hline \end{array} \\ \oplus \begin{array}{|c|c|c|c|c|} \hline & \cdots & & \cdots & \\ \hline & \cdots & & \cdots & \\ \hline \end{array} \oplus \begin{array}{|c|c|c|c|c|} \hline & \cdots & & \cdots & \\ \hline & \cdots & & \cdots & \\ \hline \end{array} \end{array} \tag{1.20}$$

$\underbrace{\hspace{1.5cm}}_m \quad \underbrace{\hspace{1.5cm}}_n \quad \underbrace{\hspace{1.5cm}}_{n+1} \quad \underbrace{\hspace{1.5cm}}_{m+1} \quad \underbrace{\hspace{1.5cm}}_{n-1} \quad \underbrace{\hspace{1.5cm}}_{m-1} \quad \underbrace{\hspace{1.5cm}}_n$

where the triple box in a column corresponds to the trivial representation and can be omitted. The relation between these representations is encapsulated in the $su(3)$ weight lattice, which is shown in Fig 2 for the case of level $l = m + n = 5$.

The Bethe ansatz equations (1.15) can be thought of as matrix equations in $\mathbf{T}(u)$ and an auxiliary matrix family $\mathbf{Q}(u)$ which commutes with $\mathbf{T}(u)$. These matrix equations imply the fusion hierarchy

$$\mathbf{T}_0^{(n,0)} \mathbf{T}_n^{(1,0)} = \mathbf{T}_0^{(n-1,1)} + \mathbf{T}_0^{(n+1,0)} \tag{1.21}$$

$$\mathbf{T}_1^{(0,m)} \mathbf{T}_0^{(0,1)} = \mathbf{T}_0^{(0,m+1)} + f_0 \mathbf{T}_1^{(1,m-1)} \tag{1.22}$$

$$\mathbf{T}_0^{(n,0)} \mathbf{T}_n^{(0,m)} = \mathbf{T}_0^{(n,m)} + f_{n-1} \mathbf{T}_0^{(n-1,0)} \mathbf{T}_{n+1}^{(0,m-1)} \tag{1.23}$$

where $\mathbf{T}_k^{(n,m)} = \mathbf{T}^{(n,m)}(u + 2k\lambda)$ is the row transfer matrix of the fused model of fusion type $(1, 0) \times (n, m)$ in the horizontal and vertical directions. Here we have suppressed the horizontal fusion level and in this case $\mathbf{T}_k^{(0,0)} = \mathbf{I}$, $\mathbf{T}_k^{(1,0)} = \mathbf{T}(u + 2k\lambda)$ and $f_k = f(u + 2k\lambda)$ with

$$f(u) = (-1)^N \frac{s(u - 3\lambda)s(u - 2\lambda)s(u - \lambda)s(u + 2\lambda)s(u + 3\lambda)s(u + 4\lambda)}{s(2\lambda)^3 s(3\lambda)^3}. \tag{1.24}$$

We will later show that the fusion equations (1.21)–(1.23) hold for arbitrary fusion of type $(n', m') \times (n, m)$, again with the horizontal fusion type suppressed. In this general

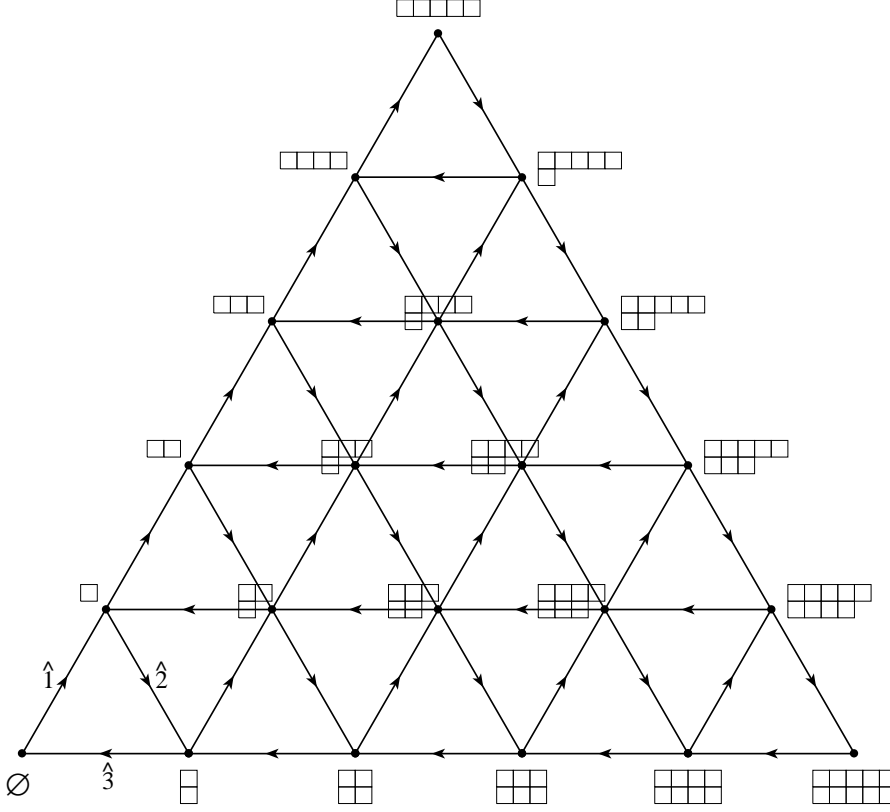


Figure 2: The $su(3)$ weight lattice at level $l = 5$. Each dot corresponds to a member of the fusion hierarchy of the dilute A_3 model labeled by the Young diagram of the respective representation of $su(3)$.

case, the fused face weights involve a rectangular block of $(n' + 2m') \times (n + 2m)$ elementary faces. For the moment, however, we only consider $(n', m') = (1, 0)$ for simplicity.

To derive the fusion hierarchy, we use semi-standard Young tableaux [23, 13, 24] and set

$$\begin{aligned}
 \boxed{1}^k &= \omega \frac{s(u + 2k\lambda - 2\lambda)s(u + 2k\lambda - 3\lambda)}{s(2\lambda)s(3\lambda)} \frac{Q(u + 2k\lambda + \lambda)}{Q(u + 2k\lambda - \lambda)} \\
 \boxed{2}^k &= \frac{s(u + 2k\lambda)s(u + 2k\lambda - 3\lambda)}{s(2\lambda)s(3\lambda)} \frac{Q(u + 2k\lambda)Q(u + 2k\lambda - 3\lambda)}{Q(u + 2k\lambda - \lambda)Q(u + 2k\lambda - 2\lambda)} \\
 \boxed{3}^k &= \omega^{-1} \frac{s(u + 2k\lambda)s(u + 2k\lambda - \lambda)}{s(2\lambda)s(3\lambda)} \frac{Q(u + 2k\lambda - 4\lambda)}{Q(u + 2k\lambda - 2\lambda)}
 \end{aligned} \tag{1.25}$$

so that

$$T_0^{(1,0)} = \boxed{1}^0 + \boxed{2}^0 + \boxed{3}^0 = \sum \square^0 \tag{1.26}$$

where such summations are performed over all allowed numberings of the boxes using the numbers 1, 2, and 3. For a general Young tableau, the numbers must not decrease

moving to the right along a row and must strictly increase moving down a column:

$$\begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 2 & 2 & 3 \\ \hline 2 & 3 & 3 & & \\ \hline \end{array} \tag{1.27}$$

Such a Young tableau denotes the product of the eight labeled boxes as given by (1.25) where it is understood that the relative shifts in the arguments are given by

$$\begin{array}{|c|c|c|c|c|} \hline u+8\lambda & u+6\lambda & u+4\lambda & u+2\lambda & u \\ \hline u+10\lambda & u+8\lambda & u+6\lambda & & \\ \hline \end{array} \tag{1.28}$$

and the zero superscript gives the shift in the top right box.

Using this notation, the eigenvalues of the fused row transfer matrix at level (n, m) can be written as

$$T_0^{(n,m)} = \sum \underbrace{\begin{array}{|c|c|c|} \hline \cdots & \cdots & \cdots \\ \hline \cdots & \cdots & \cdots \\ \hline \end{array}}_m \underbrace{\begin{array}{|c|c|c|} \hline \cdots & \cdots & \cdots \\ \hline \cdots & \cdots & \cdots \\ \hline \end{array}}_n \tag{1.29}$$

where the number of terms in the sum is given by the dimension of the irreducible representations of $su(3)$

$$\dim(n, m) = (n + 1)(m + 1)(n + m + 2)/2. \tag{1.30}$$

It is straightforward to show that these satisfy the fusion equations (1.21)–(1.23) with

$$f_0 = \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}^0, \quad T_0^{(0,1)} = \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array}^0 + \begin{array}{|c|} \hline 1 \\ \hline 3 \\ \hline \end{array}^0 + \begin{array}{|c|} \hline 2 \\ \hline 3 \\ \hline \end{array}^0 = (-1)^N \frac{s(u+2\lambda)s(u-3\lambda)}{s(2\lambda)s(3\lambda)} T_{1/2}^{(1,0)} \tag{1.31}$$

The product of factors on the left side of the fusion equations can be precisely partitioned into the two terms on the right side. Although it is not evident from these equations, the $T_0^{(n,m)}$ inductively defined are in fact entire functions of u . Furthermore, these equations close at level $n + m = 2L$ with

$$T_0^{(n,m)} = 0 \quad \text{if } n + m \geq 2L. \tag{1.32}$$

These facts will be established for the dilute A_L lattice models by carrying out the fusion procedure directly at the level of the face weights in the sequel.

The fusion equations completely determine the fused transfer matrices. Indeed, the solution of (1.21)–(1.23) for the fused transfer matrices $T_0^{(n,m)}$ can be written in the

determinantal form

$$T_0^{(n,m)} = \left| \begin{array}{cccccccc}
 T_{n+m-1}^{(0,1)} & f_{n+m-2} & & & & & & \\
 T_{n+m-1}^{(1,0)} & \ddots & \ddots & & & & & \\
 1 & \ddots & \ddots & \ddots & & & & \\
 & \ddots & \ddots & \ddots & f_n & & & \\
 & & 1 & T_{n+1}^{(1,0)} & T_n^{(0,1)} & f_{n-1} & & \\
 \hline
 & & & & 1 & T_{n-1}^{(1,0)} & T_{n-2}^{(0,1)} & f_{n-3} \\
 & & & & & 1 & \ddots & \ddots & \ddots \\
 & & & & & & \ddots & \ddots & \ddots & f_0 \\
 & & & & & & & 1 & T_1^{(1,0)} & T_0^{(0,1)} \\
 & & & & & & & & 1 & T_0^{(1,0)}
 \end{array} \right| \quad (1.33)$$

This can be directly verified using standard properties of determinants. In particular, it immediately follows from (1.15) and (1.31) that $T_0^{(n,m)}$ are entire functions of u .

2 Elementary Fusion

Fusion is a process [16] to build up new solutions of the Yang-Baxter equation. The essential idea is to form Z -invariant [25, 3] $p \times q$ blocks of elementary face weights. Then new solutions of the Yang-Baxter equation with distinct critical behavior are obtained by applying suitable projectors. To fuse the dilute A_L models, we will follow the detailed methods of Zhou and Pearce [20]. However, we present the basic example of 1×2 fusion in some detail to allow us to introduce our notation properly and to keep the paper self-contained.

In this section we consider the elementary fusion of a row of two and three faces corresponding to fusion levels $(2, 0)$, $(0, 1)$ and $(0, 0)$, the latter corresponding to a Young diagram with three vertically arranged boxes, which in the $su(3)$ case reduces to the trivial representation. Thereafter, in the following section, we treat the more general case of level $(n, 0)$ and $(0, n)$ fusion, in both horizontal and vertical directions. Subsequently, in section 4, we present the general case of fusion level (n, m) .

2.1 Projectors

Let us define local face transfer operators $X_j(u)$ with elements

$$\langle \sigma | X_j(u) | \sigma' \rangle = W \left(\begin{array}{cc|c} \sigma_{j-1} & \sigma'_j & u \\ \sigma_j & \sigma_{j+1} & \end{array} \right) \prod_{k \neq j} \delta_{\sigma_k, \sigma'_k} \quad (2.1)$$

where σ and σ' are allowed paths. The matrix $X_j(u)$ is block diagonal and we denote the blocks for fixed j by

$$X^{(b,d)}(u) = \begin{array}{c} d \\ \diagdown \quad \diagup \\ u \\ \diagup \quad \diagdown \\ b \end{array} \quad (2.2)$$

The dimension of this block is given by the number of allowed two-step paths from d to b which is $[(I + A)^2]_{b,d} \leq 3$.

In addition to the Yang-Baxter equation, the face weights of the dilute A_L models also satisfy the local inversion relation

$$\sum_{g=1}^L W \left(\begin{array}{cc|c} d & g & u \\ a & b & \end{array} \right) W \left(\begin{array}{cc|c} d & c & -u \\ g & b & \end{array} \right) = \begin{array}{c} d & & d \\ \diagdown & & \diagup \\ u & \bullet & -u \\ \diagup & & \diagdown \\ b & & b \end{array} = \rho(u)\rho(-u)\delta_{a,c} \quad (2.3)$$

where

$$\rho(u) = \frac{\vartheta_1(2\lambda - u)\vartheta_1(3\lambda - u)}{\vartheta_1(2\lambda)\vartheta_1(3\lambda)}. \quad (2.4)$$

In terms of matrix multiplication of the face transfer operators, this relation takes the form

$$X^{(b,d)}(u)X^{(b,d)}(-u) = \rho(u)\rho(-u)I. \quad (2.5)$$

It follows that the four block matrices

$$\begin{array}{c} d \\ \diagdown \quad \diagup \\ 3\lambda \\ \diagup \quad \diagdown \\ b \end{array}, \quad \begin{array}{c} d \\ \diagdown \quad \diagup \\ -3\lambda \\ \diagup \quad \diagdown \\ b \end{array}, \quad \begin{array}{c} d \\ \diagdown \quad \diagup \\ 2\lambda \\ \diagup \quad \diagdown \\ b \end{array}, \quad \begin{array}{c} d \\ \diagdown \quad \diagup \\ -2\lambda \\ \diagup \quad \diagdown \\ b \end{array} \quad (2.6)$$

are singular. In fact, after appropriate normalization, $X^{(b,d)}(3\lambda)$ and $X^{(b,d)}(2\lambda)$ are projection operators. Although the other two are not strictly projectors, we will subsequently refer to these four singular operators as projectors. Within each pair, these operators are orthogonal as a consequence of the inversion relation.

Either of the above pairs of projectors can be used to construct new solvable models by the fusion procedure [26]. These two types of fusion are very different in nature. The first pair of projectors leads to $su(2)$ type fusion with adjacency matrices $A^{(n)}$ of the fused lattice models at level n given by the $su(2)$ fusion rules

$$\begin{aligned} A^{(0)} &= I, & A^{(1)} &= I + A \\ A^{(n)}A^{(1)} &= A^{(n-1)} + A^{(n+1)} & n &= 1, 2, 3, \dots \end{aligned} \quad (2.7)$$

without any closure. The second pair of projectors leads to $su(3)$ type fusion with adjacency matrices $A^{(n,m)}$ of the fused lattice models at level (n, m) given by $su(3)$ fusion rules

$$\begin{aligned} A^{(n,m)} &= 0 \quad \text{if } n < 0 \text{ or } m < 0, & A^{(n,m)} &= A^{(m,n)} \\ A^{(0,0)} &= I, & A^{(1,0)} &= I + A \\ A^{(n,m)}A^{(1,0)} &= A^{(n+1,m)} + A^{(n-1,m+1)} + A^{(n,m-1)}, & n, m &= 0, 1, 2, \dots \end{aligned} \quad (2.8)$$

These equations close with

$$A^{(n,m)} = 0 \quad \text{for } n + m \geq 2L. \quad (2.9)$$

Note that the fusion hierarchy equations (1.21)–(1.23) yield valid adjacency equations if the fused transfer matrices are replaced by the fused adjacency matrices, the shifts are discarded and the functions f_k are set to one. The elements of the fused adjacency matrices can in general be nonnegative integers greater than one. In this case we distinguish the edges of the adjacency diagram joining two given sites by bond variables $\alpha = 1, 2, \dots$. If there is just one edge then the corresponding bond variable is $\alpha = 1$. More generally, the entries $A_{a,b}^{(n,m)}$ of the fused adjacency matrices give the number of admissible bonds joining states a and b in the fused models at level (n, m) . Explicitly, the fused adjacency matrices $A^{(n,m)} = A^{(m,n)}$ for A_3 are given by

$$\begin{aligned} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} &= A^{(0,0)} = A^{(5,0)} = A^{(0,5)} \\ \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} &= A^{(1,0)} = A^{(4,1)} = A^{(0,4)} = A^{(4,0)} = A^{(1,4)} = A^{(0,1)} \\ \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix} &= A^{(2,0)} = A^{(3,2)} = A^{(0,3)} = A^{(3,0)} = A^{(2,3)} = A^{(0,2)} \\ \begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 2 & 1 \end{pmatrix} &= A^{(1,1)} = A^{(3,1)} = A^{(1,3)} \end{aligned}$$

$$\begin{pmatrix} 1 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 1 \end{pmatrix} = A^{(1,2)} = A^{(2,1)} = A^{(2,2)} \quad (2.10)$$

where the \mathbb{Z}_3 symmetry of the weight lattice about the fusion level (m, n) is apparent, see Fig 2.

Let us list some properties of the projectors useful for the implementation of $su(3)$ fusion. These either follow from the explicit form of the face weights or from the inversion relation (2.3) with $u = -2\lambda$. The first group is obtained by inserting the explicit face weights at $u = -2\lambda$ for an given value of a . This gives

$$\begin{aligned} b_{\pm 1} \begin{array}{c} b_{\pm 2} \\ \diagup \quad \diagdown \\ 2\lambda \\ \diagdown \quad \diagup \\ b \end{array} b_{\pm 1} &= 0 \\ b \begin{array}{c} b_{\pm 1} \\ \diagup \quad \diagdown \\ 2\lambda \\ \diagdown \quad \diagup \\ b \end{array} c &= \sqrt{\frac{\vartheta_4(2b\lambda \mp \lambda)}{\vartheta_4(2b\lambda \pm 3\lambda)}} b_{\pm 1} \begin{array}{c} b_{\pm 1} \\ \diagup \quad \diagdown \\ 2\lambda \\ \diagdown \quad \diagup \\ b \end{array} c \\ b_{\pm 1} \begin{array}{c} b \\ \diagup \quad \diagdown \\ 2\lambda \\ \diagdown \quad \diagup \\ b \end{array} c &= A_{b_{\pm 1}, b} \frac{\vartheta_1(\lambda)}{\vartheta_1(2\lambda)} \frac{S(b_{\pm 1}, b)}{S(b, b)} b \begin{array}{c} b \\ \diagup \quad \diagdown \\ 2\lambda \\ \diagdown \quad \diagup \\ b \end{array} c \end{aligned} \quad (2.11)$$

for any value of c . Analogously, we obtain a second group of relations by inserting the explicit face weights at $u = 2\lambda$ for a given value of c . This gives

$$\begin{aligned} b_{\pm 1} \begin{array}{c} b_{\pm 2} \\ \diagup \quad \diagdown \\ -2\lambda \\ \diagdown \quad \diagup \\ b \end{array} b_{\pm 1} &= \frac{\vartheta_1(4\lambda)\vartheta_1(5\lambda)}{\vartheta_1(2\lambda)\vartheta_1(3\lambda)} \\ a \begin{array}{c} b_{\pm 1} \\ \diagup \quad \diagdown \\ -2\lambda \\ \diagdown \quad \diagup \\ b \end{array} b &= -\sqrt{\frac{\vartheta_4(2b\lambda \pm 3\lambda)}{\vartheta_4(2b\lambda \mp \lambda)}} a \begin{array}{c} b_{\pm 1} \\ \diagup \quad \diagdown \\ -2\lambda \\ \diagdown \quad \diagup \\ b \end{array} b_{\pm 1} \end{aligned} \quad (2.12)$$

$$A_{b, b-1} \frac{S(b-1, b)}{S(b+1, b)} a \begin{array}{c} b \\ \diagup \quad \diagdown \\ -2\lambda \\ \diagdown \quad \diagup \\ b \end{array} b-1 + \frac{\vartheta_1(2\lambda)}{\vartheta_1(\lambda)} \frac{S(b, b)}{S(b+1, b)} a \begin{array}{c} b \\ \diagup \quad \diagdown \\ -2\lambda \\ \diagdown \quad \diagup \\ b \end{array} b + A_{b, b+1} a \begin{array}{c} b \\ \diagup \quad \diagdown \\ -2\lambda \\ \diagdown \quad \diagup \\ b \end{array} b+1 = 0$$

where a is arbitrary. Here we introduced the compact notation

$$S(a, b) = \sqrt{S_a} \vartheta_4(2b\lambda + (b-a)\lambda). \quad (2.13)$$

2.2 Level (2,0) Fusion

The projector of symmetric 1×2 fusion is the local face transfer operator $X_j(-2\lambda)$, or more explicitly, the blocks $X^{(b,d)}(-2\lambda)$ in (2.2). The action of these projectors is to project out certain two-step paths from d to b . The number of remaining paths are then given by the entries of the fused adjacency matrix $A^{(2,0)}$.

To be more precise, let us denote the set of two-step paths (a, a', b) from a to b by $\text{path}(a, b; 2)$. We refer to a path (a, a', b) as *dependent* on a set of paths $\{(a, a'_i, b)\}$ (with respect to $X^{(b,a)}(-2\lambda)$) if there exist $\phi(a, a'_i, b)$ such that

$$c \begin{array}{c} a \\ \diagdown \quad \diagup \\ -2\lambda \\ \diagup \quad \diagdown \\ b \end{array} a' = \sum_i \phi(a, a'_i, b) c \begin{array}{c} a \\ \diagdown \quad \diagup \\ -2\lambda \\ \diagup \quad \diagdown \\ b \end{array} a'_i \quad (2.14)$$

holds for any c . In other words, a set of paths $\{(a, a'_i, b)\}$ is called *independent* if

$$\sum_i \phi(a, a'_i, b) c \begin{array}{c} a \\ \diagdown \quad \diagup \\ -2\lambda \\ \diagup \quad \diagdown \\ b \end{array} a'_i = 0 \quad (2.15)$$

implies that all coefficients vanish, i.e., $\phi(a, a'_i, b) = 0$. Note that this definition of dependent and independent paths obviously depends on the projector under consideration.

Given a dependent set of paths, the choice of a maximal independent subset is by no means unique. However, this does not matter for our purpose as different choices result in equivalent fused models related to each other by a local gauge transformation. This allows us to use arbitrary sets of independent paths in the construction of the fused face weights. We denote such a set of independent two-step paths from a to b (w.r.t. $X^{(b,a)}(-2\lambda)$) by $\text{indpath}_{(2,0)}[a, b]$. The actual number of these paths is of course determined by the number of non-zero eigenvalues⁵ of the block $X^{(b,a)}(-2\lambda)$. The eigenvectors for the zero eigenvalue of each block $X^{(b,a)}(-2\lambda)$ are given in the second and third equations in (2.12). As shown before, these immediately follow from the inversion relation. Note, however, that we do *not* have to use the explicit form of the eigenvectors for non-zero eigenvalues in what follows.

For our present discussion, this means that there is only one case where we have more than one independent path to consider, namely if $a = b$ and $\text{val}(a) = 3$, i.e., if $2 \leq a \leq L - 1$. In this case there are three allowed paths $(a, a - 1, a)$, (a, a, a) and

⁵Remember that $X^{(b,a)}(-2\lambda)$ is not strictly a projector, hence its eigenvalues do not have to be 0 or 1.

$(a, a + 1, a)$ which are dependent by (2.12), so one is left with two independent paths in this case. For the fused face weights, this implies that one has to consider two different kinds of bonds labeled by a *bond variable* α which takes two values $\alpha = 1, 2$ corresponding to the two independent paths.

The allowed two-step paths on the adjacency diagram of the dilute A_L models shown in Fig 1(b) are

$$\text{path}(a, b; 2) = \begin{cases} \{(a, a \pm 1, a \pm 2)\} & \text{if } b = a \pm 2 \\ \{(a, a, a \pm 1), (a, a \pm 1, a \pm 1)\} & \text{if } b = a \pm 1 \\ \{(1, 2, 1), (1, 1, 1)\} & \text{if } a = b = 1 \\ \{(L, L, L), (L, L - 1, L)\} & \text{if } a = b = L \\ \{(a, a + 1, a), (a, a, a), (a, a - 1, a)\} & \text{if } a = b, \text{val}(a) = 3 \end{cases} \quad (2.16)$$

From these we choose the independent paths for the construction of the $(2, 0)$ fused face weights as follows

$$\text{indpath}_{(2,0)}[a, b] = \begin{cases} \{(a, a \pm 1, a \pm 2)\} & \text{if } b = a \pm 2 \\ \{(a, a + 1, a + 1)\} & \text{if } b = a + 1 \\ \{(a, a, a - 1)\} & \text{if } b = a - 1 \\ \{(1, 2, 1)\} & \text{if } a = b = 1 \\ \{(L, L - 1, L)\} & \text{if } a = b = L \\ \{(a, a + 1, a), (a, a - 1, a)\} & \text{if } a = b, \text{val}(a) = 3 \end{cases} \quad (2.17)$$

or, in other words,

$$\text{indpath}_{(2,0)}[a, b] = \{(a, a', b) \in \text{path}(a, b; 2) \mid a' \neq \min(a, b)\} . \quad (2.18)$$

Obviously, the number of independent paths is

$$|\text{indpath}_{(2,0)}[a, b]| = A_{a,b}^{(2,0)} \quad (2.19)$$

where $A^{(2,0)}$ is the fused adjacency matrix of (2.8). We define $\phi_{(2,0)}(a, a', b|\alpha)$ as the coefficients of the path (a, a', b) in terms of the ‘‘basis’’ of independent paths $\text{indpath}_{(2,0)}[a, b]$, where $1 \leq \alpha \leq A_{a,b}^{(2,0)}$ labels the elements of $\text{indpath}_{(2,0)}[a, b]$, respectively. This means

$$\begin{array}{c} a \\ \diagdown \quad \diagup \\ c \quad -2\lambda \quad a' \\ \diagup \quad \diagdown \\ b \end{array} = \sum_{\alpha=1}^{A_{a,b}^{(2,0)}} \phi_{(2,0)}(a, a', b|\alpha) \begin{array}{c} a \\ \diagdown \quad \diagup \\ c \quad -2\lambda \quad a'_\alpha \\ \diagup \quad \diagdown \\ b \end{array} \quad (2.20)$$

where (a, a'_α, b) is the corresponding independent path. The coefficients can be read off

from (2.12), explicitly they are given by

$$\phi_{(2,0)}(a, a', b|\alpha) = \begin{cases} \delta_{\alpha,1} & \text{if } |a - b| = 2 \\ \left(-\sqrt{\frac{\vartheta_4(2a'_\alpha\lambda + \lambda)}{\vartheta_4(2a'_\alpha\lambda - 3\lambda)}}\right)^{a'_\alpha - a'} \delta_{\alpha,1} & \text{if } |a - b| = 1 \\ \left(-\frac{\vartheta_1(\lambda)}{\vartheta_1(2\lambda)} \frac{S(a'_\alpha, a)}{S(a, a)}\right)^{\delta_{\alpha, a'}} \delta_{\alpha,1} & \text{if } a = b, \text{val}(a) = 2 \\ \left(-\frac{\vartheta_1(\lambda)}{\vartheta_1(2\lambda)} \frac{S(a'_\alpha, a)}{S(a, a)}\right)^{\delta_{\alpha, a'}} & \text{if } a = b, \text{val}(a) = 3, |a'_\alpha - a'| < 2 \\ 0 & \text{otherwise} \end{cases} \quad (2.21)$$

where it is understood that $\phi_{(2,0)}(a, a', b|\alpha) = 0$ if $(a, a', b) \notin \text{path}(a, b; 2)$ and a'_α is defined as above.

To phrase it differently, and maybe more clearly, the blocks $X^{(b,a)}(-2\lambda)$ are square matrices in the basis given by $\text{path}(a, b; 2)$. From the inversion relation, one obtains eigenvectors of $X^{(b,a)}(-2\lambda)$ with eigenvalue 0 (if $|b - a| < 2$), (2.12) give their components in the basis of the paths up to an arbitrary normalization. The $\phi_{(2,0)}(a, a', b|\alpha)$ defined above are nothing else than the components of α linearly independent vectors which span the orthogonal complement of the zero eigenvalue eigenspace of $X^{(b,a)}(-2\lambda)$. Note that these are in general not eigenvectors of $X^{(b,a)}(-2\lambda)$ and therefore the vectors for different α are not necessarily mutually orthogonal; for instance, for $a = b$, $\text{val}(a) = 3$, the three vectors $|\alpha\rangle$ ($\alpha = 0, 1, 2$, $\alpha = 0$ denoting the eigenvector with zero eigenvalue) have the following components in the basis $\text{path}(a, a; 2)$ given in (2.16)

$$\begin{aligned} |0\rangle &= \left(S(a+1, a), \frac{\vartheta_1(2\lambda)}{\vartheta_1(\lambda)} S(a, a), S(a-1, a) \right) \\ |1\rangle &= \left(1, -\frac{\vartheta_1(\lambda)}{\vartheta_1(2\lambda)} \frac{S(a+1, a)}{S(a, a)}, 0 \right) \\ |2\rangle &= \left(0, -\frac{\vartheta_1(\lambda)}{\vartheta_1(2\lambda)} \frac{S(a-1, a)}{S(a, a)}, 1 \right) \end{aligned} \quad (2.22)$$

where $|1\rangle$ and $|2\rangle$ are both orthogonal to $|0\rangle$ but not orthogonal to each other. We can split any summation over paths (a, a', b) into a summation over the zero eigenvalue eigenvectors of $X^{(b,a)}(-2\lambda)$ and its orthogonal complement which yields a sum over α with coefficients $\phi_{(2,0)}(a, a', b|\alpha)$ by choosing a suitable basis. Clearly, (2.11) are just

$$\sum_{a'} \phi_{(2,0)}(a, a', b|\alpha) \begin{array}{c} a \\ \diagdown \quad \diagup \\ \quad 2\lambda \\ \diagup \quad \diagdown \\ b \end{array} = \alpha \begin{array}{c} a \\ \diagdown \quad \diagup \\ \quad 2\lambda \\ \diagup \quad \diagdown \\ b \end{array} = 0 \quad (2.23)$$

for $\alpha = 1, 2$ and any value of c . We refer to the sum with coefficients $\phi_{(2,0)}(a, a', b|\alpha)$ as the *symmetric sum* and denote it by a cross with label α . In particular, we find the following decomposition (“split property”)

$$\begin{aligned}
\begin{array}{c} d \\ \diagup \quad \diagdown \\ a \quad u \quad c' \\ \diagdown \quad \diagup \\ e \quad -2\lambda \quad a' \quad u+2\lambda \quad c \\ \diagdown \quad \diagup \\ b \quad \quad \quad b \end{array} &= \sum_{a'} e \begin{array}{c} a \\ \diagup \quad \diagdown \\ -2\lambda \\ \diagdown \quad \diagup \\ b \end{array} \begin{array}{c} d \quad c' \quad c \\ \diagdown \quad \diagup \\ u \quad u+2\lambda \\ \diagdown \quad \diagup \\ a \quad a' \quad b \end{array} \\
&= \sum_{a'} \sum_{\alpha} \phi_{(2,0)}(a, a', b|\alpha) e \begin{array}{c} a \\ \diagup \quad \diagdown \\ -2\lambda \\ \diagdown \quad \diagup \\ b \end{array} \begin{array}{c} d \quad c' \quad c \\ \diagdown \quad \diagup \\ u \quad u+2\lambda \\ \diagdown \quad \diagup \\ a \quad a' \quad b \end{array} \\
&= \sum_{\alpha} e \begin{array}{c} a \\ \diagup \quad \diagdown \\ -2\lambda \\ \diagdown \quad \diagup \\ b \end{array} \begin{array}{c} d \quad c' \quad c \\ \diagdown \quad \diagup \\ u \quad u+2\lambda \\ \diagdown \quad \diagup \\ a \quad a' \quad b \end{array} \quad (2.24)
\end{aligned}$$

where in the last step we performed the summation over a' . Here, the sum on the RHS includes only the independent paths (a, a', b) , $\alpha \in \{1, 2\}$.

After these preliminary remarks, we are finally in a position to define the fused face weights for the elementary symmetric fusion. These are basically the symmetric sums on the RHS of (2.24), but we still have to make a choice how to relate the bond variable β to the value of c' . In the case $c = d$ and $\text{val}(c) = 3$ this choice is not completely free. Due to our selection of independent paths we have to exclude $c' = c$, because the path (c, c, c) has non-zero coefficients in terms of the two independent paths. For definiteness and simplicity, we choose c' such that (d, c', c) is path labeled by β in the set $\text{indpath}_{(2,0)}[d, c]$.

Lemma 2.1 (Elementary Symmetric Fusion) *If (a, b) and (d, c) are admissible edges at fusion level $(2, 0)$, we define the 1×2 fused weights by*

$$\begin{aligned}
W_{(2,0)} \left(\begin{array}{c|c} d & \beta & c \\ a & \alpha & b \end{array} \middle| u \right) &= \begin{array}{c} d \quad \beta \quad c \\ \diagdown \quad \diagup \\ u \quad u+2\lambda \\ \diagdown \quad \diagup \\ a \quad \alpha \quad b \end{array} = \begin{array}{c} d \quad c'_{\beta} \quad c \\ \diagdown \quad \diagup \\ u \quad u+2\lambda \\ \diagdown \quad \diagup \\ a \quad \alpha \quad b \end{array} \\
&= \sum_{a'} \phi_{(2,0)}(a, a', b|\alpha) W \left(\begin{array}{c|c} d & c'_{\beta} \\ a & a' \end{array} \middle| u \right) W \left(\begin{array}{c|c} c'_{\beta} & c \\ a' & b \end{array} \middle| u+2\lambda \right) \quad (2.25)
\end{aligned}$$

where the sum is over all allowed spins a' , the bond variables take values $1 \leq \alpha \leq A_{a,b}^{(2,0)}$, $1 \leq \beta \leq A_{c,d}^{(2,0)}$, and where the coefficients $\phi_{(2,0)}(a, a', b|\alpha)$ are those of (2.21). Furthermore, the value of c' on the RHS is chosen such that $(d, c', c) \in \text{indpath}_{(2,0)}[d, c]$, with β being the label of this particular element of $\text{indpath}_{(2,0)}[d, c]$. The weights so

defined satisfy the Yang-Baxter equation. In particular, we note that

$$W_{(2,0)}\left(\begin{array}{ccc|c} d & \beta & c & 0 \\ a & \alpha & b & \end{array}\right) = W_{(2,0)}\left(\begin{array}{ccc|c} d & \beta & c & \lambda \\ a & \alpha & b & \end{array}\right) = 0 \quad (2.26)$$

for all $a, b, c, d, \alpha, \beta$.

To show that the Yang-Baxter equation is indeed satisfied, we proceed as follows. From the Yang-Baxter equation of the elementary weights and (2.23), one obtains

$$\begin{array}{c} d & & d \\ \diagdown & & / \\ a & u & c' \\ / & \bullet & \diagdown \\ \alpha' & & 2\lambda \\ \diagup & & \diagdown \\ & u+2\lambda & c \\ & & b \end{array} = \begin{array}{c} d & & \\ \diagdown & & / \\ a & u+2\lambda & e \\ / & \bullet & \diagdown \\ \alpha & & 2\lambda \\ \diagup & & \diagdown \\ & u & c \\ & & b \end{array} = 0 = \begin{array}{c} d & & d \\ \diagdown & & / \\ a & -2\lambda & c' \\ / & \bullet & \diagdown \\ & & c \\ \diagup & & \diagdown \\ & & c \end{array} \quad (2.27)$$

This establishes the ‘‘push-through’’ property, i.e., the dependence of the symmetric sum in (2.27) on the path (d, c', c) is obviously the same as that of the projector $X^{(c,d)}(-2\lambda)$, see (2.12). Explicitly, this yields

$$\begin{array}{c} c \pm 1 & & c \\ \diagdown & & / \\ a & u & c \\ / & \bullet & \diagdown \\ & & \alpha \\ \diagup & & \diagdown \\ & u+2\lambda & b \end{array} = -\sqrt{\frac{\vartheta_4(2c\lambda \pm 3\lambda)}{\vartheta_4(2c\lambda \mp \lambda)}} \begin{array}{c} c \pm 1 & & c \\ \diagdown & & / \\ a & u & c \\ / & \bullet & \diagdown \\ & & \alpha \\ \diagup & & \diagdown \\ & u+2\lambda & b \end{array} \quad (2.28)$$

$$A_{c,c-1} \frac{S(c-1, c)}{S(c+1, c)} \begin{array}{c} c-1 & & c \\ \diagdown & & / \\ a & u & c \\ / & \bullet & \diagdown \\ & & \alpha \\ \diagup & & \diagdown \\ & u+2\lambda & b \end{array} + \frac{\vartheta_1(2\lambda)}{\vartheta_1(\lambda)} \frac{S(c, c)}{S(c+1, c)} \begin{array}{c} c & & c \\ \diagdown & & / \\ a & u & c \\ / & \bullet & \diagdown \\ & & \alpha \\ \diagup & & \diagdown \\ & u+2\lambda & b \end{array} + A_{c,c+1} \begin{array}{c} c+1 & & c \\ \diagdown & & / \\ a & u & c \\ / & \bullet & \diagdown \\ & & \alpha \\ \diagup & & \diagdown \\ & u+2\lambda & b \end{array} = 0$$

or, for short,

$$\begin{array}{c} d & & c' \\ \diagdown & & / \\ a & u & c \\ / & \bullet & \diagdown \\ & & \alpha \\ \diagup & & \diagdown \\ & u+2\lambda & b \end{array} = \sum_{\beta} \phi_{(2,0)}(d, c', c|\beta) \begin{array}{c} d & & c' \\ \diagdown & & / \\ a & u & c \\ / & \bullet & \diagdown \\ & & \alpha \\ \diagup & & \diagdown \\ & u+2\lambda & b \end{array} \quad (2.29)$$

In complete analogy to the decomposition in (2.24), one finds

$$\begin{array}{c} f & & e' & & e \\ \diagdown & & / & & \\ d & v & c' & v+2\lambda & c \\ / & \bullet & \diagdown & & \\ & & u & u+2\lambda & \\ \diagup & & \diagdown & & \\ a & & & & b \end{array} = \sum_{\beta} \begin{array}{c} d & & c' \\ \diagdown & & / \\ a & u & c \\ / & \bullet & \diagdown \\ & & \alpha \\ \diagup & & \diagdown \\ & u+2\lambda & b \end{array} \times \begin{array}{c} f & & e' & & e \\ \diagdown & & / & & \\ d & v & & v+2\lambda & c \\ / & \bullet & \diagdown & & \\ & & & & \beta \\ \diagup & & \diagdown & & \\ & & & & c \end{array} \quad (2.30)$$

and therefore the Yang-Baxter equation for the fused weights follows immediately from that for the elementary faces (1.13). This also shows why it is most convenient to choose the c' in the definition of the fused weights (2.25) to correspond to an independent path and that one has to be careful if there is more than one independent path.

From the initial condition (1.5), the fused weights (2.25) at $u = 0$ reduce to the symmetric sum of the elementary face weight at $u = 2\lambda$ which vanishes due to (2.23). Similarly, one obtains the second part of (2.26) using the crossing symmetry (1.7). This implies that the fused weights contain an overall factor of $\vartheta_1(u)\vartheta_1(u - \lambda)$.

2.3 Level (0,1) Fusion

The antisymmetric 1×2 fusion is constructed by using the blocks $X^{(c,d)}(2\lambda)$ as projectors. This means that the fused weights are basically given by the products

Here, things are somewhat simpler than for the $(2, 0)$ fusion, since the blocks $X^{(c,d)}(2\lambda)$, up to normalization, really are projectors onto at most one-dimensional spaces. In particular, this implies that we do not need to introduce a bond variable.

Note that we define the $(0, 1)$ fusion by summing over the variable c' instead a' . This convention allows us to use the same product of elementary face weights in both cases (otherwise we would have to interchange u and $u + 2\lambda$). This also means that the definition of independent paths $\text{indpath}_{(0,1)}[d, c]$ apparently involves “the other side” of the projector which however makes no difference since the weights (1.2) are symmetric under reflection, see (1.6). Let us choose the following sets of independent paths

$$\text{indpath}_{(0,1)}[d, c] = \{(d, c', c) \in \text{path}(d, c; 2) \mid c' = \min(c, d)\} \quad (2.32)$$

which is just the complement of the set of independent paths for level $(2, 0)$ fusion, see (2.17)–(2.18). Clearly, the corresponding adjacency matrix $A^{(0,1)}$ for the fused weights coincides with that of the elementary face weights, i.e., $A^{(0,1)} = A^{(1,0)}$ (2.8). From (2.11), we obtain

$$\phi_{(0,1)}(d, c', c) = \begin{cases} \left(\frac{\vartheta_4(2c'_1\lambda + 3\lambda)}{\vartheta_4(2c'_1\lambda - \lambda)} \right)^{(c'-c'_1)/2} & \text{if } |c - d| = 1 \\ \left(\frac{\vartheta_1(\lambda)}{\vartheta_1(2\lambda)} \frac{S(c', d)}{S(d, d)} \right)^{|c'-d|} & \text{if } c = d \\ 0 & \text{otherwise} \end{cases} \quad (2.33)$$

for all $(d, c', c) \in \text{path}(d, c; 2)$ and zero otherwise, where $c'_1 = \min(c, d)$.

We refer to the sum over c' with coefficients $\phi_{(0,1)}(d, c', c)$ as an *antisymmetric sum* and denote it by a circle (without any further label) in our diagrammatical notation.

Obviously, (2.12) becomes

$$\sum_c \phi_{(0,1)}(d, c', c) \begin{array}{c} d \\ \diagdown \quad \diagup \\ a \quad -2\lambda \quad c' \\ \diagup \quad \diagdown \\ c \end{array} = \begin{array}{c} d \\ \diagdown \quad \diagup \\ a \quad -2\lambda \quad \bullet \\ \diagup \quad \diagdown \\ c \end{array} = 0 \quad (2.34)$$

and the split relation analogous to (2.24) is simply

$$\begin{array}{c} d \quad d \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \\ a \quad u \quad c' \quad 2\lambda \quad e \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ a' \quad u+2\lambda \quad c \\ b \end{array} = \begin{array}{c} d \\ \diagdown \quad \diagup \\ c' \quad 2\lambda \quad e \\ \diagup \quad \diagdown \\ c \end{array} \times \begin{array}{c} d \quad c \\ \diagdown \quad \diagup \\ a \quad u \quad a' \quad u+2\lambda \quad b \end{array} \quad (2.35)$$

where (d, c'_1, c) is the corresponding independent path, i.e., $c'_1 = \min(c, d)$. As before, we define the fused weights to be the object on the RHS of the above equation, up to a choice on the values of a' . By the same arguments as above, the dependence on a' can be read off directly from (2.11), yielding

$$\begin{array}{c} d \quad c \\ \diagdown \quad \diagup \\ a \quad u \quad a \pm 1 \quad u+2\lambda \quad a \pm 1 \\ \diagup \quad \diagdown \\ a \end{array} = \sqrt{\frac{\vartheta_4(2a\lambda \mp \lambda)}{\vartheta_4(2a\lambda \pm 3\lambda)}} \begin{array}{c} d \quad c \\ \diagdown \quad \diagup \\ a \quad u \quad a \pm 1 \quad u+2\lambda \quad a \pm 1 \\ \diagup \quad \diagdown \\ a \end{array} \\ \begin{array}{c} d \quad c \\ \diagdown \quad \diagup \\ a \quad u \quad a \pm 1 \quad u+2\lambda \quad a \\ \diagup \quad \diagdown \\ a \pm 1 \end{array} = A_{a \pm 1, a} \frac{\vartheta_1(\lambda)}{\vartheta_1(2\lambda)} \frac{S(a \pm 1, a)}{S(a, a)} \begin{array}{c} d \quad c \\ \diagdown \quad \diagup \\ a \quad u \quad a \quad u+2\lambda \quad a \\ \diagup \quad \diagdown \\ a \end{array} \quad (2.36)$$

which is just

$$\begin{array}{c} d \quad c \\ \diagdown \quad \diagup \\ a \quad u \quad a' \quad u+2\lambda \quad b \\ \diagup \quad \diagdown \\ a \end{array} = \phi_{(0,1)}(a, a', b) \begin{array}{c} d \quad c \\ \diagdown \quad \diagup \\ a \quad u \quad a'_1 \quad u+2\lambda \quad b \\ \diagup \quad \diagdown \\ a \end{array} \quad (2.37)$$

with $a'_1 = \min(a, b)$. Again, we choose a' in (2.35) such that $(a, a', b) \in \text{indpath}_{(0,1)}[a, b]$ to define the fused weights.

Lemma 2.2 (Elementary Antisymmetric Fusion) *Define*

$$\begin{aligned} W_{(0,1)} \left(\begin{array}{cc|c} d & c & u \\ a & b & \end{array} \right) &= \begin{array}{c} d \quad c \\ \diagdown \quad \diagup \\ a \quad u \quad a' \quad u+2\lambda \quad b \\ \diagup \quad \diagdown \\ a \end{array} = \begin{array}{c} d \quad c \\ \diagdown \quad \diagup \\ a \quad u \quad a'_1 \quad u+2\lambda \quad b \\ \diagup \quad \diagdown \\ a \end{array} \\ &= \sum_{c'} \phi_{(0,1)}(d, c', c) W \left(\begin{array}{cc|c} d & c' & u \\ a & a' & \end{array} \right) W \left(\begin{array}{cc|c} c' & c & u+2\lambda \\ a' & b & \end{array} \right) \quad (2.38) \end{aligned}$$

where we sum over all allowed values of c' and where $a' = \min(a, b)$. Then the fused weights satisfy

$$\begin{aligned} W_{(0,1)} \left(\begin{array}{cc|c} d & c & u \\ a & b & \end{array} \right) &= -s_1^1 s_{-3/2}^1 \frac{g(d, c)}{g(a, b)} W \left(\begin{array}{cc|c} d & c & u+\lambda \\ a & b & \end{array} \right) \\ &= -r_1^1 \frac{g(d, c)}{g(a, b)} W \left(\begin{array}{cc|c} d & c & u+\lambda \\ a & b & \end{array} \right) \quad (2.39) \end{aligned}$$

where the gauge factors $g(a, b)$ are given by

$$g(a, b) = \left(\frac{\vartheta_1(2\lambda)}{\vartheta_1(\lambda)} \right)^{|a-b|} \times \begin{cases} \sqrt{\frac{\vartheta_4(2b\lambda + \lambda)}{\vartheta_4(2b\lambda - \lambda)}} & \text{for } a > b \\ \sqrt{\frac{\vartheta_4(2b\lambda - \lambda)}{\vartheta_4(2b\lambda - 3\lambda)}} & \text{for } a < b \\ \sqrt{\frac{\vartheta_4(2b\lambda + \lambda)\vartheta_4(2b\lambda - \lambda)}{\vartheta_4^2(2b\lambda)}} & \text{for } a = b \end{cases} \quad (2.40)$$

and s_k^m and r_k^m are functions of u defined as

$$s_k^m = \prod_{j=0}^{m-1} \frac{\vartheta_1(u + 2(k-j)\lambda)}{\sqrt{\vartheta_1(2\lambda)\vartheta_1(3\lambda)}} \quad (2.41)$$

$$r_k^m = s_k^m s_{k-5/2}^m = \prod_{j=0}^{m-1} \frac{\vartheta_1(u + 2(k-j)\lambda)\vartheta_1(u + (2(k-j)-5)\lambda)}{\vartheta_1(2\lambda)\vartheta_1(3\lambda)}. \quad (2.42)$$

In particular, this implies that

$$W_{(0,1)} \left(\begin{array}{cc|c} d & c & -2\lambda \\ a & b & \end{array} \right) = W_{(0,1)} \left(\begin{array}{cc|c} d & c & 3\lambda \\ a & b & \end{array} \right) = 0 \quad (2.43)$$

for all a, b, c, d .

In other words, up to a gauge and some overall factors the $(0, 1)$ fused weights are nothing but the elementary face weights (1.2) shifted by λ . This of course already implies that the Yang-Baxter equation is fulfilled which alternatively follows from the push-through properties (2.36) as in the symmetric case. We omit the proof of the lemma since it reduces to the explicit computation of the fused weights. However, the appearance of the overall factor $r_1^1 = s_1^1 s_{-3/2}^1$ in (2.39) can be understood directly from (2.43) which follows from (2.34) using the initial condition (1.5) and crossing symmetry (1.7).

The symmetric and antisymmetric fusion are ‘‘orthogonal’’ in the sense that

$$\begin{array}{c} d \\ \circ \\ \square \\ \alpha \\ a \quad u \quad u+2\lambda \quad b \\ c \end{array} = 0 \quad (2.44)$$

for all a, b, c, d and α . This follows, for example from (2.27), which basically is just the inversion relation (2.5) for $u = \pm 2\lambda$. But this is not all, the action of the two projectors is also complementary in the sense that there is no non-trivial subspace that is annihilated by both the blocks $X^{(b,d)}(2\lambda)$ and $X^{(b,d)}(-2\lambda)$, provided that $|b - d| \leq 2$. This follows from the inversion relation (2.5) since the RHS has only simple zeros in the spectral

parameter u . If both blocks really were projectors, this would mean that they add up to the identity matrix. Although this is not the case here, we can choose the independent paths such that the two sets of independent paths $\text{indpath}_{(2,0)}[a, b]$ and $\text{indpath}_{(0,1)}[a, b]$ are disjoint and that their union consists of all allowed paths, i.e.,

$$\begin{aligned} \text{indpath}_{(2,0)}[a, b] \cap \text{indpath}_{(0,1)}[a, b] &= \emptyset \\ \text{indpath}_{(2,0)}[a, b] \cup \text{indpath}_{(0,1)}[a, b] &= \text{path}(a, b; 2) \end{aligned} \quad (2.45)$$

which is exactly what we did, see (2.17)–(2.18) and (2.32). In particular, this implies the relation

$$\left(A^{(1,0)}\right)^2 = A^{(2,0)} + A^{(0,1)} \quad (2.46)$$

for the fused adjacency matrices.

2.4 Antisymmetric 1×3 Fusion: Level $(0,0)$

Before we move on to the higher level symmetric fusion, we first have a look at the completely antisymmetric 1×3 fused weights. The corresponding projector is the following product of elementary face weights

$$\text{diamond}(d, d'', c', c, e, 2\lambda, 2\lambda, 4\lambda, u) = e_1 \text{diamond}(d, d'', c'', e_1, 2\lambda) \times \text{square}(d, c'', c, d', e, 2\lambda, 4\lambda) \quad (2.47)$$

where the above equality is just (2.35) with $u = 2\lambda$ and e_1 is defined accordingly.

Of course, (2.47) means that we can equally well regard the antisymmetric sum on the RHS as the projector defining the fused weights. By Lemma (2.2), it is proportional to the elementary face weights at $u = 3\lambda$ which in turn is related by crossing symmetry to $u = 0$ and hence proportional to $\delta_{c,d}$. Furthermore, the local face operators (2.1) at $u = 3\lambda$ represent projectors onto at most one dimension. This implies that the same is true for our projector (2.47), viewed as a matrix acting from path (d, d', c', c) to (d, d'', c'', c) , hence we have to deal with only one independent path here. Let us choose this path to be (c, c, c, c) . Then the only non-vanishing coefficients $\phi_{(0,0)}(d, d', c', c)$ are given by $\phi_{(0,0)}(c, c, c, c) = 1$ and

$$\begin{aligned} \phi_{(0,0)}(c, c \pm 1, c, c) &= \phi_{(0,0)}(c, c, c \pm 1, c) \\ &= \sqrt{\frac{\vartheta_4(2c\lambda \mp \lambda)}{\vartheta_4(2c\lambda \pm 3\lambda)}} \phi_{(0,0)}(c, c \pm 1, c \pm 1, c) = \frac{\vartheta_1(\lambda)}{\vartheta_1(2\lambda)} \frac{S(c \pm 1, c)}{S(c, c)} \end{aligned} \quad (2.48)$$

The fused weights are deduced from the product of elementary faces with the projector of (2.47). From the above remarks, we find

$$\begin{aligned}
& \begin{array}{c} d \quad d \quad d \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ a \quad u \quad \epsilon \quad 2\lambda \quad f \quad 2\lambda \quad c'' \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ a' \quad u+2\lambda \quad 4\lambda \quad c' \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \\ a'' \quad u+4\lambda \quad c \\ \diagup \quad \diagdown \\ b \end{array} = f_1 \begin{array}{c} d \\ \diagdown \quad \diagup \\ c' \end{array} \times \begin{array}{c} d \quad d \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \\ a \quad u \quad \epsilon \quad 2\lambda \quad c' \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ a' \quad u+2\lambda \quad 4\lambda \quad c \\ \diagdown \quad \diagup \\ a'' \quad u+4\lambda \quad c \\ \diagup \quad \diagdown \\ b \end{array} \\
& = h(u) \phi_{(0,1)}(a', a'', b) \frac{g(d, c')}{g(a', b)} f_1 \begin{array}{c} d \\ \diagdown \quad \diagup \\ c' \end{array} \times \begin{array}{c} d \quad d \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \\ a \quad u \quad \epsilon \quad 3\lambda \quad c' \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ a' \quad u+3\lambda \quad c \\ \diagdown \quad \diagup \\ b \end{array} \quad (2.49)
\end{aligned}$$

where

$$h(u) = - \frac{\vartheta_1(\lambda)\vartheta_1(4\lambda)\vartheta_1(u-\lambda)\vartheta_1(u+4\lambda)}{\vartheta_1^2(2\lambda)\vartheta_1^2(3\lambda)}. \quad (2.50)$$

Hence the fused weights are basically the same as those of antisymmetric $su(2)$ type fusion with the projector $X^{(c,d)}(3\lambda)$. This shows that the fused weights are essentially trivial, in particular they vanish unless $c = d$ and also $a = b$, which follows using the Yang-Baxter equation (1.13). We obtain the following result for the fused weights.

Lemma 2.3 (Antisymmetric 1×3 Fusion) *The non-zero fused weights defined by*

$$\begin{aligned}
W_{(0,0)} \left(\begin{array}{c|c} c & c \\ a & a \end{array} \middle| u \right) &= \begin{array}{c} c \quad \overline{\quad\quad\quad} \quad c \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \\ a \quad u \quad u+2\lambda \quad u+4\lambda \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ a \quad a \end{array} = \begin{array}{c} c \quad \overline{\quad\quad\quad} \quad c \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \\ a \quad u \quad u+2\lambda \quad u+4\lambda \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ a \quad a \end{array} \\
&= \sum_{d', c'} \phi_{(0,0)}(c, d', c', c) W \left(\begin{array}{c|c} c & d' \\ a & a \end{array} \middle| u \right) W \left(\begin{array}{c|c} d' & c' \\ a & a \end{array} \middle| u+2\lambda \right) W \left(\begin{array}{c|c} c' & c \\ a & a \end{array} \middle| u+4\lambda \right) \quad (2.51)
\end{aligned}$$

have the following simple form

$$W_{(0,0)} \left(\begin{array}{c|c} c & c \\ a & a \end{array} \middle| u \right) = f_0^1 \frac{g(c)}{g(a)} \quad (2.52)$$

where we define functions f_k^m of u by

$$f_k^m = (-1)^m r_{k+1}^m r_{k+3/2}^m r_{k+2}^m = (-1)^m s_{k-3/2}^m s_{k-1}^m s_{k-1/2}^m s_{k+1}^m s_{k+3/2}^m s_{k+2}^m \quad (2.53)$$

and $g(a)$ is given by

$$g(a) = \sqrt{\frac{\vartheta_4(2a\lambda + \lambda)\vartheta_4(2a\lambda - \lambda)}{\vartheta_4^2(2a\lambda)}}. \quad (2.54)$$

Clearly, one has again the split and push-through properties for the fused weights as in the two previous cases. However, we do not list them here as there is nothing to prove apart from an explicit calculation of the weights.

Note that the $g(a)$ act as a simple gauge. This means that the fused weights are trivial, in the sense that the transfer matrix constructed from these weights is a multiple of the identity matrix. This is in agreement with the expected $su(3)$ structure of the fusion hierarchy of the dilute A_L models (1.31).

3 Fusion at Levels $(n,0)$ and $(0,n)$

We now want to apply the fusion procedure to higher levels, starting with symmetric fusion. To do so we start by generalizing our notation for paths and independent paths.

3.1 Projectors and Paths

Let us define graphically

$$P_{(n,0)}(u)_{(a,b_1,b_2,\dots,b_n,b)}^{(a,a_1,a_2,\dots,a_n,b)} = \begin{array}{c} \begin{array}{c} a \\ \swarrow \quad \searrow \\ b_1 \quad u \quad a_1 \\ \swarrow \quad \searrow \quad \swarrow \quad \searrow \\ b_2 \quad -2(n-1)\lambda \quad u+2\lambda \quad a_2 \\ \swarrow \quad \searrow \quad \swarrow \quad \searrow \quad \swarrow \quad \searrow \\ b_{n-1} \quad -4\lambda \quad -4\lambda \quad -4\lambda \quad u+2(n-2)\lambda \quad a_{n-1} \\ \swarrow \quad \searrow \quad \swarrow \quad \searrow \quad \swarrow \quad \searrow \quad \swarrow \quad \searrow \\ b_n \quad -2\lambda \quad -2\lambda \quad -2\lambda \quad -2\lambda \quad u+2(n-1)\lambda \quad a_n \\ \swarrow \quad \searrow \quad \swarrow \quad \searrow \quad \swarrow \quad \searrow \\ b \quad b \quad b \quad b \quad b \end{array} \end{array} \quad (3.1)$$

where we regard $P_{(n,0)}(u)$ as an operator acting on the $(n+1)$ -step path $(a, a_1, a_2, \dots, a_n, b)$ to produce the path $(a, b_1, b_2, \dots, b_n, b)$. As we will see in what follows, $P_{(n,0)}(u)$ will give the weights of level $(n,0)$ fusion, and $P_{(n,0)}(-2n\lambda)$ corresponds to the “projector” of symmetric level $(n+1, 0)$ fusion.

Clearly, $P_{(1,0)}(u)$ is just an elementary block, and $P_{(2,0)}(u)$ is related to the 1×2 symmetric fusion presented in section 2.2. In particular, (2.24) becomes

$$P_{(2,0)}(u)_{(d,a,\epsilon,b)}^{(d,c',c,b)} = \sum_{\alpha} P_{(1,0)}(-2\lambda)_{(a,\epsilon,b)}^{(a,a'_\alpha,b)} W_{(2,0)} \left(\begin{array}{ccc|c} d & \beta & c & \\ a & \alpha & b & u \end{array} \right) \quad (3.2)$$

where the sum labeled by the bond variable $\alpha \in \{1, 2\}$ is over all independent paths $(a, a'_\alpha, b) \in \text{indpath}_{(2,0)}[a, b]$.

Now consider $P_{(n,0)}(u)$. By re-arranging elementary faces using the Yang-Baxter equation (1.13), it is easy to see that any two adjacent faces with the spectral parameters $u + 2j\lambda$ and $u + 2(j+1)\lambda$ in (3.1) can be considered as an instance of level $(2, 0)$ fusion. Therefore the properties (2.24) and (2.28) imply the relations

$$P_{(n,0)}(u)_{(a,b_1,\dots,b_j,b_{j+1},b_{j+2},\dots,b_n,b)}^{(a,a_1,\dots,a_j,a_{j+2},a_{j+2},\dots,a_n,b)} = - \sqrt{\frac{\vartheta_4(2a_{j+2}\lambda \pm 3\lambda)}{\vartheta_4(2a_{j+2}\lambda \mp \lambda)}} P_{(n,0)}(u)_{(a,b_1,\dots,b_j,b_{j+1},b_{j+2},\dots,b_n,b)}^{(a,a_1,\dots,a_j,a_j,a_{j+2},\dots,a_n,b)} \quad (3.3)$$

for $a_j - a_{j+2} = \pm 1$ and

$$A_{a_j, a_{j-1}} \frac{S(a_j - 1, a_j)}{S(a_j + 1, a_j)} P_{(n,0)}(u)_{(a,b_1,\dots,b_j,b_{j+1},b_{j+2},\dots,b_n,b)}^{(a,a_1,\dots,a_j,a_j-1,a_{j+2},\dots,a_n,b)} + \frac{\vartheta_1(2\lambda)}{\vartheta_1(\lambda)} \frac{S(a_j, a_j)}{S(a_j + 1, a_j)} P_{(n,0)}(u)_{(a,b_1,\dots,b_j,b_{j+1},b_{j+2},\dots,b_n,b)}^{(a,a_1,\dots,a_j,a_j,a_{j+2},\dots,a_n,b)} + A_{a_j, a_{j+1}} P_{(n,0)}(u)_{(a,b_1,\dots,b_j,b_{j+1},b_{j+2},\dots,b_n,b)}^{(a,a_1,\dots,a_j,a_{j+1},a_{j+2},\dots,a_n,b)} = 0 \quad (3.4)$$

for $a_j = a_{j+2}$. These two equations take over both the role of (2.12) in level $(n+1, 0)$ fusion (with $u = -2n\lambda$) and, via a generalized split property (2.24), of (2.28) in level $(n, 0)$ fusion. Of course, (3.3) and (3.4) only express that any antisymmetric sum of $P_{(n,0)}(u)_{(a,b_1,\dots,b_n,b)}^{(a,a_1,\dots,a_n,b)}$ over any of the variables a_j vanishes. Therefore we can summarize them by

$$\sum_{a_j} \phi_{(0,1)}(a_{j-1}, a_j, a_{j+1}) P_{(n,0)}(u)_{(a,b_1,\dots,b_n,b)}^{(a,a_1,\dots,a_n,b)} = 0 \quad (3.5)$$

for $1 \leq j \leq n$, where we set $a_0 = a$ and $a_{n+1} = b$.

As before, we denote the set of n -step paths from a to b on the effective adjacency diagram of Fig 1(b) by $\text{path}(a, b; n)$, or $(a, b; n)$ for short. The number of such paths is given by

$$|\text{path}(a, b; n)| = |(a, b; n)| = [(I + A)^n]_{a,b} \quad (3.6)$$

i.e., by the corresponding element of the n -th power of the effective adjacency matrix $A^{(1,0)} = I + A$. In this basis, $P_{(n,0)}(u)$ becomes a square matrix which we can choose to be block-diagonal by an appropriate ordering of paths. Let us introduce $(a, b; n|j)$ as a short-hand notation for the j -th path in the set $\text{path}(a, b; n)$.

The notion of independent paths also generalizes immediately from the discussion of section 2. Here, a set of $(n+1)$ -step paths $\{(a, a_1^i, \dots, a_n^i, b)\}$ is *independent* (w.r.t.

$P_{(n,0)}(-2n\lambda)$ if, for any path (a, b_1, \dots, b_n, b) ,

$$\sum_i \phi(a, a_1^i, \dots, a_n^i, b) P_{(n,0)}(-2n\lambda)_{(a,b_1,\dots,b_n,b)}^{(a,a_1^i,\dots,a_n^i,b)} = 0 \quad (3.7)$$

implies that all coefficients $\phi(a, a_1^i, \dots, a_n^i, b)$ vanish. We denote by $\text{indpath}_{(n+1,0)}[a, b] = [a, b; (n+1, 0)]$ a maximal set of independent paths from a to b , by $|[a, b; (n+1, 0)]|$ the number of its elements, and abbreviate its α -th element by $[a, b; (n+1, 0)|\alpha]$ (in order to avoid confusion, we will always use Greek letters for indices referring to independent paths).

As in section 2, we define coefficients $\phi_{(n+1,0)}$ by

$$P_{(n,0)}(-2n\lambda)_{(a,b;n+1|k)}^{(a,b;n+1|j)} = \sum_{\alpha=1}^{|[a,b;(n+1,0)]|} \phi_{(n+1,0)}[a, b](j|\alpha) P_{(n,0)}(-2n\lambda)_{(a,b;n+1|k)}^{[a,b;(n+1,0)|\alpha]} \quad (3.8)$$

so the $\phi_{(n+1,0)}[a, b](j|\alpha)$ are the ‘‘coordinates’’ of the path $(a, b; n+1|j)$ in the basis of independent paths $[a, b; (n+1, 0)|\alpha]$.

From these definitions, we immediately obtain the generalization of the split property (2.24) to the $(n, 0)$ case. By definition, we have

$$\begin{aligned} & P_{(n+1,0)}(u)_{(a,b_1,\dots,b_n,b_{n+1},b)}^{(a,a_1,\dots,a_n,a_{n+1},b)} \\ &= \sum_{j=1}^{|(b_1,b,n+1)|} P_{(n,0)}(-2n\lambda)_{(b_1,\dots,b_{n+1},b)}^{(b_1,b;n+1|j)} \quad \begin{array}{c} a \qquad a_1 \qquad a_2 \qquad \dots \qquad a_n \qquad a_{n+1} \\ \boxed{u \quad | \quad u+2\lambda \quad | \quad \dots \quad | \quad u+2n\lambda} \\ b_1 \qquad c_1^j \qquad c_2^j \qquad \dots \qquad c_n^j \qquad b \end{array} \\ &= \sum_{j=1}^{|(b_1,b,n+1)|} \sum_{\alpha=1}^{|[b_1,b;(n+1,0)]|} P_{(n,0)}(-2n\lambda)_{(b_1,\dots,b_{n+1},b)}^{[b_1,b;(n+1,0)|\alpha]} \phi_{(n+1,0)}[b_1, b](j|\alpha) \quad \begin{array}{c} a \qquad a_1 \qquad a_2 \qquad \dots \qquad a_n \qquad a_{n+1} \\ \boxed{u \quad | \quad u+2\lambda \quad | \quad \dots \quad | \quad u+2n\lambda} \\ b_1 \qquad c_1^j \qquad c_2^j \qquad \dots \qquad c_n^j \qquad b \end{array} \\ &= \sum_{\alpha=1}^{|[b_1,b;(n+1,0)]|} P_{(n,0)}(-2n\lambda)_{(b_1,\dots,b_{n+1},b)}^{[b_1,b;(n+1,0)|\alpha]} \quad \begin{array}{c} a \qquad a_1 \qquad a_2 \qquad \dots \qquad a_n \qquad a_{n+1} \\ \boxed{u \quad | \quad u+2\lambda \quad | \quad \dots \quad | \quad u+2n\lambda} \\ b_1 \qquad \underbrace{\hspace{2cm}}_{\alpha} \qquad b \end{array} \quad (3.9) \end{aligned}$$

where the c_k^j are given by

$$(b_1, c_1^j, \dots, c_n^j, b) = (b_1, b; n+1|j). \quad (3.10)$$

In the last line of (3.9), we have introduced a symbol for the ‘‘symmetric sum’’ over c_1^j, \dots, c_n^j labeled by α , which we will use in what follows to define the $(n+1, 0)$ fused weights.

We also need to generalize the push-through property (2.29). We use the recursive nature of the fusion procedure to prove the push-through property by induction and at

the same time obtain an expression for the number of independent paths. Suppose that the push-through property holds for the symmetric sum of n elementary faces

$$\begin{aligned}
 & \begin{array}{c} d \\ \hline \begin{array}{|c|c|c|c|} \hline u & u+2\lambda & \cdots & u+2(n-1)\lambda \\ \hline \end{array} \\ \hline a \end{array} \begin{array}{c} e_1^j \\ e_2^j \\ \cdots \\ e_{n-1}^j \\ \hline c \end{array} \\
 &= \sum_{\beta=1}^{|[d,c;(n,0)]|} \phi_{(n,0)}[d,c](j|\beta) \begin{array}{c} d \\ \hline \begin{array}{|c|c|c|c|} \hline u & u+2\lambda & \cdots & u+2(n-1)\lambda \\ \hline \end{array} \\ \hline a \end{array} \begin{array}{c} e_1^\beta \\ e_2^\beta \\ \cdots \\ e_{n-1}^\beta \\ \hline c \end{array} \quad (3.11)
 \end{aligned}$$

where the paths are labeled by $(d,c;n|j) = (d, e_1^j, \dots, e_{n-1}^j, c)$ and $[d,c;(n,0)|\beta] = (d, e_1^\beta, \dots, e_{n-1}^\beta, c)$. As in (3.9), but splitting the product in a different place, we have

$$\begin{aligned}
 & P_{(n+1,0)}(u)_{(a,b_1,\dots,b_n,b_{n+1},b)}^{(a,a_1,\dots,a_n,a_{n+1},b)} \\
 &= \sum_{j=1}^{|(b_2,b;n)|} P_{(n-1,0)}(-2(n-1)\lambda)_{(b_2,\dots,b_{n+1},b)}^{(b_2,b;n|j)} \\
 &= \sum_{\beta=1}^{|[b_2,b;(n,0)]|} P_{(n-1,0)}(-2(n-1)\lambda)_{(b_2,\dots,b_{n+1},b)}^{[b_2,b;(n,0)|\beta]} \\
 &= \sum_{\beta=1}^{|[b_2,b;(n,0)]|} \sum_{\gamma=1}^{|[b_1,d;(n,0)]|} P_{(n-1,0)}(-2(n-1)\lambda)_{(b_2,\dots,b_{n+1},b)}^{[b_2,b;(n,0)|\beta]} \\
 &= \sum_{\gamma=1}^{|[b_1,d^\delta;(n,0)]|} \sum_{d^\delta} P_{(n,0)}(-2n\lambda)_{(b_1,\dots,b_{n+1},b)}^{([b_1,d^\delta;(n,0)|\gamma],b)} \\
 &= \sum_{\alpha=1}^{|[b_1,b;(n+1,0)]|} P_{(n,0)}(-2n\lambda)_{(b_1,\dots,b_{n+1},b)}^{[b_1,b;(n+1,0)|\alpha]} \quad (3.12)
 \end{aligned}$$

where we used the push-through property (3.11), the split property and the elementary symmetric fusion of Lemma 2.1. The last line is just equation (3.9) again. This clearly reflects the recursive nature of the construction of fused weights: the projector of fusion at level $(n+1, 0)$ is basically a face weight of level $(n, 0)$ fusion. We also see that in order to find the independent paths at level $(n+1, 0)$, we only have to consider paths which are independent at level $(n, 0)$ and append one step to those. This also establishes the push-through property for the symmetric sum of $n+1$ elementary faces by using the push-through properties (3.11) for n faces and (2.29) for the elementary fusion of two faces.

The number of independent paths is of course given by the difference of the total number of paths and the number of independent equations obtained from (3.3) and (3.4). This is just the matrix element $A_{a,b}^{(n+1,0)}$ of the fused adjacency matrix defined in (2.8), i.e.,

$$\left| \text{indpath}_{(n+1,0)}[a, b] \right| = |[a, b; (n+1, 0)]| = A_{a,b}^{(n+1,0)}. \quad (3.13)$$

Of course, we in fact have to prove that the recursive formula (2.8) is true and that it defines the adjacency matrices of the fused weights which we construct. This can be done by induction. However, even for the special case we have considered so far we need to be able to say something about fusion levels $(n, 1)$, because these enter in (2.8). Although these fusion levels will not be discussed until section 4, we briefly sketch the argument and it will become clear that the definitions of section 4 ensure that it is correct.

Assume that we know the set $\text{indpath}_{(n,0)}[a, a']$, $n \geq 1$, and that it contains $A_{a,a'}^{(n,0)}$ elements. This is clearly fulfilled for $n = 1$. Then the number of all allowed $(n+1)$ -step paths from a to b which are obtained by appending one step to the paths in $\text{indpath}_{(n,0)}[a, a']$, is obviously given by

$$\sum_{a'} A_{a,a'}^{(n,0)} A_{a',b}^{(1,0)} = \left(A^{(n,0)} \cdot A^{(1,0)} \right)_{a,b}. \quad (3.14)$$

However, not all of these are independent. How many relations are there? To see this, we have to look at the last two steps of the paths, and count how many of these are dependent as two-step paths, keeping track of the number of paths in $\text{indpath}_{(n,0)}[a, a']$ which terminate in the corresponding manner. To count these directly appears to be complicated, but remember that for two-step paths one could choose the independent paths for antisymmetric fusion as the complement of those for symmetric fusion. This means that we arrive at a set of independent paths $\text{indpath}_{(n+1,0)}[a, a']$ if we exclude all those paths which are independent w.r.t. fusion containing $(n-1)$ symmetric and one antisymmetric sum at the end. We will see later that this is exactly the definition of level

$(n-1, 1)$ fused weights, and hence there are $A_{a,b}^{(n-1,1)}$ such paths. Therefore

$$\left| \text{indpath}_{(n+1,0)}[a, b] \right| = \left(A^{(n,0)} \cdot A^{(1,0)} - A^{(n-1,1)} \right)_{a,b} = A_{a,b}^{(n+1,0)} \quad (3.15)$$

in agreement with (2.8).

3.2 Level $(n, 0)$ and $(0, n)$ Fusion

We start by giving the $(n+1, 0)$ fused weights.

Lemma 3.1 (Symmetric Fusion of a Single Row) *Define the fused weights at level $(n+1, 0)$ by*

$$\begin{aligned} W_{(n+1,0)} \left(\begin{array}{ccc|c} d & \beta & c & u \\ a & \alpha & b & \end{array} \right) &= \begin{array}{c} \begin{array}{ccccccc} d & e_0^\beta & e_1^\beta & \dots & e_n^\beta & c \\ \hline u & u+2\lambda & \dots & u+2n\lambda \\ \hline a & & & & & b \end{array} \\ \alpha \end{array} \\ &= \sum_{j=1}^{|(a,b;n+1)|} \phi_{(n+1,0)}[a, b](j|\alpha) \begin{array}{c} \begin{array}{ccccccc} d & e_0^\beta & e_1^\beta & \dots & e_n^\beta & c \\ \hline u & u+2\lambda & \dots & u+2n\lambda \\ \hline a & e_1^j & e_2^j & \dots & e_n^j & b \end{array} \end{array} \\ &= \sum_{j=1}^{|(a,b;n+1)|} \phi_{(n+1,0)}[a, b](j|\alpha) \prod_{k=0}^n W \left(\begin{array}{cc|c} e_k^\beta & e_{k+1}^\beta & \\ e_k^j & e_{k+1}^j & u+2k\lambda \end{array} \right) \end{array} \quad (3.16)$$

where

$$\begin{aligned} (a \equiv e_0^j, e_1^j, \dots, e_n^j, e_{n+1}^j \equiv b) &= (a, b; n+1|j), \\ (d \equiv e_0^\beta, e_1^\beta, \dots, e_n^\beta, e_{n+1}^\beta \equiv c) &= [d, c; (n+1, 0)|\beta], \end{aligned}$$

and the bond variables α and β take values

$$1 \leq \alpha \leq |[a, b; (n+1, 0)]| = A_{a,b}^{(n+1,0)}, \quad 1 \leq \beta \leq |[d, c; (n+1, 0)]| = A_{c,d}^{(n+1,0)}.$$

These weights satisfy the Yang-Baxter equation (3.26).

This lemma follows in the same way as Lemma 2.1 for the case of level $(2, 0)$ fusion by using the split (3.9) and push-through (3.11) properties. For a formal proof one should use induction over n .

The fused weights for level $(0, n)$ are constructed as follows. We consider a row of $2n$ elementary faces with spectral parameters arranged according to the sequence $(u, u+2\lambda,$

$u + 2\lambda, u + 4\lambda, u + 4\lambda, \dots, u + 2(n - 1)\lambda, u + 2(n - 1)\lambda, u + 2n\lambda$). We then perform an antisymmetric fusion on each pair of adjacent faces with spectral parameters differing by 2λ , i.e.,

$$(3.17)$$

By Lemma 2.2 this yields, apart from gauge and overall factors, a row of n elementary faces with spectral parameters $(u + \lambda, u + 3\lambda, \dots, u + (2n - 3)\lambda, u + (2n - 1)\lambda)$. These are then fused by the symmetric $(n, 0)$ fusion process described above. Altogether, this means that the $(0, n)$ fused weights differ from the weights at fusion level $(n, 0)$ only by a shift in the spectral parameter, a gauge transformation and some overall factors. Keeping track of the accumulated factors, one obtains

$$W_{(0,n)}\left(\begin{matrix} d & \beta & c \\ a & \alpha & b \end{matrix} \middle| u\right) = \frac{g(d, c|\beta)}{g(a, b|\alpha)} \left(\prod_{k=1}^n (-r_k^1)\right) W_{(n,0)}\left(\begin{matrix} d & \beta & c \\ a & \alpha & b \end{matrix} \middle| u + \lambda\right) \quad (3.18)$$

where $g(a, b|\alpha)$ denotes the product of the gauge factors of (2.40) along the independent path $[a, b; (n, 0)|\alpha]$. Strictly speaking, α and β have different meaning on the two sides of equation (3.18), labeling independent paths at level $(0, n)$ on the left and independent paths at level $(n, 0)$ on the right. However, our construction of the fused weights gives a one-to-one correspondence between these paths which is implied in equation (3.18).

The fusion of a single column of elementary faces is basically the same as that of a single row. Again we can make a choice on which side of the column we perform the symmetric sum, here we choose the left side. The operator in (3.1) is then replaced by

$$P^{(n,0)}(u)_{(a,b_1,b_2,\dots,b_n,b)}^{(a,a_1,a_2,\dots,a_n,b)} = \quad (3.19)$$

where here and in what follows we use upper indices to denote the fusion level in the vertical direction. Of course, $P^{(n,0)}(-2n\lambda)$ is now the projector of level $(n + 1, 0)$ fusion of $n + 1$ elementary faces in a single column. But, from the Yang-Baxter equation, one

finds

$$\begin{aligned}
 P_{(n,0)}(-2n\lambda)_{(a,b_1,\dots,b_n,b)}^{(a,a_1,\dots,a_n,b)} &= \\
 &= \begin{array}{c} \begin{array}{c} a \\ b_1 \quad -2n\lambda \quad a_1 \\ b_2 \quad -2(n-1)\lambda \quad -2(n-1)\lambda \quad a_2 \\ \vdots \\ b_{n-1} \quad -4\lambda \quad -4\lambda \quad -4\lambda \quad -4\lambda \quad a_{n-1} \\ b_n \quad -2\lambda \quad -2\lambda \quad -2\lambda \quad -2\lambda \quad -2\lambda \quad a_n \\ b \quad b \quad b \quad b \quad b \end{array} \\ \\ \begin{array}{c} a \quad a \quad a \quad a \quad a \\ b_1 \quad -2\lambda \quad -2\lambda \quad -2\lambda \quad -2\lambda \quad -2\lambda \quad a_1 \\ b_2 \quad -4\lambda \quad -4\lambda \quad -4\lambda \quad -4\lambda \quad a_2 \\ \vdots \\ b_{n-1} \quad -2(n-1)\lambda \quad -2(n-1)\lambda \quad a_{n-1} \\ b_n \quad -2n\lambda \quad a_n \\ b \end{array} \end{array} = P_{(n,0)}(-2n\lambda)_{(a,b_1,\dots,b_n,b)}^{(a,a_1,\dots,a_n,b)} \quad (3.20)
 \end{aligned}$$

so they are in fact identical.

This means we can use the same paths and independent paths to construct the vertically fused weights at level $(n,0)$ as in the horizontal case. Of course, this also holds true for the above discussion about the level $(0,n)$ fused weights. Therefore, we will not go into more detail here and instead move on to discuss directly the more general case of symmetric fusion of rectangular blocks of elementary faces.

3.3 Symmetric Fusion of Rectangular Blocks

So far, we have considered fusion of a single row and a single column of weights with fusion types labeled by the irreducible representations $(n,0)$ and $(0,n)$ of $su(3)$. In fact, we want to fuse the dilute models in both the horizontal and vertical direction simultaneously. The corresponding fused weights are labeled by two irreducible representations of $su(3)$ (respectively their Young tableaux) where we use the convention that the lower index of the weights corresponds to the horizontal and the upper index to the vertical fusion level. In general, these fused weights have bond variables not only on the horizontal, but also

on the vertical bonds. Let us now consider the symmetric fusion of a rectangular m by n block of elementary faces. Given two positive integers $m, n > 1$ the fused face weights are defined by

$$\begin{aligned}
W_{(n,0)}^{(m,0)} \left(\begin{array}{ccc|c} d & \beta & c & \\ \mu & & \nu & u \\ a & \alpha & b & \end{array} \right) &= \begin{array}{c} \begin{array}{ccc} d & \beta & c \\ \mu & u & \nu \\ a & \alpha & b \end{array} \\ \end{array} \\
&= \sum_{j=1}^{|(a,d;m)|} \phi_{(m,0)}^{(a,d)}(j|\mu) \sum_{\alpha_2, \dots, \alpha_m} \prod_{k=1}^m W_{(n,0)} \left(\begin{array}{ccc|c} e_{k+1}^j & \alpha_{k+1} & e_{k+1}^\nu & \\ e_k^j & \alpha_k & e_k^\nu & u - 2(m-k)\lambda \end{array} \right), \quad (3.21)
\end{aligned}$$

where

$$\begin{aligned}
(a \equiv e_1^j, e_2^j, \dots, e_m^j, e_{m+1}^j \equiv d) &= (a, d; m|j), \\
(b \equiv e_1^\nu, e_2^\nu, \dots, e_m^\nu, e_{m+1}^\nu \equiv c) &= [b, c; (m, 0)|\nu],
\end{aligned}$$

and the four bond variables α, β, μ and ν take values

$$\begin{aligned}
1 \leq \alpha \leq |[a, b; (n, 0)]| &= A_{a,b}^{(n,0)}, \quad 1 \leq \beta \leq |[d, c; (n, 0)]| = A_{c,d}^{(n,0)} \\
1 \leq \mu \leq |[a, d; (m, 0)]| &= A_{a,d}^{(m,0)}, \quad 1 \leq \nu \leq |[b, c; (m, 0)]| = A_{b,c}^{(m,0)}.
\end{aligned}$$

For the bond variables $\alpha_1, \alpha_2, \dots, \alpha_{m+1}$ in (3.21) we have that $\alpha = \alpha_1, \beta = \alpha_{m+1}$ and the other internal bond variables α_k are summed over $1, \dots, A_{e_k^j, e_k^\nu}^{(n,0)}$. The $(n, 0)$ fusion of $1 \times n$ faces is given by (3.16). Of course these rectangular fused face weights could be obtained by fusing columns together instead of rows.

The fused face weights (3.21) are shown diagrammatically in Fig 3. These weights depend on both the spin variables a, b, c, d and the bond variables α, β, μ, ν . Moreover, according to section 3.2 the fused weights of symmetric $(m, 0) \times (0, n)$ fusion and $(0, m) \times (0, n)$ fusion can be simply derived from those of symmetric $(m, 0) \times (n, 0)$ fusion

$$W_{(0,n)}^{(m,0)} \left(\begin{array}{ccc|c} d & \beta & c & \\ \mu & & \nu & u \\ a & \alpha & b & \end{array} \right) \sim \left(\prod_{k=1}^n (-1)^m r_k^m \right) W_{(n,0)}^{(m,0)} \left(\begin{array}{ccc|c} d & \beta & c & \\ \mu & & \nu & u + \lambda \\ a & \alpha & b & \end{array} \right) \quad (3.22)$$

$$W_{(0,n)}^{(0,m)} \left(\begin{array}{ccc|c} d & \beta & c & \\ \mu & & \nu & u \\ a & \alpha & b & \end{array} \right) \sim \left(\prod_{k=1}^n f_{k-2}^m \right) W_{(n,0)}^{(m,0)} \left(\begin{array}{ccc|c} d & \beta & c & \\ \mu & & \nu & u \\ a & \alpha & b & \end{array} \right) \quad (3.23)$$

where \sim means equality up to gauge factors, compare equation (3.18) above, and the functions r_k^m and f_k^m are defined in (2.42) and (2.53), respectively.

The fused blocks (3.21) are built up from rows of fused faces. All properties held by single row of fused faces carry over to the fused blocks (3.21). From the push-through property (3.11) we have the following lemma.

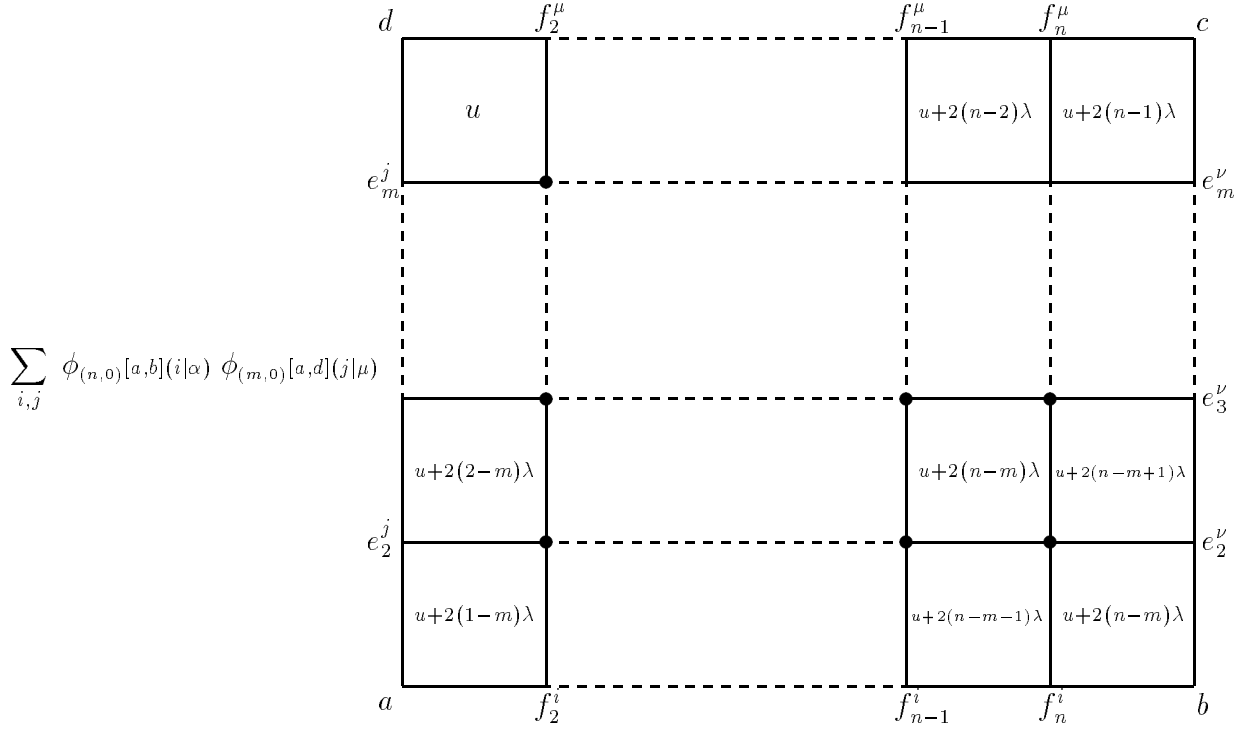


Figure 3: Diagrammatic representation of the face weights of the $n \times m$ fully symmetrically fused models. Sites indicated with a solid circle are summed over all possible spin states and the spins along the outer edges are given by $(a, e_2^j, \dots, e_m^j, d) = (a, d; m|j)$, $(b, e_2^\nu, \dots, e_m^\nu, c) = [b, c; (m, 0)|\nu]$, $(a, f_2^i, \dots, f_n^i, b) = (a, b; n|i)$ and $(d, f_2^\mu, \dots, f_n^\mu, c) = [d, c; (n, 0)|\mu]$.

Lemma 3.2 *Decomposing the path $(d, c; n|j)$ in terms of the independent paths $[d, c; (n, 0)|\beta]$ gives*

$$W_{(n,0)}^{(m,0)} \left(\begin{array}{ccc|c} d & j & c & u \\ \mu & & \nu & \\ a & \alpha & b & \end{array} \right) = \sum_{\beta=1}^{A_{(d,c)}^{(n,0)}} \phi_{(n,0)}[d, c](j|\beta) W_{(n,0)}^{(m,0)} \left(\begin{array}{ccc|c} d & \beta & c & u \\ \mu & & \nu & \\ a & \alpha & b & \end{array} \right) \quad (3.24)$$

Similarly, decomposing the path $(b, c; m|j)$ in terms of the independent paths $[b, c; (m, 0)|\nu]$ gives

$$W_{(n,0)}^{(m,0)} \left(\begin{array}{ccc|c} d & \beta & c & u \\ \mu & & j & \\ a & \alpha & b & \end{array} \right) = \sum_{\nu=1}^{A_{(b,c)}^{(m,0)}} \phi_{(m,0)}[b, c](j|\nu) W_{(n,0)}^{(m,0)} \left(\begin{array}{ccc|c} d & \beta & c & u \\ \mu & & \nu & \\ a & \alpha & b & \end{array} \right). \quad (3.25)$$

This lemma implies the following theorem.

Theorem 3.3 For a triple of positive integers m, n, l , the fused face weights (3.21) satisfy the Yang-Baxter equation

$$\begin{aligned}
& \sum_{(\eta_1, \eta_2, \eta_3)} \sum_g W_{(n,0)}^{(m,0)} \left(\begin{array}{c|c} e & \mu & d \\ \nu & \eta_3 & u \\ f & \eta_1 & g \end{array} \right) W_{(l,0)}^{(m,0)} \left(\begin{array}{c|c} d & \gamma & c \\ \eta_3 & \beta & u-v \\ g & \eta_2 & b \end{array} \right) W_{(n,0)}^{(l,0)} \left(\begin{array}{c|c} f & \eta_1 & g \\ \rho & \eta_2 & v \\ a & \alpha & b \end{array} \right) \\
&= \sum_{(\eta_1, \eta_2, \eta_3)} \sum_g W_{(n,0)}^{(m,0)} \left(\begin{array}{c|c} g & \eta_1 & c \\ \eta_3 & \beta & u \\ a & \alpha & b \end{array} \right) W_{(l,0)}^{(m,0)} \left(\begin{array}{c|c} e & \eta_2 & g \\ \nu & \eta_3 & u-v \\ f & \rho & a \end{array} \right) W_{(n,0)}^{(l,0)} \left(\begin{array}{c|c} e & \mu & d \\ \eta_2 & \gamma & v \\ g & \eta_1 & c \end{array} \right).
\end{aligned} \tag{3.26}$$

4 Fusion at Level (n, m)

The fused face weights at fusion level (n, m) are more complicated to construct. For clarity let us first consider the case $n = m = 1$. Then we will generalize the construction to the general fusion level (n, m) .

4.1 Level $(1, 1)$ Fusion

Consider the following product of elementary face weights

$$\begin{array}{c} d \\ / \quad \backslash \\ a \quad u \quad c' \\ / \quad \backslash \\ a'' \quad -2\lambda \quad u+4\lambda \quad c'' \\ / \quad \backslash \\ e \quad -4\lambda \quad 2\lambda \quad u+2\lambda \quad c \\ / \quad \backslash \\ b \quad b \quad b \end{array} = \sum_f \begin{array}{c} d \\ / \quad \backslash \\ a \quad u \quad c' \\ / \quad \backslash \\ a'' \quad -2\lambda \quad u+2\lambda \quad g \quad 2\lambda \quad c'' \\ / \quad \backslash \\ e \quad -4\lambda \quad f \quad f \quad u+4\lambda \quad c \\ / \quad \backslash \\ b \quad b \quad b \end{array}. \tag{4.1}$$

This product can be simplified using the elementary antisymmetric fusion of section 2.3. Applying equations (2.35) and (2.36), and subsequently Lemma 2.2, the product (4.1) reduces to

$$\begin{array}{c} c' \\ / \quad \backslash \\ c_1 \quad 2\lambda \quad c'' \\ / \quad \backslash \\ c \end{array} \sum_f \begin{array}{c} d \\ / \quad \backslash \\ a \quad u \quad c' \\ / \quad \backslash \\ a'' \quad -2\lambda \quad u+2\lambda \quad g \quad 2\lambda \quad c'' \\ / \quad \backslash \\ e \quad -4\lambda \quad f \quad f \quad u+4\lambda \quad c \\ / \quad \backslash \\ b \quad b \quad b \end{array} = \begin{array}{c} c' \\ / \quad \backslash \\ c_1 \quad 2\lambda \quad c'' \\ / \quad \backslash \\ c \end{array} \begin{array}{c} d \\ / \quad \backslash \\ a \quad u \quad c' \\ / \quad \backslash \\ a'' \quad -2\lambda \quad u+2\lambda \quad g \quad 2\lambda \quad c'' \\ / \quad \backslash \\ e \quad -4\lambda \quad f_1 \quad f_1 \quad u+4\lambda \quad c \\ / \quad \backslash \\ b \quad b \quad b \end{array}$$

$$= h'(u) \frac{g(c', c)}{g(a, e)} \phi_{(0,1)}(a, a'', e) \quad c_1 \begin{array}{c} \text{---} c' \\ \diagdown \quad \diagup \\ 2\lambda \\ \diagup \quad \diagdown \\ \text{---} c \end{array} \quad \begin{array}{c} d \\ \diagdown \quad \diagup \\ a \quad u \quad c' \\ \diagup \quad \diagdown \\ e \quad -3\lambda \quad u+3\lambda \\ \diagdown \quad \diagup \\ b \quad a' \quad b \end{array} \quad (4.2)$$

where $c_1 = \min(c, c')$, the gauge factors $g(a, b)$ are given by (2.40), the coefficients of independent paths $\phi_{(0,1)}(a, a', b)$ are listed in (2.33) and

$$h'(u) = \frac{\vartheta_1(2\lambda)\vartheta_1(7\lambda)\vartheta_1(u-\lambda)\vartheta_1(u+4\lambda)}{\vartheta_1^2(2\lambda)\vartheta_1^2(3\lambda)}.$$

So we see that the fusion of three faces essentially reduces to that of two faces, similar to what we observed in section 2.4. Hence the fused weights at level $(1, 1)$ are basically those of elementary symmetric $su(2)$ type fusion [26] with the projector $X^{(b,a)}(-3\lambda)$.

Since it is our primary interest to derive relations between fused row transfer matrices, we do not care about the gauge factors in what follows. That means we can use (4.2) to *define* the fused weights using independent paths w.r.t. the projector $X^{(b,a)}(-3\lambda)$. Clearly, these are two-step paths whereas the fusion with the $\pm 2\lambda$ projectors is built on three-step paths, but they are of course in one-to-one correspondence which can be seen from (4.2). Although, to be precise, we should distinguish the two sets by using another notation, we will not do so because we do not want to introduce more symbols here.

As the complimentary projector $X^{(b,a)}(3\lambda)$ projects onto at most one-dimensional subspaces, we only have to exclude at most one path from all possible paths to obtain an independent set of paths for the projector $X^{(b,a)}(-3\lambda)$. Let us choose

$$\text{indpath}_{(1,1)}[a, b] = \begin{cases} \text{path}(a, b; 2) & \text{if } a \neq b \\ \{(a, a', a) \in \text{path}(a, a; 2) \mid a' \neq a\} & \text{if } a = b. \end{cases} \quad (4.3)$$

The corresponding coefficients $\phi_{(1,1)}(a, a', b|\alpha)$ can be obtained from the analogues of (2.12) which follow from the inversion relations (2.5) and the face weights at spectral parameter $u = 3\lambda$. Explicitly, they read

$$\phi_{(1,1)}(a, a', b|\alpha) = \begin{cases} \delta_{a', a_\alpha} & \text{if } a \neq b \\ A_{a', a'_\alpha}^{(1,0)} (-1)^{\delta_{a, a'}} \left(\frac{S(a'_\alpha)}{S(a')} \right)^{1/2} & \text{if } a = b \end{cases} \quad (4.4)$$

where (a, a'_α, b) is the element of $\text{indpath}_{(1,1)}[a, b]$ labeled by α . As before, the independent paths can again be used to split the projector from the fused faces

$$\begin{array}{c} d \\ \diagdown \quad \diagup \\ a \quad u \quad c' \\ \diagup \quad \diagdown \\ e \quad -3\lambda \quad u+3\lambda \\ \diagdown \quad \diagup \\ b \quad a' \quad b \end{array} = \sum_{a'} \phi_{(1,1)}(a, a', b|\alpha) \quad \begin{array}{c} a \\ \diagdown \quad \diagup \\ -3\lambda \\ \diagup \quad \diagdown \\ b \end{array} \quad \begin{array}{c} d \\ \text{---} c' \\ \text{---} u \quad \text{---} u+3\lambda \\ \text{---} a' \\ \text{---} b \end{array} \quad (4.5)$$

The fused weights are defined in the following Lemma.

Lemma 4.1 *If (a, b) and (d, c) are admissible edges at fusion level $(1, 1)$, then*

$$\begin{aligned}
W_{(1,1)}\left(\begin{array}{ccc|c} d & \beta & c & u \\ a & \alpha & b & \end{array}\right) &= -r_2^1 \begin{array}{c} d \qquad \beta \qquad c \\ \boxed{\begin{array}{cc} u & u+3\lambda \\ \alpha & \alpha \end{array}} \\ a \qquad \qquad b \end{array} = -r_2^1 \begin{array}{c} d \qquad c'_\beta \qquad c \\ \boxed{\begin{array}{cc} u & u+3\lambda \\ \alpha & \alpha \end{array}} \\ a \qquad \qquad b \end{array} \\
&= -r_2^1 \sum_{a'} \phi_{(1,1)}(a, a', b|\alpha) W\left(\begin{array}{cc|c} d & c'_\beta & u \\ a & a' & \end{array}\right) W\left(\begin{array}{cc|c} c'_\beta & c & u+3\lambda \\ a' & b & \end{array}\right) \quad (4.6)
\end{aligned}$$

where the sum is over all allowed spins a' , the bond variables take values $1 \leq \alpha \leq A_{a,b}^{(1,1)}$, $1 \leq \beta \leq A_{c,d}^{(1,1)}$, and where the coefficients $\phi_{(1,1)}(a, a', b|\alpha)$ are those of (4.4). Furthermore, the value of c'_β on the RHS is chosen such that $(d, c'_\beta, c) \in \text{indpath}_{(1,1)}[d, c]$, with β labeling the particular element of $\text{indpath}_{(1,1)}[d, c]$. The such defined weights satisfy the Yang-Baxter equation (4.17). In particular, we note that for all fixed $a, b, c, d, \alpha, \beta$ we have

$$W_{(1,1)}\left(\begin{array}{ccc|c} d & \beta & c & u \\ a & \alpha & b & \end{array}\right) = 0 \quad \text{for } u = 0, \lambda, -4\lambda. \quad (4.7)$$

Here we also used the same symbol for symmetric sum, which now refers to the symmetric sum defined by the projector $X^{(b,a)}(-3\lambda)$. Note that we kept the spectral parameter-dependent function $-r_2^1$ in the definition of the fused weights, which directly explains two of the zeros of (4.7). The zero at $u = 0$ is obvious from the orthogonality of the projectors $X^{(b,a)}(\pm 3\lambda)$.

4.2 Fusion at Level (n, m)

Fusion at level (n, m) can be constructed by generalizing the fusion procedure of level $(1, 1)$ fusion. Firstly, let us consider the one row fusion. Let $Y = (n, m)$ be a Young diagram with $n + 2m$ nodes in Fig 4. We use $2m$ faces with spectral parameter shifts listed in the left double rows of the Young diagram in Fig 4. The weights for level $(0, m)$ antisymmetric fusion represented by (3.17) are given as (3.18)

$$W_{(0,m)}\left(\begin{array}{ccc|c} d & \beta & c & u + 2n\lambda \\ a & \alpha & b & \end{array}\right) \sim \left(\prod_{k=1}^m (-r_k^1)\right) W_{(m,0)}\left(\begin{array}{ccc|c} d & \beta & c & u + 2n\lambda + \lambda \\ a & \alpha & b & \end{array}\right). \quad (4.8)$$

Using the remaining faces with the other spectral parameter shifts in Fig 4, we can construct the fusion $W_{(n,0)}\left(\begin{array}{ccc|c} d & \beta & c & u \\ a & \alpha & b & \end{array}\right)$. The one-row symmetric (n, m) fusion or the

$u+2(m+n-1)\lambda$	$u+2(m+n-2)\lambda$	$u+2n\lambda$	$u+2(n-1)\lambda$...	u
$u+2(m+n)\lambda$	$u+2(m+n-1)\lambda$	$u+2(n+1)\lambda$			

Figure 4: The sequence of the spectral parameter shifts for $Y = (n, m)$.

symmetric $(1, 0) \times (n, m)$ fusion can be constructed by studying the product

$$\begin{array}{c}
 \begin{array}{ccc}
 d & & c \\
 \hline
 & W_{(n,0)}(u) & W_{(m,0)}(u+(2n+1)\lambda) \\
 \hline
 a & \xrightarrow{\alpha} & c' \\
 & \searrow^{\alpha} & \swarrow_{\beta} \\
 a & & b \\
 & W_{(n,0)}^{(m,0)}(-2n+1)\lambda & \\
 & \swarrow & \searrow \\
 & a & b
 \end{array} \\
 (4.9)
 \end{array}$$

The projector in (4.9) is given by (3.21) with the spectral parameter $u = -2n\lambda - \lambda$ and we sum over α, β and c' . Following the discussion in previous sections, the fused face weight is found by introducing the coefficients $\phi_{(n,m)}[a, b](j|\mu)$ and then splitting the projector from the fused faces. To do so we again need to know the decomposition of paths into independent paths with respect to the projector in (4.9). For the case of level $(1, 1)$ fusion this is gained by studying the single face $X^{(b,a)}(-3\lambda)$ in (2.2) and thus we obtain the coefficients (4.4).

Let $(a, b; n, m)$ be the set of paths from a to b through α, c, β such that

$$\begin{array}{c}
 a \\
 \gamma \swarrow \quad \searrow \alpha \\
 d \quad (n,m) \quad c \\
 \rho \swarrow \quad \searrow \beta \\
 b
 \end{array}
 := W_{(n,0)}^{(m,0)} \left(\begin{array}{ccc} a & \alpha & c \\ \gamma & & \beta \\ d & \rho & b \end{array} \middle| -2n\lambda - \lambda \right) \neq 0. \quad (4.10)$$

This means that

$$(a, b; n, m) = \{(a, \alpha, c, \beta, b) | 1 \leq \alpha \leq A_{a,c}^{(n,0)}, 1 \leq \beta \leq A_{c,b}^{(m,0)} \text{ and } A_{a,c}^{(n,0)} A_{c,b}^{(m,0)} \neq 0\}. \quad (4.11)$$

It is obvious that the number $|(a, b; n, m)|$ of paths in the set $(a, b; n, m)$ is $[A^{(n,0)} A^{(m,0)}]_{a,b}$. Similar to the elementary fusion procedure at level $(2, 0)$ the fusion coefficients

$\phi_{(n,m)}[a, b](i|\mu)$ are introduced by

$$\begin{array}{c} a \\ \gamma \quad \alpha^i \\ \diagdown \quad / \\ (n,m) \\ \diagup \quad \backslash \\ \rho \quad \beta^i \\ b \end{array} = \sum_{\mu=1}^{|[a,b;(n,m)]|} \phi_{(n,m)}[a, b](i|\mu) \begin{array}{c} a \\ \gamma \quad \alpha^\mu \\ \diagdown \quad / \\ (n,m) \\ \diagup \quad \backslash \\ \rho \quad \beta^\mu \\ b \end{array} \quad (4.12)$$

where the independent paths $(a, \alpha^\mu, c^\mu, \beta^\mu, b) = [a, b; (n, m)|\mu] \in \text{indpath}_{(n,m)}[a, b]$ and $|[a, b; (n, m)]|$ is the number of the independent paths. Obviously, the number of independent paths, $|\text{indpath}_{(n,m)}[a, b]| = |[a, b; (n, m)]| = A_{a,b}^{(n,m)}$, is equal to $|(a, b; n, m)|$ minus the number of independent equations in (4.12). Notice that (4.12) is over determined if $n + m \geq 2L$. This means that there is no projector for the fusion when $n + m \geq 2L$ and so no independent paths. Therefore $A^{(n,m)} = 0$ for $n + m \geq 2L$.

To calculate the level (n, m) fused face weights it is necessary to find the coefficients $\phi_{(n,m)}[a, b](j|\mu)$ and the set $\text{indpath}_{(n,m)}[a, b]$. This can be done from the explicit weights $W_{(n,0)}^{(m,0)}$. The projector is given by the fused face weights of $(m, 0) \times (n, 0)$. The decomposition of paths with respect to this projector follows from (3.24)–(3.25) and the face $X(-3\lambda)$ which sits in the right-hand corner of the projector. One does not have a simple or unified form for the coefficients. It is tedious but straightforward to calculate them. Instead of doing so we formulate the fused weights of symmetric $1 \times (n, m)$ fusion in terms of the coefficients as follows

$$\begin{aligned}
 & W_{(n,m)} \left(\begin{array}{ccc|c} d & \nu & c & u \\ a & \mu & b & \end{array} \right) \\
 &= \prod_{k=1}^m \left(-s_{n+k}^1 s_{n+k-5/2}^1 \right) \sum_{i=1}^{|(a,b;n,m)|} \phi_{(n,m)}[a, b](i|\mu) \\
 & \times W_{(n,0)} \left(\begin{array}{ccc|c} d & \gamma^\nu & e^\nu & u \\ a & \gamma^i & e^i & \end{array} \right) W_{(m,0)} \left(\begin{array}{ccc|c} e^\nu & \eta^\nu & c & u + 2n\lambda + \lambda \\ c^i & \eta^i & b & \end{array} \right) \quad (4.13)
 \end{aligned}$$

where

$$\begin{aligned}
 (d, \gamma^\nu, e^\nu, \eta^\nu, c) &= [d, c; (n, m)|\nu] \in \text{indpath}_{(n,m)}[d, c] \\
 (a, \gamma^i, e^i, \eta^i, b) &= (a, b; (n, m)|i) \in (a, b; n, m)
 \end{aligned}$$

and the notation $(a, b; (n, m)|i)$ denotes the i -th path in the set $(a, b; n, m)$. These bond variables ν, μ are restricted by $1 \leq \nu \leq A_{(d,c)}^{(n,m)}$ and $1 \leq \mu \leq A_{(a,b)}^{(n,m)}$. The push-through property follows from (4.9). To see this we push the projector from the bottom through to the top by using the Yang-Baxter equation (3.26). It is easy to see that the path along the top from d to c satisfies the same property as the projector. So from (4.12) we have the push-through property for $W_{(n,m)}$.

Based on one row fusion $W_{(n,m)}$ we can easily build up multi-row fusion. Let m, n, \bar{m}, \bar{n} be positive integers and define

$$\begin{aligned}
W_{(\bar{n}, \bar{m})}^{(n,m)} \left(\begin{array}{ccc|c} d & \beta & c & \\ \mu & & \nu & u \\ a & \alpha & b & \end{array} \right) &= \begin{array}{|c|} \hline d & \beta & c \\ \hline \mu & u & \nu \\ \hline a & \alpha & b \\ \hline \end{array} \\
&= \prod_{k=1}^{\bar{n}} (-s_{k+1/2}^m s_{k-2}^m) \prod_{k=1}^{\bar{m}} (-s_{k+\bar{n}+1}^m s_{k+\bar{n}-3/2}^m) \sum_{i=1}^{|(a,d;n,m)|} \phi_{(n,m)}[a, d](i|\mu) \\
&\quad \sum_{\rho} W_{(\bar{n}, \bar{m})}^{(m,0)} \left(\begin{array}{ccc|c} d & \beta & c & \\ \eta^i & & \eta^\nu & u \\ e^i & \rho & e^\nu & \end{array} \right) W_{(\bar{n}, \bar{m})}^{(n,0)} \left(\begin{array}{ccc|c} e^i & \rho & e^\nu & \\ \gamma^i & & \gamma^\nu & u-2m\lambda-\lambda \\ a & \alpha & b & \end{array} \right), \quad (4.14)
\end{aligned}$$

where

$$\begin{aligned}
(b, \gamma^\nu, e^\nu, \eta^\nu, c) &= [b, c; (n, m)|\nu] \in \text{indpath}_{(n,m)}[b, c] \\
(a, \gamma^i, e^i, \eta^i, d) &= (a, d; (n, m)|i) \in (a, d; n, m).
\end{aligned}$$

The sum over ρ is over $1, 2, \dots, |[e^i, e^\nu; (\bar{n}, \bar{m})]|$. The push-through property of each row carries over to the multi-row fusion. Therefore we have the following Lemma.

Lemma 4.2 *Decomposing the path $(d, c; \bar{n}, \bar{m}|j)$ in terms of the independent paths $[d, c; (\bar{n}, \bar{m})|\beta]$ gives*

$$W_{(\bar{n}, \bar{m})}^{(n,m)} \left(\begin{array}{ccc|c} d & j & c & \\ \mu & & \nu & u \\ a & \alpha & b & \end{array} \right) = \sum_{\beta=1}^{A_{(d,c)}^{(\bar{n}, \bar{m})}} \phi_{(n,0)}[d, c](j|\beta) W_{(\bar{n}, \bar{m})}^{(n,m)} \left(\begin{array}{ccc|c} d & \beta & c & \\ \mu & & \nu & u \\ a & \alpha & b & \end{array} \right) \quad (4.15)$$

Similarly, decomposing the path $(b, c; n, m|j)$ in terms of the independent paths $[b, c; (n, m)|\nu]$ gives

$$W_{(\bar{n}, \bar{m})}^{(n,m)} \left(\begin{array}{ccc|c} d & \beta & c & \\ \mu & & j & u \\ a & \alpha & b & \end{array} \right) = \sum_{\nu=1}^{A_{(b,c)}^{(n,m)}} \phi_{(m,0)}[b, c](j|\nu) W_{(\bar{n}, \bar{m})}^{(n,m)} \left(\begin{array}{ccc|c} d & \beta & c & \\ \mu & & \nu & u \\ a & \alpha & b & \end{array} \right). \quad (4.16)$$

Then the lemma implies that the fused faces satisfy the Yang-Baxter relation (4.17).

Theorem 4.3 *For a group of positive integers $m, n, l, \bar{m}, \bar{n}, \bar{l}$, the fused face weights (4.15) satisfy the Yang-Baxter equation*

$$\begin{aligned}
&\sum_{(\eta_1, \eta_2, \eta_3)} \sum_g W_{(\bar{n}, \bar{n})}^{(m, \bar{m})} \left(\begin{array}{ccc|c} e & \mu & d & \\ \nu & & \eta_3 & u \\ f & \eta_1 & g & \end{array} \right) W_{(\bar{l}, \bar{l})}^{(m, \bar{m})} \left(\begin{array}{ccc|c} d & \gamma & c & \\ \eta_3 & & \beta & u-v \\ g & \eta_2 & b & \end{array} \right) \times W_{(\bar{n}, \bar{n})}^{(l, \bar{l})} \left(\begin{array}{ccc|c} f & \eta_1 & g & \\ \rho & & \eta_2 & v \\ a & \alpha & b & \end{array} \right) \\
&= \sum_{(\eta_1, \eta_2, \eta_3)} \sum_g W_{(\bar{n}, \bar{n})}^{(m, \bar{m})} \left(\begin{array}{ccc|c} g & \eta_1 & c & \\ \eta_3 & & \beta & u \\ a & \alpha & b & \end{array} \right) \times W_{(\bar{l}, \bar{l})}^{(m, \bar{m})} \left(\begin{array}{ccc|c} e & \eta_2 & g & \\ \nu & & \eta_3 & u-v \\ f & \rho & a & \end{array} \right) W_{(\bar{n}, \bar{n})}^{(l, \bar{l})} \left(\begin{array}{ccc|c} e & \mu & d & \\ \eta_2 & & \gamma & v \\ g & \eta_1 & c & \end{array} \right). \quad (4.17)
\end{aligned}$$

5 $su(3)$ Fusion Hierarchy

5.1 Functional Equations

The $su(3)$ fusion rule (2.8) relates the adjacency matrices of fused models. We will see in this section that the similar relations carry over to row transfer matrices.

Suppose that \mathbf{a} ($\boldsymbol{\alpha}$) and \mathbf{b} ($\boldsymbol{\beta}$) are allowed spin (bond) configurations of two consecutive rows of an N (even) column lattice with periodic boundary conditions. The elements of the fused row transfer matrix $\mathbf{T}(u)$ are given by

$$\langle \mathbf{a}, \boldsymbol{\alpha} | \mathbf{T}_{(\bar{n}, \bar{m})}^{(n, m)}(u) | \mathbf{b}, \boldsymbol{\beta} \rangle = \prod_{j=1}^N \sum_{\eta_j} W_{(\bar{n}, \bar{m})}^{(n, m)} \left(\begin{array}{cc|c} a_{j+1} & \eta_{j+1} & b_{j+1} \\ \alpha_j & & \beta_j \\ a_j & \eta_j & b_j \end{array} \middle| u \right) = \begin{array}{c} \begin{array}{ccc} a_{j+1} & & b_{j+1} \\ \alpha_j & u & \beta_j \\ a_j & & b_j \end{array} \\ \hline \end{array} \quad (5.1)$$

where the periodic boundary conditions $a_{N+1} = a_1$, $b_{N+1} = b_1$ and $\eta_{N+1} = \eta_1$ are imposed. The Yang-Baxter equations (4.17) imply that

$$[\mathbf{T}_{(\bar{n}, \bar{m})}^{(n, m)}(u), \mathbf{T}_{(\bar{n}', \bar{m}')}^{(n', m')}(v)] = 0. \quad (5.2)$$

Thus for each fixed pair (n, m) we obtain a hierarchy of commuting transfer matrices. Moreover, the fusion procedure implies various relations among these transfer matrices. Let us define

$$\begin{aligned} \mathbf{T}_k^{(n, m)} &= \mathbf{T}_{(n, m)}^{(n', m')}(u + 2k\lambda) \\ \mathbf{T}^{(n, m)} &= 0 \quad \text{if } n < 0 \text{ or } m < 0 \\ \mathbf{T}^{(0, 0)} &= \mathbf{I} \end{aligned} \quad (5.3)$$

and

$$s_n^Y = \prod_{j \in Y} \frac{\vartheta_1(u - u_j + 2n\lambda)}{\sqrt{\vartheta_1(2\lambda)\vartheta_1(3\lambda)}} \quad (5.4)$$

$$f_n^Y = (-1)^{nN} [s_{n-3/2}^Y s_{n-1}^Y s_{n-1/2}^Y s_{n+1}^Y s_{n+3/2}^Y s_{n+2}^Y]^{nN} \quad (5.5)$$

where the shifts u_i are listed in Fig 4 with $u = 0$. We summarize the functional relations in the following theorems.

Theorem 5.1 ($su(3)$ Fusion Hierarchy) *The fused transfer matrices of the dilute A_L lattice models satisfy the functional relations*

$$\mathbf{T}_0^{(n,0)} \mathbf{T}_n^{(1,0)} = \mathbf{T}_0^{(n-1,1)} + \mathbf{T}_0^{(n+1,0)} \quad (5.6)$$

$$\mathbf{T}_0^{(n,0)} \mathbf{T}_n^{(0,1)} = \mathbf{T}_0^{(n,1)} + f_{n-1}^Y \mathbf{T}_0^{(n-1,0)} \quad (5.7)$$

where $Y = (n', m')$ refers to the fixed vertical fusion level. This hierarchy closes and the fused transfer matrices vanish for $m + n \geq 2L$

$$\mathbf{T}^{(n,m)} = 0 \quad \text{if } m + n \geq 2L. \quad (5.8)$$

Theorem 5.2 (Symmetry) *The fused transfer matrices $\mathbf{T}_0^{(n,m)}$ are given by*

$$\mathbf{T}_0^{(n,0)} \mathbf{T}_n^{(0,m)} = \mathbf{T}_0^{(n,m)} + f_{n-1}^Y \mathbf{T}_0^{(n-1,0)} \mathbf{T}_{n+1}^{(0,m-1)} \quad (5.9)$$

and satisfy the symmetry

$$\left[\prod_{k=0}^{n-1} s_{k-2}^Y s_{k+1/2}^Y \right]^N \mathbf{T}^{(n,m)}(u) = (-1)^{Nn(m-n)} \left[\prod_{k=0}^{m-1} s_{n+k-3/2}^Y s_{n+k+1}^Y \right]^N \left[\mathbf{T}^{(m,n)}(-u - 2(n+m-2)\lambda) \right]^T \quad (5.10)$$

where the superscript T denotes transpose and $Y = (n', m')$.

These theorems reduce to (1.21)–(1.23) for $Y = (n', m') = (1, 0)$.

5.2 Bethe Ansatz Equations

In section 1.3 we discussed the connection of the fusion hierarchy (1.21)–(1.23) with the Bethe ansatz equations for $(n', m') = (1, 0)$. We now generalize the Bethe ansatz equations to the fused transfer matrices with $Y = (n', m')$.

First let us consider the transfer matrix $\mathbf{T}_{(1,0)}^{(n',m')}(u) = \mathbf{T}_{(1,0)}^Y(u)$. The eigenvalues $\Lambda_{(1,0)}^Y(u)$ and the Bethe ansatz equations of this transfer matrix are given by

$$\begin{aligned} T_{(1,0)}^Y(u) &= \omega [s_{-1}^Y(u) s_{-3/2}^Y(u)]^N \frac{Q^Y(u + \lambda)}{Q^Y(u - \lambda)} + \omega^{-1} [s_0^Y(u) s_{-1/2}^Y(u)]^N \frac{Q^Y(u - 4\lambda)}{Q^Y(u - 2\lambda)} \\ &+ [(-1)^{n'} s_0^Y(u) s_{-3/2}^Y(u)]^N \frac{Q^Y(u) Q^Y(u - 3\lambda)}{Q^Y(u - \lambda) Q^Y(u - 2\lambda)} \end{aligned} \quad (5.11)$$

where

$$Q^Y(u) = \prod_{j=1}^{(n+2m)N} \vartheta_1(u - iu_j) \quad (5.12)$$

and the zeros $\{u_j\}$ satisfy the Bethe ansatz equations

$$\omega \left[\frac{s_{1/2}^Y(iu_j)}{s_{-1/2}^Y(iu_j)} \right]^N = (-1)^{n'N+1} \prod_{k=1}^{(n+2m)N} \frac{\vartheta_1(iu_j - iu_k + 2\lambda)\vartheta_1(iu_j - iu_k - \lambda)}{\vartheta_1(iu_j - iu_k - 2\lambda)\vartheta_1(iu_j - iu_k + \lambda)} \quad (5.13)$$

with $j = 1, \dots, N$ and $\omega = \exp(i\pi\ell/(L+1))$, $\ell = 1, \dots, L$.

As in section 1.3 we now set

$$\begin{aligned} \boxed{1}^k &= \omega [s_{k-1}^Y(u)s_{k-3/2}^Y(u)]^N \frac{Q^Y(u + k\lambda_2 + \lambda)}{Q^Y(u + k\lambda_2 - \lambda)} \\ \boxed{2}^k &= [-s_k^Y(u)s_{k-3/2}^Y(u)]^N \frac{Q^Y(u + k\lambda_2)Q^Y(u + k\lambda_2 - 3\lambda)}{Q^Y(u + k\lambda_2 - \lambda)Q^Y(u + k\lambda_2 - 2\lambda)} \\ \boxed{3}^k &= \omega^{-1} [s_k^Y(u)s_{k-1/2}^Y(u)]^N \frac{Q^Y(u + k\lambda_2 - 4\lambda)}{Q^Y(u + k\lambda_2 - 2\lambda)} \end{aligned} \quad (5.14)$$

so that

$$T_{(1,0)}^Y(u) = \boxed{1}^0 + \boxed{2}^0 + \boxed{3}^0 = \sum \boxed{}^0 \quad (5.15)$$

The $su(3)$ functional equations described in Theorem 5.1 imply the eigenvalues of the fused row transfer matrix at level (n, m) can be written as

$$T_{(n,m)}^Y(u) = \sum \underbrace{\left[\begin{array}{|c|c|c|} \hline \cdots & & \\ \hline \cdots & & \\ \hline \end{array} \right]}_m \underbrace{\left[\begin{array}{|c|c|c|} \hline \cdots & & \\ \hline \cdots & & \\ \hline \end{array} \right]}_n^0 \quad (5.16)$$

where the number of terms in the sum is given by the dimension of the irreducible representations of $su(3)$

$$(n+1)(m+1)(n+m+2)/2. \quad (5.17)$$

Such a Young tableau denotes the product of the $(n+2m)$ labeled boxes as given by (5.14) where it is understood that the relative shifts in the arguments are given by Figure 4. These zeros $\{u_j\}$ satisfy the Bethe ansatz equations (5.13). Thus $T_{(n,m)}^Y(u)$ can also be represented in the determinantal form (1.33) where f is given by (5.5).

6 Concluding Remarks

We applied the fusion procedure to the dilute A_L lattice models. For these models, there are two types of fusion, related to the values $u = \pm 2\lambda$ and $u = \pm 3\lambda$ of the spectral parameter u where the local face operator $X_j(u)$ (2.1) becomes singular. These two types of fusion are very different in nature, one showing an $su(2)$ structure, the other to an $su(3)$ structure. Here we concentrated on the $su(3)$ type fusion, the $su(2)$ fusion hierarchies of the dilute A_L models are studied in [26].

The fused models which we construct are labeled by altogether four integers, one pair denoting the horizontal and the other pair the vertical fusion level. Even in simplest non-trivial case, the actual fused face weights become rather cumbersome which is the reason why we avoid to show any explicit fused weights throughout this paper.

As the main result of our investigation, we derived the $su(3)$ fusion hierarchy (Theorems 5.1 and 5.2). We also discuss the connection to the Bethe ansatz equations for the dilute A_L models. The hierarchy closes (5.8) and thereby yields functional equations for the row transfer matrices of the dilute A_L models. These equations can be written in a determinantal form (1.33) and in principle can be solved for the eigenvalue spectra. The solution of these equations will be presented in a subsequent paper [15].

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