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Exact Solvability of the Calogero and Sutherland Models

by

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ABSTRACT

Translationally invariant symmetric polynomials as coordinates for n-body problems with identical particles are proposed. It is shown that in those coordinates the Calogero and Sutherland n-body Hamiltonians after appropriate gauge transformations can be presented as a *quadratic* polynomial in the generators of the algebra sl_n in finite-dimensional degenerate representations. The exact solvability of these models follows from the existence of the infinite flags of such representation spaces, preserved by above Hamiltonians.

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The Calogero and Sutherland models are remarkable n-body classical and quantum problems with a pairwise interaction that are completely integrable and at the same time are exactly solvable in their quantum versions [1, 2] (see also an excellent review [3]). However, unlike the notion of integrability, which had been well-established, for a long time the meaning of exact solvability remained intuitive, intrinsically folkloric. Just recently, an attempt to establish a certain definition of exact solvability was done [4].

Let us take an infinite set of finite-dimensional spaces $V_0, V_1, V_2, \ldots$ with an explicit basis. Assume that they are embedded one into another, forming the infinite flag of spaces: $V_0 \subset V_1 \subset V_2 \subset \ldots$. The simplest example of such a flag of spaces V_n is given by a linear space of polynomials in one variable of degree not higher than n :

$$V_n = \mathrm{span}\{1, x, x^2, \ldots x^n\} \ . \tag{1}$$

Then one can give a definition that *a linear operator h is named* **exactly solvable**, *if it preserves the infinite flag* of the spaces V_n :

$$h : V_n \mapsto V_n \ , \ n = 0, 1, 2, \ldots \tag{2}$$

Such a definition immediately leads to the conclusion that in the basis, where all spaces V_n are naturally defined, the operator h in matrix form is given by a triangular matrix. This does not contradict h being selfadjoint.

Now one can state that our task is to find the flags of different spaces V_n, where the Hamilton operators act. Once we succeed in this task, we end up with the *exactly solvable Hamilton operators.*

There are different ways to choose the spaces V_n. One of the most natural choices is to identify the space V_n with a finite-dimensional representation space of a certain semisimple Lie algebra. Then it is evident that generically an exactly solvable operator h should be a nonlinear combination of generators of the Cartan and negative root generators. For instance, the space (1) corresponds to the $(n+1)$-dimensional representation of the algebra sl_2 realized by first-order differential operators acting on the real line.

Of course, such a choice of V_n is very particular. However, it is quite amazing that all known one-dimensional exactly solvable quantum mechanical problems like the harmonic oscillator, Coulomb problem etc can be interpreted in such a manner as well as all exactly solvable discrete operators having Hahn, Kravchuk, Pollachek, Charlier and Meixner polynomials as the eigenfunctions [5]. The purpose of the present paper is to show that the $N-$body Calogero and Sutherland models also belong to this type being matched to finite-dimensional represention of the algebra sl_N. Thus the list of finite-dimensional exactly solvable problems is exhausted and we arrived at the conclusion that all of them are of the same type being presented by a quadratic combination of the generators of a certain Lie algebra.

1. The Calogero model

The Calogero model [1] is a quantum mechanical system of N particles on a line interacting via a pairwise potential and is defined by the Hamiltonian

$$H_{Cal} = \frac{1}{2}\sum_{i=1}^{N}\left[-d_i^2 + \omega^2 x_i^2\right] + \sum_{j<i}^{N}\frac{g}{(x_i - x_j)^2} \qquad , \tag{3}$$

where $d_i \equiv \frac{\partial}{\partial x_i}$, ω is the harmonic oscillator frequency and hereafter normalized to $\omega = 1$, and $g = \nu(\nu - 1)$ is the coupling constant.

As it was found by F. Calogero [1], this model is completely integrable and possesses some very remarkable properties. Firstly, in order to be normalizable, all eigenfunctions must (up to a certain function as a common factor) be either totally symmetric or totally antisymmetric. Secondly, it turned out that the energy spectrum is that of N bosons or fermions interacting via harmonic forces only, but with a total energy shift proportional to ν (for a review see [3]). Furthermore, the multiplicity of degeneracies of the states is the same as in the pure oscillator problem without repulsion. It is also worth noting that for $N = 3$ (and for this case only) the problem can be solved by separation of variables.

Thus the problem (3) is not only completely integrable, but also exactly solvable: the spectrum can be found explicitly, in a closed analytic form. All eigenfunctions have a form $\beta^{\nu^\star} e^{-\frac{X^2}{2}} P(x)$, where β is the Vandermonde determinant, $X^2 = \sum_{i=1}^{N} {x_i}^2$, $\nu^\star$ equals either ν or $1 - \nu$ and $P(x)$ is a completely symmetric polynomial under the permutation of any two coordinates. Thirdly, after extracting the center-of-mass motion, the remaining operator is translationally invariant under: $x_i \to x_i + a$.

Now let us make a rotation of the Hamiltonian (3) [6]

$$
\begin{aligned}
h \equiv -2\beta^{-\nu^\star} e^{\frac{X^2}{2}} H_{Cal} \beta^{\nu^\star} e^{-\frac{X^2}{2}} = \\
= \sum_{i=1}^{N} d_i^2 \ - \ 2\sum_{i=1}^{N} x_i d_i \ + \ \nu^\star \sum_{j\neq i}^{N} \frac{1}{x_i - x_j}[d_i - d_j] \\
- N - \nu^\star N(N-1) \ ,
\end{aligned}
\tag{4}
$$

where $g = \nu^\star(\nu^\star - 1)$, thereafter the constant term in (4) will be omitted. Introduce the center-of-mass

$$
Y \ = \ \sum_{j=1}^{N} x_j \ ,
$$

and translationally invariant Jacobi coordinates [7] (see also [6])

$$
y_i \ = \ x_i - \frac{1}{N}\sum_{j=1}^{N} x_j \ , \quad i = 1, 2, \ldots N \tag{5}
$$

fulfilling the constraint $\sum_{i=1}^{N} y_i = 0$.

It is known that the appropriate eigenfunctions of the operator (4) are completely symmetric under the permutations of x-coordinates [1] and, consequently, of y-coordinates. Now, instead of the explicit symmetrization of the eigenfunctions of (4), we encode this feature in a coordinate system consisting of functions in the $\{x_i\}$ and $\{y_i\}$ which are completely symmetric under the permutations. The eigenfunctions, depending on these coordinates, are then automatically symmetric.

There exist many different types of symmetric polynomials. For our purpose, the most adequate set of symmetric polynomials is given by the elementary symmetric (ES) polynomials [8]:

$$
\begin{aligned}
\sigma_1(x) &= \sum_{1\leq i\leq N} x_i \ , \\
\sigma_2(x) &= \sum_{1\leq i<j\leq N} x_i x_j \ , \\
\sigma_3(x) &= \sum_{1\leq i<j<k\leq N} x_i x_j x_k \ , \\
&\vdots \\
\sigma_N(x) &= x_1 x_2 x_3 \ldots x_N \ ,
\end{aligned}
\tag{6}
$$

where $x \in \mathbf{R}$. We notice that $\sigma_1(x)$ coincides with the center-of-mass coordinate Y. Those polynomials can be defined also by a generating function

$$\prod_{i=1}^{N}(1 + x_i t) = \sum_{n=0}^{N} \sigma_n(x) t^n \tag{7}$$

with $\sigma_0 = 1$. The ES-functions can be used as coordinates on the subset $\mathbf{E} \subset \mathbf{R}^N$

$$0 < x_1 < x_2 < x_3 \ldots < x_N < \infty$$

of $\mathbf{R}^N$. In turn, differential operators acting on $\mathbf{E}$ can be rewritten in terms of ES-coordinates. If these differential operators are themselves symmetric, they can be extended from $\mathbf{E}$ to $\mathbf{R}^N$.

It is worth emphasizing that the coordinates (6) allow to avoid the problem of over-completeness of the basis, which appears if the Newton polynomials

$$s_k = \sum_{1}^{N} x_i^k \tag{8}$$

are used as the coordinates [7].

Moreover, the Jacobian in the volume form

$$d\sigma_1 \wedge d\sigma_2 \wedge \ldots \wedge d\sigma_N = J_N dx_1 \wedge dx_2 \wedge \ldots \wedge dx_N \tag{9}$$

is

$$J_N = (-1)^{\left[\frac{N}{2}\right]} \beta(x_1, x_2, \ldots, x_N) \tag{10}$$

with β the Vandermonde determinant as before and $[a]$ means the integer part of a. In turn we can express J_N^2 in terms of the polynomials s_k (8)

$$J_N^2 = \begin{vmatrix} N & s_1 & s_2 & s_3 & \ldots & s_{N-1} \\ s_1 & s_2 & s_3 & & & s_N \\ s_2 & s_3 & & & & \\ s_3 & & & & & \\ \vdots & & & & & \\ s_{N-1} & s_N & & & & s_{2N-2} \end{vmatrix} \tag{11}$$

These s_k (including those with $k > N$) can be expressed by the σ_n with the help of the generating function (7)

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} s_n(x) t^n = \log \left(\sum_{n=0}^{N} \sigma_n(x) t^n \right) \tag{12}$$

The coordinates (6) are not translationally invariant, but this property can also be included by considering the ES-polynomials of the arguments (5)

$$\tau_n(x) = \sigma_n((y(x)) \ , n = 2, 3, \ldots N \tag{13}$$

as the relative coordinates and with remaining $\sigma_1(x) = Y$ as the coordinate of the center of mass. Certainly, once $n > N$, all $\tau_n(x) = 0$. So τ_n implicitly contains the information of the value of N.

Making quite sophisticated but straightforward calculations, one can get an explicit expression for the Laplacian in τ-coordinates (7)

$$\Delta \equiv \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} = N \frac{\partial^2}{\partial \sigma_1^2} + \sum_{j,k=2}^{N} A_{jk} \frac{\partial^2}{\partial \tau_j \partial \tau_k} + \sum_{i=2}^{N} B_i \frac{\partial}{\partial \tau_i} \ , \tag{14}$$

where

$$A_{jk} = \frac{(N-j+1)(k-1)}{N} \tau_{j-1}\tau_{k-1} + \sum_{\ell \geq \max(1,k-j)} (k-j-2\ell)\tau_{j+\ell-1}\tau_{k-\ell-1}$$

$$-B_i = \frac{(N-i+2)(N-i+1)}{N} \tau_{i-2}$$

Here we put $\tau_0 = 1, \tau_1 = 0$ and $\tau_p = 0$, if $p < 0$ and $p > N$.

Finally, after the extraction of the center-of-mass motion and omitting the constant terms, the operator (4) reduces to h_{rel} for the relative motion

$$h_{rel} = \sum_{j,k=2}^{N} A_{jk} \frac{\partial^2}{\partial \tau_j \partial \tau_k} - 2 \sum_{i=2}^{N} i \tau_i \frac{\partial}{\partial \tau_i} - (\frac{1}{N} + \nu^\star) \sum_{i=2}^{N} (N-i+2)(N-i+1) \tau_{i-2} \frac{\partial}{\partial \tau_i} \tag{15}$$

Now one can pose a question: would it be possible to rewrite (15) as an element of the universal enveloping algebra of a certain Lie algebra in a finite-dimensional representation?

Let us take the algebra $gl_N(\mathbf{R})$. One of the simplest representations of this algebra in terms of first-order-differential operators is the following

$$J_i^- = \frac{\partial}{\partial t_i}\ , \quad i = 2,3,\dots N\ ,$$

$$J_{i,j}^0 = t_i J_j^- = t_i \frac{\partial}{\partial t_j}\ , \quad i,j = 2,3,\dots N\ ,$$

$$J^0 = n - \sum_{p=2}^{N} t_p \frac{\partial}{\partial t_p}\ ,$$

$$J_i^+ = t_i J^0\ , \quad i = 2,3,\dots N\ . \tag{16}$$

which acts on functions of $t \in \mathbf{R^{N-1}}$. One of the generators, namely $J^0 + \sum_{p=2}^{N} J_{p,p}^0$ is proportional to a constant and if it is extracted, we end up with the algebra $sl_N(\mathbf{R})$. The generators $J_{i,j}^0$ form the algebra of the vector fields $sl_{N-1}(\mathbf{R})$. The parameter n in (16) can be any real number. If n is a non-negative integer, the representation (16) becomes the finite-dimensional representation acting on the space of polynomials

$$V_n(t) \ = \ \mathrm{span}\{t_2^{n_2} t_3^{n_3} t_4^{n_4} \dots t_N^{n_N} : 0 \le \sum n_i \le n\} \tag{17}$$

This representation corresponds to a Young tableau of one row and n blocks and is irreducible.

It is easy to see that the operator h_{rel} for the relative motion can be rewritten in terms of the generators (16). The representation of h_{rel} in terms of generators (16) is

$$\begin{aligned} h_{rel} \ &= \ \sum_{j=2}^{N} \left\{ \frac{(N-j+1)(j-1)}{N} (J_{j-1,j}^0)^2 - 2 \sum_{\ell=1}^{j-1} \ell J_{j+\ell-1,j}^0 J_{j-\ell-1,j}^0 \right\} \\ &+ \ 2 \sum_{2 \le k < j \le N} \left\{ \frac{(N-j+1)(k-1)}{N} J_{j-1,j}^0 J_{k-1,k}^0 - \right. \\ &- \ \left. \sum_{\ell=1}^{k-1} (j-k+2\ell) J_{j+\ell-1,j}^0 J_{k-\ell-1,k}^0 \right\} - 2 \sum_{k=2}^{N} k J_{k,k}^0 \\ &- \ (\frac{1}{N} + \nu^\star) \sum_{k=2}^{N} (N-k+2)(N-k+1) J_{k-2,k}^0 \end{aligned} \tag{18}$$

where we identify $J^0_{0,k}$ with J^-_k and put $J^0_{1,k}$ equal to zero. Moreover the t_i in (16) are replaced by the τ_i.

Such a rewriting (18) can be performed when the parameter n takes any value. Hence, the operator h_{rel} possesses infinitely many finite-dimensional invariant subspaces $V_n(t)$, $n = 0, 1, 2, \ldots$ and, correspondingly, preserves an infinite flag of the spaces $V_n(t)$. Therefore this operator is exactly solvable according to the definition given above. The case $N = 3$ confirms the general hypothesis stated at [4] about non-existence of exactly- and quasi-exactly solvable problems in two-dimensional flat space without separability of variables (see p.3). Also the operator (18) at $N = 3$ is not contained in the lists of the (quasi)-exactly-solvable operators in R^2 presented in the papers [9].

2. The Sutherland model

The Sutherland model [2] (for a review see [3]) is a quantum mechanical system of N particles on a line interacting via a pairwise potential and defined by the Hamiltonian (cf.(3)) ($d_i = \frac{\partial}{\partial x_i}$)

$$H_{Suth} = -\frac{1}{2}\sum_{i=1}^{N} d_i^2 + \sum_{1\leq i<j\leq N} \frac{g}{\sin^2\frac{1}{2}(x_i - x_j)} \tag{19}$$

which is defined on the Hilbert space of functions over the torus $(S_1)^{\times N}$. Like in (3), here $g = \nu(\nu - 1)$ is the coupling constant.

Similarly to the Calogero model, this model is completely integrable and exactly-solvable – the spectrum can be found explicitly, in a closed analytic form. All eigenfunctions, in order to be normalizable, must (up to a certain function as a common factor) be either totally symmetric or totally antisymmetric and then have a form $\beta^{\nu^\star} P(x)$, where

$$\beta = \prod_{1\leq i<j\leq N} |\sin\frac{1}{2}(x_i - x_j)|$$

is a natural modification of the Vandermonde determinant, $\nu^\star$ equals either ν or $1 - \nu$, while $P(x)$ is a completely symmetric polynomial under the permutation of any two coordinates. It is evident, that the Hamiltonian is translationally invariant under $x_i \to x_i + a$.

Now we make the gauge transformation of the Hamiltonian (19) (see [2])

$$\begin{aligned} h &= -2\beta^{-\nu^\star} H_{Suth} \beta^{\nu^\star} \\ &= \sum_{i=1}^{N} d_i^2 + i\nu^\star \sum_{1\le i<j\le N} \cot \tfrac{1}{2}(x_i - x_j)(d_i - d_j) + \text{const} \end{aligned} \tag{20}$$

with the same relations between $g, \nu, \nu^\star$ as in the Calogero model.

In order to realize a possible hidden algebraic structure one should introduce a torus analogue of symmetric polynomial coordinates (7). Let us consider the torus

$$T_N = (S_1)^{\times N}$$

with standard coordinates

$$z_i(x) = e^{ix_i}, \qquad i = 1, \dots N \ , \tag{21}$$

$$0 \le x_i < 2\pi \ .$$

Introduce the ES polynomials (see (6))

$$\begin{aligned} \xi_n(x) &= \sigma_n(z(x)) \ , \\ \eta_n(x) &= \sigma_n(w) \ , \end{aligned} \tag{22}$$

where

$$w_i = e^{iy_i}(x), \qquad y_i = x_i - \frac{1}{N}\sigma_1(x) \ . \tag{23}$$

Obviously the coordinates (22) are translationally invariant under $x_i \to x_i + a$ and

$$\eta_n(x) = \frac{\xi_n(x)}{\Big(\xi_N(x)\Big)^{\frac{n}{N}}} \ .$$

Finally, as a complete system of the coordinates we take

$$\{\xi_N \ , \eta_n, 1 \le n \le (N-1)\} \ , \tag{24}$$

where ξ_N describes the motion of the center-of-mass. In these coordinates the Laplacian has the form

$$\begin{aligned} -\Delta &= N(\xi_N \frac{\partial}{\partial \xi_N})^2 + \sum_{j,k=1}^{N-1} B_{jk} \frac{\partial^2}{\partial \eta_j \partial \eta_k} \\ &+ \frac{1}{N} \sum_{l=1}^{N-1} l(N-l)\eta_l \frac{\partial}{\partial \eta_l} \end{aligned} \tag{25}$$

with

$$B_{jk} = \frac{k(N-j)}{N}\eta_j\eta_k + \sum_{l\geq \max(1,k-j)} (k-j-2l)\eta_{j+l}\eta_{k-l} \tag{26}$$

(cf.(14)). The center-of-mass motion can be separated and after omitting the constant term the Hamiltonian of the relative motion takes the form

$$\begin{aligned} -h_{rel} &= \sum_{j,k=1}^{N-1} B_{jk}\frac{\partial^2}{\partial\eta_j\partial\eta_k} \\ &+ (\nu^\star + \frac{1}{N})\sum_{l=1}^{N-1} l(N-l)\eta_l\frac{\partial}{\partial\eta_l} \end{aligned} \tag{27}$$

With t_j replaced by η_{j-1} in (16) we obtain

$$\begin{aligned} -h_{rel} &= \sum_{j=1}^{N-1}\left\{\frac{j(N-j)}{N}(J^0_{j,j})^2 - 2\sum_{l=1}^{j} lJ^0_{j+l,j}J^0_{j-l,j}\right\} \\ &+ 2\sum_{1\leq k<j\leq N-1}\left\{\frac{k(N-j)}{N}J^0_{j,j}J^0_{k,k} - \sum_{l=1}^{k}(j-k+2l)J^0_{j+l,j}J^0_{k-l,k}\right\} \\ &+ \nu^\star\sum_{l=1}^{N-1} l(N-l)J^0_{l,l} \end{aligned} \tag{28}$$

Here we identified $J^0_{0,k}$ with J^-_k. Like (18), the operator (28) at $N=3$ is not contained in the lists of the (quasi)-exactly-solvable operators in R^2 presented in the papers [9].

In the conclusion it is worth emphasizing that the Laplace operators (18), (25) (from physical point of view it means that we study a free motion without an interparticle interaction) after extraction of the center-of-mass motion have the representation as a quadratic combination in the generators (16) of the algebra $sl_N(\mathbf{R})$.

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References

[1] F. Calogero,

*J. Math. Phys.***10** (1969) 2191, 2197
*J. Math. Phys.***12** (1971) 419

[2] B. Sutherland,

Phys. Rev. **A 4** (1971) 2019
Phys. Rev. **A 5** (1972) 1372

[3] M. A. Olshanetsky and A. M. Perelomov,

Phys. Rep. **94** (1983) 313.

[4] A. V. Turbiner,

"Lie algebras and linear operators with invariant subspace", in "Lie Algebras, Cohomologies and New Findings in Quantum Mechanics", *Contemporary Mathematics*, vol. 160, AMS, 1994, N. Kamran and P. Olver (eds.), pp.263-310

[5] Yu. F. Smirnov, A. V. Turbiner,

Preprint IF-UNAM 95-68, Mexico (1995)
Intern. Journal of Modern Physics **A** (1995) (in press)

[6] A. V. Turbiner,

Phys. Lett. **B320** (1994) 281

[7] A. M. Perelomov,

Theoret. i Matem. Fizika **6** (1971)364

English translation: *Sov.Phys.– Theor. and Math. Phys.* **6** (1971)263

[8] I. G. Macdonald,

Symmetric functions and Hall polynomials, Oxford 1979, p. 12.

[9] M. A. Shifman, A. V. Turbiner,

Comm. Math. Phys. **126**, 347-365 (1989)

A. González-Lopéz, N. Kamran and P.J. Olver,
Comm.Math.Phys., **159** (1994) 503-538

R. Milson , “On the construction of Q.E.S. Schroedinger operators on homogeneous spaces”,
Journal of Math. Physics (1995) (in press)