Small Angle Bhabha Scattering for LEP †‡

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Abstract

We present the results of our calculations to a one, two, and three loop approximation of the $e^+e^-\to e^+e^-$ Bhabha scattering cross-section at small angles. All terms contributing to the radiatively corrected cross-section, within an accuracy of $\delta\sigma/\sigma = 0.1\%$, are explicitely evaluated and presented in an analytic form. $O(\alpha)$ and $O(\alpha^2)$ contributions are kept up to next-to-leading logarithmic accuracy, and $O(\alpha^3)$ terms are taken into account to the leading logarithmic approximation. We define an experimentally measurable cross-section by integrating the calculated distributions over a given range of final-state energies and angles. The cross-sections for exclusive channels as well as for the totally integrated distributions are also given.

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1 Introduction

An accurate verification of the Standard Model is one of the primary aims of LEP [1]. While electroweak radiative corrections to the s-channel annihilation process and to large angle Bhabha scattering allow a direct extraction of the Standard Model parameters, small angle Bhabha cross-section affects, as an overall normalization condition, all observable cross-sections and represents an equally unavoidable condition towards a precise determination of the Standard Model parameters. The small angle Bhabha scattering process is used to measure the luminosity of electron positron colliders. At LEP an experimental accuracy on the luminosity of

$$|\frac{\delta\sigma}{\sigma}| < 0.001 \qquad (1)$$

will soon be reached [2]. However, to obtain the total accuracy, a systematic theoretical error must also be added. This precision calls for an equally accurate theoretical expression for the Bhabha scattering cross-section in order to extract the Standard Model parameters from the observed distributions. An accurate determination of the small angle Bhabha cross-section and of the luminosity directly affects the determination of absolute cross-sections such as, for example, the determination of the invisible width and of the number of massless neutrino species N_ν [3].

In recent years a considerable attention has been devoted to the Bhabha process [4, 5]. However, the accuracy reached, is still inadequate. According to these evaluations the theoretical estimates are still incomplete, moreover, are in disagreement with each other up to 0.5%, far from the required theoretical and experimental accuracy [2].

The process that will be considered in this work is that of Bhabha scattering when electrons and positrons are emitted at small angles with respect to the initial electron and positron directions. We have examined the radiative processes inclusively accompanying the main $e^+e^-\to e^+e^-$ reaction at high energies, when both the scattered electron and positron are tagged within the counter aperture.

We assume that the center-of-mass energies are within the range of the LEP collider $2\epsilon = \sqrt{s} = 90-200$ GeV and the scattering angles are within the range $\theta \simeq 10-150$ mrad. We assume that the charged particle detectors have the following polar angle cuts:

$$\theta_1 < \theta_- = \widehat{\vec{p_1}\vec{p}_{1'}} \equiv \theta < \theta_3 \quad , \qquad \theta_2 < \theta_+ = \widehat{\vec{p_2}\vec{p}_{2'}} < \theta_4 \quad , \qquad 0.01 \lesssim \theta_i \lesssim 0.1 \text{ rad} \;, \qquad (2)$$

where $\vec{p}_1$, $\vec{p}_{1'}$, ($\vec{p}_2$, $\vec{p}_{2'}$) represent the momenta of the initial and of the scattered electron (positron) in the center-of-mass frame.

In this paper we present the results of our calculations of the Bhabha scattering cross-section with an accuracy of $O(0.1\%)$. The squared matrix elements of the various exclusive processes inclusively contributing to the $e^+e^-\to e^+e^-$ reaction are integrated in order to define an experimentally measurable cross-section according to suitable restrictions on the angles and energies of the detected particles. The various contributions to the electron and positron distributions, needed for the required accuracy, are presented using analytical expressions.

In order to define the angular range of interest and the implications on the required accuracy, let us first briefly discuss, in a general way, the angle-dependent corrections to the cross-section.

We consider e^+e^- scattering at angles as defined in Eq. (2). Within this region, if one expresses the cross-section by means of a series expansion in terms of angles, the main contribution to the cross-section $d\sigma/d\theta^2$ comes from the diagrams for the scattering amplitudes containing one exchanged photon in the t-channel. These diagrams, as it is well known, show a singularity of the type θ^{-4} for $\theta \to 0$, e.g.

$$\frac{d\sigma}{d\theta^2} \sim \theta^{-4} \quad .$$

Let us now estimate the correction of order θ^2 to this contribution. If

$$\frac{d\sigma}{d\theta^2} \sim \theta^{-4}(1 + c_1\theta^2) \quad , \tag{3}$$

then, after integration over θ^2 in the angular range as Eq. (2), one obtains:

$$\int\limits_{\theta^2_{\rm min}}^{\theta^2_{\rm max}} \frac{d\sigma}{d\theta^2}\, d\theta^2 \sim \theta^{-2}_{\rm min}(1 + c_1\theta^2_{\rm min} \ln \frac{\theta^2_{\rm max}}{\theta^2_{\rm min}}). \tag{4}$$

We see that, for $\theta_{\rm min} = 50$ mrad and $\theta_{\rm max} = 150$ mrad (we have taken the case where the θ^2 corrections are maximal), the relative contribution of the θ^2 terms is about $5\text{x}10^{-3}c_1$. Therefore, the terms of relative order θ^2 must only be kept in the Born cross-section where the coefficient c_1 is not small. In higher orders of the perturbative expansion the coefficient c_1 contains at least one factor α/π and therefore these terms can safely be omitted. This implies that, within our accuracy, only radiative corrections from the scattering type diagrams contribute. Furthermore, one should take into account only diagrams with one photon exchanged in the t-channel, since, according to the generalized eikonal representation, the large logarithmic terms from the diagrams with the multi-photon exchange are cancelled.

Having, as a final goal for the experimental cross-section, the relative accuracy as in Eq. (1), and by taking into account that the minimal value of the squared momentum transfer $Q^2 = 2\epsilon^2(1-\cos\theta)$ in the region (2) is of the order of 1 GeV^2, we may omit in the following also the terms appearing in the radiative corrections of the type m^2/Q^2 with m equal to the electron (m_e), or the muon (m_μ) mass.

The contents of this paper can be outlined as follows. In Section 2 we discuss the Born cross-section $d\sigma^B$, taking the Z^0 boson exchange into account, and compute the corrections to it caused by the virtual and real soft photon emission. We present the results, as discussed above, in the form of an expansion in terms of the scattering angle θ. We then define an experimentally measurable cross-section, $\sigma_{\rm exp}$, which is obtained by tagging the scattered electron and positron within a suitable range of polar angles and energies. We introduce the ratio $\Sigma = \sigma_{\rm exp}/\sigma_0$ by normalizing $\sigma_{\rm exp}$ with respect to the cross-section $\sigma_0 = 4\pi\alpha^2/\epsilon^2\theta_1^2$. In Section 2, by using a simplified version of the differential cross-section for the small angle scattering, we discuss the contribution to $\sigma_{\rm exp}$ from the single bremsstrahlung process. In Section 3 we find all corrections of $O(\alpha^2)$ to $\sigma_{\rm exp}$ caused by two virtual and real photon emissions. In Section 4 we consider $O(\alpha^2)$ corrections caused by e^+e^- pair emission. In Section 5 we derive the leading logarithmic corrections to the α^3 order by using the structure function method. In Section 6 we estimate the contributions of the neglected terms. Finally, in Section 7 we give the results obtained in terms of the ratio Σ as functions of the experimental parameters.

A more detailed derivation of these results will be presented elsewhere [6].

2 Born and one-loop soft and virtual corrections

The Born cross-section for Bhabha scattering within the Standard Model is well known [7]:

$$\frac{d\sigma^B}{d\Omega} = \frac{\alpha^2}{8s}\ \{4B_1 + (1-c)^2 B_2 + (1+c)^2 B_3\}\ , \tag{5}$$

where

$$\begin{aligned}
B_1 &= (\frac{s}{t})^2|1+(g_v^2-g_a^2)\xi|^2\ , \qquad B_2 = |1+(g_v^2-g_a^2)\chi|^2\ , \\
B_3 &= \frac{1}{2}|1+\frac{s}{t}+(g_v+g_a)^2(\frac{s}{t}\xi+\chi)|^2 + \frac{1}{2}|1+\frac{s}{t}+(g_v-g_a)^2(\frac{s}{t}\xi+\chi)|^2\ , \\
\chi &= \frac{\Lambda s}{s-m_z^2+iM_Z\Gamma_Z}\ , \qquad \xi = \frac{\Lambda t}{t-M_Z^2}\ , \\
\Lambda &= \frac{G_F M_Z^2}{2\sqrt{2}\pi\alpha} = (\sin 2\theta_w)^{-2}\ , \quad g_a = -\frac{1}{2}\ g_v = -\frac{1}{2}(1-4\sin^2\theta_w)\ , \\
s &= (p_1+p_2)^2 = 4\varepsilon^2\ , \qquad t = -Q^2 = (p_1-p_{1'})^2 = -\frac{1}{2}\ s\ (1-c)\ , \\
c &= \cos\theta\ , \qquad \theta = \widehat{\vec{p_1}\vec{p}_{1'}}\ .
\end{aligned}$$

Here θ_w is the Weinberg angle. In the small angle limit

$$c = \cos\theta = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} - \frac{\theta^6}{720} + \ldots\ .$$

Expanding the result (5) we have

$$\frac{d\sigma^B}{\theta d\theta} = \frac{8\pi\alpha^2}{\epsilon^2\theta^4}\ (1 - \frac{\theta^2}{2} + \frac{9}{40}\ \theta^4 + \delta_{\rm weak}), \tag{6}$$

where $\epsilon = \sqrt{s}/2$ is the electron or positron initial energy. The contribution from weak interactions $\delta_{\rm weak}$, connected with diagrams with Z^0 boson exchange, is given by the expression:

$$\delta_{\rm weak} = 2g_v^2\ \xi - \frac{\theta^2}{4}\ (g_v^2+g_a^2)\ Re\chi + \frac{\theta^4}{32}\ (g_v^4+g_a^4+6g_v^2g_a^2)\ |\chi|^2\ . \tag{7}$$

From Eq. (7) one can see that the contribution c_1^w of the weak correction $\delta_{\rm weak}$ into the coefficient c_1 in Eq. (3) is

$$c_1^w \lesssim 2g_v^2 + \frac{(g_v^2+g_a^2)}{4}\ \frac{M_Z}{\Gamma_Z} + \theta_{\rm max}^2\ \frac{(g_v^4+g_a^4+6g_v^2g_a^2)}{32}\ \frac{M_Z^2}{\Gamma_Z^2} \sim 1\ . \tag{8}$$

According to our previous discussion after Eq. (4) this means that the contribution connected with the Z^0 boson exchange diagrams does not exceed 0.3%. We shall therefore neglect the radiative corrections to weak contributions, since they could contribute at most with terms $\lesssim 10^{-4}$.

In the pure QED case one-loop radiative corrections to the Bhabha cross-section were calculated a long time ago [8]. Taking into account the contribution coming from the emission of soft photons with energy less than a given finite threshold $\Delta\epsilon$ as well, one obtains for $d\sigma^{(1)}_{\rm QED}/dc$ in one-loop approximation

$$\frac{d\sigma^{(1)}_{\rm QED}}{dc} = \frac{d\sigma^{B}_{\rm QED}}{dc}\,(1+\delta_{\rm virtual}+\delta_{\rm soft}) \ , \tag{9}$$

where $d\sigma^{B}_{\rm QED}$ is the Born cross-section in the pure QED case (i.e. it is equal to $d\sigma^B$ with $g_a = g_v = 0$) and

$$\begin{aligned}\delta_{\rm virtual}+\delta_{\rm soft} \ = \ & 2\,\frac{\alpha}{\pi}\,\{2\,[1-\ln(\frac{4\epsilon^2}{m^2})+2\ln ctg\,\frac{\theta}{2}]\ln\frac{\epsilon}{\Delta\epsilon}+\int_{\cos^2\frac{\theta}{2}}^{\sin^2\frac{\theta}{2}}\frac{dx}{x}\ln(1-x)\\ & -\frac{23}{9}+\frac{11}{6}\,\ln(\frac{4\epsilon^2}{m^2})\}+\frac{\alpha}{\pi}\,\frac{1}{(3+c^2)^2}\,[\frac{\pi^2}{3}\,(2c^4-3c^3-15c)\\ & +2\,(2c^4-3c^3+9c^2+3c+21)\,\ln^2\sin\frac{\theta}{2}\\ & -4\,(c^4+c^2-2c)\,\ln^2\cos\frac{\theta}{2}-4\,(c^3+4c^2+5c+6)\,\ln^2 tg\frac{\theta}{2}\\ & +\frac{2}{3}\,(11c^3+33c^2+21c+111)\,\ln\sin\frac{\theta}{2}+2\,(c^3-3c^2+7c-5)\,\ln\cos\frac{\theta}{2}\\ & +2\,(c^3+3c^2+3c+9)\,\delta_t-2\,(c^3+3c)(1-c)\,\delta_s]\ .\end{aligned}$$

The value δ_t (δ_s) is defined by the contributions to the photon vacuum polarization function $\Pi(t)$ [$\Pi(s)$] as follows:

$$\Pi(t) = \frac{\alpha}{\pi}\,(\delta_t+\frac{1}{3}L-\frac{5}{9})+\frac{1}{4}\,(\frac{\alpha}{\pi})^2 L \ , \tag{10}$$

where

$$L = \ln\frac{Q^2}{m^2} \ , \quad Q^2 = -t = 2\epsilon^2(1-c) \ , \tag{11}$$

and we have taken into account the leading logarithmic part of the two-loop corrections to the polarization operator. In the Standard Model δ_t contains contributions of muons, tau-leptons, W-bosons, and hadrons:

$$\delta_t = \delta_t^{\mu}+\delta_t^{\tau}+\delta_t^{W}+\delta_t^{H} \ , \qquad \delta_s = \delta_t\,(Q^2\to -s) \ , \tag{12}$$

and the first three contributions are theoretically calculable:

$$\begin{aligned}\delta_t^{\mu} &= \frac{1}{3}\ln\frac{Q^2}{m_\mu^2}-\frac{5}{9} \ ,\\ \delta_t^{\tau} &= \frac{1}{2}\,v_\tau\,(1-\frac{1}{3}v_\tau^2)\,\ln\frac{v_\tau+1}{v_\tau-1}+\frac{1}{3}\,v_\tau^2-\frac{8}{9} \ , \qquad v_\tau = (1+\frac{4m_\tau^2}{Q^2})^{\frac{1}{2}} \ ;\\ \delta_t^{w} &= \frac{1}{4}\,v_w\,(v_w^2-4)\,\ln\frac{v_w+1}{v_w-1}-\frac{1}{2}\,v_w^2+\frac{11}{6} \ , \qquad v_w = (1+\frac{4m_w^2}{Q^2})^{\frac{1}{2}} \ .\end{aligned} \tag{13}$$

The contribution of the hadrons δ_t^H can be expressed through the experimentally measurable cross-section of the e^+e^- annihilation [9].

In the limit of small scattering angles we can present Eq. (9) in the following form:

$$\frac{d\sigma^{(1)}_{\rm QED}}{dc} = \frac{d\sigma^{B}_{\rm QED}}{dc}\,[1-\Pi(t)]^{-2}\,(1+\delta)\,, \tag{14}$$
$$\delta = 2\frac{\alpha}{\pi}\,[2(1-L)\ln\frac{1}{\Delta} + \frac{3}{2}L - 2] + \frac{\alpha}{\pi}\,\theta^2\,\Delta_\theta + \frac{\alpha}{\pi}\,\theta^2\,\ln\Delta\,,$$
$$\Delta_\theta = \frac{3}{16}l^2 + \frac{7}{12}l - \frac{19}{18} + \frac{1}{4}\,(\delta_t - \delta_s)\,,$$
$$\Delta = \frac{\Delta\epsilon}{\epsilon}, \qquad l = \ln\frac{Q^2}{s} \simeq \ln\,\frac{\theta^2}{4}\,.$$

This representation gives us a possibility to verify explicitly that the terms of a relative order θ^2 in the radiative corrections are small. Taking into account that the large contribution proportional to $\ln\Delta$ disappears when we add the cross-section for the hard emission, one can verify once more that such terms can be neglected. Therefore in higher orders we will omit the annihilation diagrams as well as multiple-photon exchange diagrams in the scattering channel. The second simplification is justified by the generalized eikonal representation for the amplitudes at small scattering angles [10].

Let us introduce now the dimensionless quantity

$$\Sigma = \frac{Q_1^2\,\sigma_{\rm exp}}{4\pi\alpha^2}\,,$$

with $Q_1^2 = \epsilon^2\theta_1^2$ where $\sigma_{\rm exp}$ represents the experimentally observable cross-section:

$$\Sigma = \frac{Q_1^2}{4\pi\alpha^2}\int dx_1 \int dx_2\,\theta(x_1x_2 - x_c)\int d^2q_1\,\theta_1^c\int d^2q_2\,\theta_2^c \tag{15}$$
$$\frac{d\sigma^{e^+e^-\to e^+(\vec{q}_2,x_2)\;e^-(\vec{q}_1,x_1)+X}}{dx_1d^2q_1dx_2d^2q_2}\,,$$

where $x_{1,2}$ and $\vec{q}_{1,2}$ are, respectively, the energy fractions and the transverse components of the electron and positron momenta in the final state, sx_c is the experimental cut-off on their squared invariant mass sx_1x_2, and the functions θ_i^c which take into account the angular cuts in Eq. (2) are defined as:

$$\theta_1^c = \theta(\theta_3 - \frac{|\vec{q}_1|}{x_1\epsilon})\;\theta(\frac{|\vec{q}_1|}{x_1\epsilon} - \theta_1)\;, \qquad \theta_2^c = \theta(\theta_4 - \frac{|\vec{q}_2|}{x_2\epsilon})\;\theta(\frac{|\vec{q}_2|}{x_2\epsilon} - \theta_2)\;. \tag{16}$$

We restrict ourselves further to the symmetrical case only:

$$\theta_2 = \theta_1\;, \qquad \theta_4 = \theta_3\;, \qquad \rho = \frac{\theta_3}{\theta_1} > 1\;. \tag{17}$$

Let us define Σ as a sum of exclusive contributions:

$$\Sigma = \Sigma_0 + \Sigma^{\gamma} + \Sigma^{2\gamma} + \Sigma^{e^+e^-} + \Sigma^{3\gamma} + \Sigma^{e^+e^-\gamma} \quad , \tag{18}$$

where Σ_0 stands for a modified Born contribution, Σ^{γ}, $\Sigma^{2\gamma}$, and $\Sigma^{3\gamma}$ stand for the contributions of one, two, and three photon emissions (both real and virtual), $\Sigma^{e^+e^-}$ and $\Sigma^{e^+e^-\gamma}$ represent the emission of virtual or real (soft and hard) pairs without and with the accompaining real or virtual photon.

By integrating Eq. (6) with the use of the full propagator for the t-channel photon, which takes into account the growth of the electric charge at small distances, we obtain:

$$\Sigma_0 = \theta_1^2 \int\limits_{\theta_1^2}^{\theta_2^2} \frac{d\theta^2}{\theta^4}(1 - \Pi(t))^{-2} + \Sigma_W + \Sigma_\theta \quad , \tag{19}$$

where

$$\Sigma_W = \theta_1^2 \int\limits_{\theta_1^2}^{\theta_2^2} \frac{d\theta^2}{\theta^4}\, \delta_{\mathrm{weak}} \quad ,$$

is the correction due to the weak interactions and the term

$$\Sigma_\theta = \theta_1^2 \int\limits_{1}^{\rho^2} \frac{dz}{z}\, [1 - \Pi(-zQ_1^2)]^{-2}\, (-\frac{1}{2} + z\, \theta_1^2\, \frac{9}{40}) \quad ,$$

comes from the expansion of the Born cross-section in Eq. (5) in powers of θ^2.

The one-loop contribution Σ^{γ} comes from one-photon emission (real and virtual). By adding to Eq. (14) the cross-section for hard emission calculated using a simplified version of the differential cross-section for small angle scattering [11] we obtain:

$$\Sigma^{\gamma} = \frac{\alpha}{\pi} \int\limits_{1}^{\rho^2} \frac{dz}{z^2}[1 - \Pi(-zQ_1^2)]^{-2}\{\int\limits_{x_c}^{1} dx\, [(L_z - 1)P(x) \\ [1 + \theta(x^2\rho^2 - z)] + \frac{1+x^2}{1-x}k(x,z)] - 1\} \quad , \tag{20}$$

where

$$P(x) = (\frac{1+x^2}{1-x})_+ = \lim_{\Delta\to 0}\, \{\frac{1+x^2}{1-x}\, \theta(1 - x - \Delta) + (\frac{3}{2} + 2\ln\Delta)\, \delta(1-x)\} \quad , \tag{21}$$

is the non-singlet splitting kernel and

$$k(x,z) = \frac{(1-x)^2}{1+x^2}\, [1 + \theta(x^2\rho^2 - z)] + \; L_1 + \theta(x^2\rho^2 - z)\; L_2 \\ + \theta(z - x^2\rho^2)L_3 \quad . \tag{22}$$

Here $L_z = \ln \frac{z\,Q_1^2}{m^2}$ and

$$L_1 = \ln|\frac{x^2(z-1)(\rho^2-z)}{(x-z)(x\rho^2-z)}| \ , \quad L_2 = \ln|\frac{(z-x^2)(x^2\rho^2-z)}{x^2(x-z)(x\rho^2-z)}| \ , \tag{23}$$
$$L_3 = \ln|\frac{(z-x^2)(x\rho^2-z)}{(x-z)(x^2\rho^2-z)}| \ .$$

3 Two-photon emission

Let us now consider the corrections of the order of α^2. They come from the two-photon emission as well as from the pair production, real and virtual. The virtual and soft real photon corrections can be obtained by using the results of Refs. [11]-[14].

The corresponding contributions to Σ are:

$$\Sigma^{\gamma\gamma}_{S+V} = \Sigma_{VV} + \Sigma_{VS} + \Sigma_{SS} = (\frac{\alpha}{\pi})^2 \int\limits_1^{\rho^2} dz\ z^{-2}(1-\Pi(-zQ_1^2))^{-2} R^{\gamma\gamma}_{S+V} \ . \tag{24}$$

It is convenient to present the $R^{\gamma\gamma}_{S+V}$ in the following way:

$$R^{\gamma\gamma}_{S+V} = r^{(\gamma\gamma)}_{S+V} + r_{S+V(\gamma\gamma)} + r^{\gamma}_{S+V\gamma} \tag{25}$$
$$r^{(\gamma\gamma)}_{S+V} = r_{S+V(\gamma\gamma)} = L_z^2(2\ \ln^2\ \Delta + 3\ \ln\ \Delta + \frac{9}{8}) + L_z(-4\ \ln^2\Delta - 7\ \ln\ \Delta + 3\xi_3 - \frac{3}{2}\xi_2 - \frac{45}{16})$$
$$r^{\gamma}_{S+V\gamma} = 4[(L_z-1)\ \ln\ \Delta + \frac{3}{4}L_z - 1]^2 \ .$$

Here the quantity $\Delta = \delta\epsilon/\epsilon\ << 1$ is the maximal energy fraction carried by a soft photon.

The single hard photon radiation can be accompanied by real soft or virtual photons. It is useful to separate the cases of photons emitted by the same electron or positron or by both of them.

$$d\sigma|_{H,S+V} = d\sigma^{H(S+V)} + d\sigma_{H(S+V)} + d\sigma^{H}_{(S+V)} + d\sigma^{(S+V)}_{H} \ . \tag{26}$$

In the case where one of the fermions emits the hard real photon and another interacts with a virtual or soft real photon, we find:

$$\Sigma^{H}_{(S+V)} + \Sigma^{(S+V)}_{H} = 2\frac{\alpha}{\pi}\int\limits_{x_c}^{1-\Delta} dx \frac{1+x^2}{1-x}\int\limits_1^{\rho^2} dz z^{-2}[1-\Pi(-zQ_1^2)]^{-2} \tag{27}$$
$$\{[1+\theta(x^2\rho^2-z)]\ (L_z-1) + k(x,z)\}(\frac{\alpha}{\pi})[(L_z-1)\ \ln\ \Delta + \frac{3}{4}L_z - 1] \ .$$

A more complicated expression arises when both photons interact with the same fermion. In this case the cross-section can be expressed in terms of the Compton tensor with a 'heavy photon' [15]. We will consider below the case of the photon emission from the electron. An equal contribution arises from the positron activity. For the small angle detection of the final fermions we have [7]:

$$\begin{aligned}\Sigma^{H(S+V)} &= \Sigma_{H(S+V)} \qquad (28)\\ &= \frac{1}{2}(\frac{\alpha}{\pi})^2 \int_1^{\rho^2} \frac{dz}{z^2(1-\Pi(-zQ_1^2))^2} \int_{x_c}^{1-\Delta} \frac{dx(1+x^2)}{1-x} L_z \, \{(2\ \ln\ \Delta - \ln\ x + \frac{3}{2})\\ &[(L_z-1)(1+\theta)+k(x,z)] + \frac{1}{2}\ \ln^2 x + (1+\theta)[-2+\ln\ x - 2\ \ln\ \Delta]\\ &+(1-\theta)[\frac{1}{2}L_z\ \ln\ x + 2\ \ln\ \Delta\ \ln\ x - \ln\ x\ \ln(1-x)\\ &-\ln^2 x + \int_0^{1-x} \frac{dt}{t}\ \ln\ (1-t) - \frac{x(1-x)+4x\ \ln\ x}{2(1+x^2)}] - \frac{(1-x)^2}{2(1+x^2)}\} \ ,\end{aligned}$$

where $\theta = \theta(x^2\rho^2 - z)$.

Let us consider now the contribution from the emission of two hard real photons. One can distinguish two cases: double photon bremsstrahlung, a) in opposite directions along electron and positron momenta, and b) in the same direction along electron or positron momenta.

The differential cross-section in the first case can be obtained by using the factorization of cross-sections in the impact parameter space [15]. It takes the following form [11, 7]:

$$\begin{aligned}\Sigma_H^H = \frac{1}{4}(\frac{\alpha}{\pi})^2 \int_0^{\infty} dz\ z^{-2}[1-\Pi(-zQ_1^2)]^{-2} \int_{x_c}^{1-\Delta} dx_1 \int_{\frac{x_c}{x_1}}^{1-\Delta} dx_2 \qquad (29)\\ \frac{1+x_1^2}{1-x_1}\,\frac{1+x_2^2}{1-x_2}\ \Phi(x_1,z)\Phi(x_2,z)\ ,\end{aligned}$$

where

$$\begin{aligned}\Phi(x,z) &= (L_z-1)[\theta(z-1)\theta(\rho^2-z)+\theta(z-x^2)\theta(\rho^2x^2-z)]\\ &+L_3[-\theta(x^2-z)+\theta(z-x^2\rho^2)] + [L_2 + \frac{(1-x)^2}{1+x^2}]\theta(z-x^2)\theta(x^2\rho^2-z) \qquad (30)\\ &+[L_1 + \frac{(1-x)^2}{1+x^2}]\theta(z-1)\theta(\rho^2-z) + [\theta(1-z)-\theta(z-\rho^2)]\ \ln\ |\frac{(z-x)(\rho^2-z)}{(x\rho^2-z)(z-1)}| \ .\end{aligned}$$

Let us now turn to the double hard photon emission in the same direction, and the hard e^+e^- pair production. We distinguish the cases of the collinear and semi-collinear kinematics of the final particles. In the first case all particles produced move in the cones within the polar angles $\theta_0 << \theta_1$ centred along the charged particle momenta (final or initial). The region corresponding to the case when both photons are radiated outside these cones does not contain any large logarithmic contribution. In the semi-collinear region only one of the particles produced moves inside the cones, whilst the other moves outside them.

In the totally inclusive cross-section the dependence on the auxiliary parameter θ_0 disappears, and the total contribution has the form:

$$\Sigma^{HH} = \Sigma_{HH} = \frac{1}{4}(\frac{\alpha}{\pi})^2 \int\limits_1^{\rho^2} dz\, z^{-2}[1-\Pi(-zQ_1^2)]^{-2} \int\limits_{x_c}^{1-2\Delta} dx \int\limits_{\Delta}^{1-x-\Delta} dx_1 \frac{I^{HH} L_z}{x_1(1-x-x_1)(1-x_1)^2} \; , \qquad (31)$$
$$I^{HH} = A\,\theta(x^2\rho^2 - z) + B + C\,\theta[(1-x_1)^2\rho^2 - z] \quad .$$

Here

$$A = \gamma\beta(\frac{L_z}{2} + \ln\frac{(\rho^2x^2-z)^2}{x^2(\rho^2x(1-x_1)-z)^2}) + (x^2+(1-x_1)^4)\ln\frac{(1-x_1)^2(1-x-x_1)}{xx_1} + \gamma_A \; ,$$
$$B = \gamma\beta(\frac{L_z}{2} + \ln|\frac{x^2(z-1)(\rho^2-z)(z-x^2)(z-(1-x_1)^2)^2(\rho^2x(1-x_1)-z)^2}{(\rho^2x^2-z)(z-(1-x_1))^2(\rho^2(1-x_1)^2-z)^2(z-x(1-x_1))^2}|)$$
$$+(x^2+(1-x_1)^4)\ln\frac{(1-x_1)^2x_1}{x(1-x-x_1)} + \delta_B \; ,$$
$$C = \gamma\beta(L_z + 2\ln|\frac{x(\rho^2(1-x_1)^2-z)^2}{(1-x_1)^2(\rho^2x(1-x_1)-z)(\rho^2(1-x_1)-z)}|) - 2(1-x_1)\beta - 2x(1-x_1)\gamma \; ,$$
$$\gamma = 1+(1-x_1)^2 \; , \quad \beta = x^2+(1-x_1)^2 \; , \qquad (32)$$
$$\gamma_A = xx_1(1-x-x_1) - x_1^2(1-x-x_1)^2 - 2(1-x_1)\beta \; ,$$
$$\delta_B = xx_1(1-x-x_1) - x_1^2(1-x-x_1)^2 - 2x(1-x_1)\gamma.$$

One can verify [6] that the combinations

$$(\frac{\alpha}{\pi})^2\int_1^{\rho^2}\frac{dz}{z^2[1-\Pi(-zQ_1^2)]^2} r^{\gamma\gamma}_{S+V} + \Sigma^{H(S+V)} + \Sigma^{HH} \; ,$$
$$(\frac{\alpha}{\pi})^2\int_1^{\rho^2}\frac{dz}{z^2[1-\Pi(-zQ_1^2)]^2} r^{\gamma}_{S+V\gamma} + \Sigma^{H}_{S+V} + \Sigma^{S+V}_{H} + \Sigma^{H}_{H} \; , \qquad (33)$$

do not depend on Δ for $\Delta \to 0$.

The total expression $\Sigma^{2\gamma}$ which describes the contribution to Eq. (18) from all (real and virtual) two-photon emissions is determined by the expressions in Eqs. (24), (25), (27), (28), (29), and (31).

Furthermore, it does not depend on the auxiliary parameter Δ and can be written as follows:

$$\Sigma^{2\gamma} = \Sigma^{\gamma\gamma}_{S+V} + 2\,\Sigma^{H(V+S)} + \Sigma^{H}_{(V+S)} + \Sigma^{(V+S)}_{H} + \Sigma^{H}_{H} + 2\Sigma^{HH} \qquad (34)$$
$$= \Sigma^{\gamma\gamma} + \Sigma^{\gamma}_{\gamma} + (\frac{\alpha}{\pi})^2\mathcal{L}\phi^{\gamma\gamma} \; , \qquad \mathcal{L} = \ln\frac{\epsilon^2\Theta_1^2}{m^2} \; .$$

The leading contributions $\Sigma^{\gamma\gamma}, \Sigma^{\gamma}_{\gamma}$ have the form:

$$\Sigma^{\gamma\gamma} = \frac{1}{2}(\frac{\alpha}{\pi})^2 \int\limits_1^{\rho^2} L_z^2\, dz\; z^{-2}[1-\Pi(-Q_1^2 z)]^{-2} \int\limits_{x_c}^{1} dx\; \{\frac{1}{2}P^{(2)}(x)\; [\;\theta(x^2\rho^2 - z) + 1]$$
$$+ \int\limits_x^1 \frac{dt}{t} P(t)\; P(\frac{x}{t})\; \theta(t^2\rho^2 - z)\} \;, \qquad (35)$$

where

$$P^{(2)}(x) = \int\limits_x^1 \frac{dt}{t} P(t)\; P(\frac{x}{t}) = \lim_{\Delta\to 0}\{\; [(2\ln\Delta + \frac{3}{2})^2 - 4\xi_2]\; \delta(1-x) \qquad (36)$$
$$+ 2\,[\frac{1+x^2}{1-x}(2\ln(1-x) - \ln x + \frac{3}{2}) + \frac{1}{2}(1+x)\ln x - 1 + x]\; \theta(1-x-\Delta)\} \;,$$

and

$$\Sigma^{\gamma}_{\gamma} \;=\; \frac{1}{4}(\frac{\alpha}{\pi})^2 \int\limits_0^{\infty} L_z^2 dz\; z^{-2}[1-\Pi(-Q_1^2 z)]^{-2} \int\limits_{x_c}^{1} dx_1 \int\limits_{x_c/x_1}^{1} dx_2 P(x_1)P(x_2)[\theta(z-1)\theta(\rho^2-z)$$
$$+\theta(z-x_1^2)\theta(x_1^2\rho^2-z)][\theta(z-1)\theta(\rho^2-z) + \theta(z-x_2^2)\theta(x_2^2\rho^2-z)] \;. \qquad (37)$$

We see that the leading contributions to $\Sigma^{2\gamma}$ can be expressed in terms of kernels for the evolution equation for structure functions. The function $\phi^{\gamma\gamma}$ in the expression (34) collects the next-to-leading contributions which cannot be obtained by the structure functions method [16, 13]. It has a form given explicitely in Ref. [6].

4 Pair production

In a similar way we consider also pair production. The corrections due to virtual e^+e^- pairs can be extracted from Ref. [12]. Using the expression for soft pair production cross-section [14] one obtains for the contribution of the virtual and soft real e^+e^- pairs to Σ the following result:

$$\Sigma^{e^+e^-}_{S+V} = (\frac{\alpha}{\pi})^2 \int\limits_1^{\rho^2} dz z^{-2}[1-\Pi(-zQ_1^2)]^{-2} R^{e^+e^-}_{S+V} \;, \qquad (38)$$
$$R^{e^+e^-}_{S+V} = L_z^2(\frac{2}{3}\ln\Delta + \frac{1}{2}) + L_z(-\frac{17}{6} + \frac{4}{3}\ln^2\Delta$$
$$-\frac{20}{9}\ln\Delta - \frac{4}{3}\xi_2) + O(1) \;.$$

In this expression the quantity $\Delta = \delta\epsilon/\epsilon \ << 1$ is the maximal energy fraction carried by a soft pair.

Here we have taken into account only e^+e^- pairs. An order of magnitude of the pair production correction is less than 0.5%. A rough estimate of the muon pair contribution gives less than 0.05% since $\ln Q^2/m^2 \sim 3 \ln Q^2/m_\mu^2$. Contributions of pion and tau-lepton pairs give even smaller corrections. Therefore, within the 0.1% accuracy, we can omit any pair except e^+e^- .

Let us consider now the hard pair production. In this case we can restrict ourselves only to the collinear region where the produced pair moves in the small cones within the polar angles θ_0 around the fermion momenta. Indeed, the non-leading semi-collinear region will give contributions of one order of magnitude smaller than the leading ones and therefore within the required accuracy one can neglect them [7].

The pair production contributions to Σ appear from two regions with the pair components moving respectively along initial and scattered electrons and analogously for positrons. This separation allows us to carry out the integration over the energy fraction of the pair components. The resulting contribution to Σ has, with both directions included, the form :

$$\Sigma_H^{e^+e^-} = \frac{1}{4}(\frac{\alpha}{\pi})^2 \int\limits_1^{\rho^2} dz\ z^{-2}[1-\Pi(-zQ_1^2)]^{-2} \int\limits_{x_c}^{1-\Delta} dx\{R_0(x)\,[\mathcal{L}^2\,[1+\theta(x^2\rho^2-z)] \tag{39}$$
$$+ 4\mathcal{L}\ \ln\ x] + 2\ \theta(x\rho-1)\ \theta(x^2\rho^2-z)\ \mathcal{L}\ f(x) + 2f_1(x)\ \mathcal{L}\}\ ,$$

where we put the auxiliary parameter θ_0^2 equal to $\theta^2 = z\theta_1^2$, and introduce the following notations:

$$R_0(x) = \frac{2}{3}\frac{1+x^2}{1-x} + \frac{1-x}{3x}(4+7x+4x^2) + 2(1+x)\ln x\ , \tag{40}$$
$$f(x) = -\frac{131}{9} + \frac{136}{9}x - \frac{2}{3}x^2 - \frac{4}{3x} - \frac{20}{9(1-x)} + \frac{2}{3}(-4x^2-5x+1+\frac{4}{x}$$
$$+\frac{4}{1-x})\ln\,(1-x) + \frac{1}{3}(8x^2+5x-7-\frac{13}{1-x})\ln x - 2\frac{1}{1-x}\ln^2 x$$
$$+\frac{4x^2}{1-x}\int\limits_0^{1-x} dy\frac{\ln\,(1-y)}{y} + 2(1+x)[\xi_2 + \ln x \ln\,(1-x) + \int\limits_0^x dy\frac{\ln\,(1-y)}{y}] \tag{41}$$
$$f_1(x) = -\frac{116}{9} + \frac{151}{9}x + \frac{2}{3x} + \frac{4x^2}{3} - \frac{20}{9(1-x)} + \frac{1}{3}[8x^2 - 10x - 10 + \frac{5}{1-x}]\ln x$$
$$+\frac{2}{3}[-4x^2-5x+1+\frac{4}{x}+\frac{4}{1-x}]\ln(1-x) - (1+x)\ln^2 x$$
$$+2(x+1)[\xi_2 + \int\limits_0^x dy\frac{\ln\,(1-y)}{y} + \ln x\ln\,(1-x)] - \frac{4}{1-x}\int\limits_0^{1-x} dy\frac{\ln\,(1-y)}{y} \tag{42}$$

where $f_1(x) = -xf(\frac{1}{x})$.

The total contribution of virtual, soft, and hard pairs does not depend on the auxiliary parameter Δ and can be written as follows:

$$\Sigma^{e^+e^-} = \Sigma^{e^+e^-}_{S+V} + \Sigma^{e^+e^-}_{H} = \frac{1}{4}(\frac{\alpha}{\pi})^2\mathcal{L}^2\int\limits_1^{\rho^2} dzz^{-2}[1-\Pi(-zQ_1^2)]^{-2} \tag{43}$$
$$\int\limits_{x_c}^{1} dxR(x)[\theta(x^2\rho^2-z)+1] + (\frac{\alpha}{\pi})^2\mathcal{L}\phi^{e^+e^-} \ ,$$

where

$$R(x) = 2\,(1+x)\ln x + \frac{1-x}{3x}(4+7x+4x^2) + \frac{2}{3}P(x) \ . \tag{44}$$

The explicit expression for $\phi^{e^+e^-}$ which collects the non-leading terms from Eqs. (38) and (39) is :

$$\phi^{e^+e^-} = \frac{1}{2}\int\limits_1^{\rho^2} dzz^{-2}\int\limits_{x_c}^{1} dx[\theta(x\rho-1)\theta(x^2\rho^2-z)\bar{f} + \bar{f}_1 - \frac{17}{6} - \frac{4}{3}\xi_2] \ , \tag{45}$$

where

$$\bar{f} = f + \frac{20}{9}[\frac{1}{1-x} - (\frac{1}{1-x})_+] + \frac{8}{3}(-\frac{\ln(1-x)}{1-x} + (\frac{\ln(1-x)}{1-x})_+) \ . \tag{46}$$

The same definition applies to $\bar{f}_1$. The regularization corresponding to the $(\quad)_+$ prescription is as in Ref. [16].

5 $O(\alpha^3)$ corrections

In order to evaluate the leading logarithmic contribution represented by terms of the type $(\alpha\mathcal{L})^3$ we use the iteration of the master equation obtained in Refs. [16, 13].

To simplify the analytical expressions we adopt here a realistic assumption about the smallness of the threshold for the detection of the hard subprocess energy. By neglecting the terms of the order of:

$$x_c\,(\frac{\alpha}{\pi}\mathcal{L})^3 \sim 10^{-4} \ , \tag{47}$$

one should consider only the emission by the initial particles. Three photon (virtual and real) contribution to Σ have the form:

$$\Sigma^{3\gamma} = \frac{1}{4} \; (\frac{\alpha}{\pi}\mathcal{L})^3 \int\limits_1^{\rho^2} dz \; z^{-2} \int\limits_{x_c}^{1} dx_1 \int\limits_{x_c}^{1} dx_2 \; \theta(x_1 x_2 - x_c) \; [\frac{1}{6} \; \delta(1-x_2) \; P^{(3)}(x_1) \qquad (48)$$

$$\theta(x_1\rho - 1)\theta(x_1^2\rho^2 - z) + \frac{1}{2}P^{(2)}(x_1)P(x_2)\theta_1\theta_2] \; [1 + O(x_c^3)] \;\; ,$$

where $P(x)$ and $P^{(2)}(x)$ are given by Eqs. (21) and (36) correspondingly,

$$\theta_1\theta_2 = \theta(z - \frac{x_2^2}{x_1^2}) \; \theta(\rho^2 \frac{x_2^2}{x_1^2} - z) \;\; ,$$

$$P^{(3)}(x) = \delta(1-x) \; \Delta_t + \theta(1 - x - \Delta) \; \theta_t \;\; ,$$

$$\Delta_t = 48 \; [\frac{1}{2}\xi_3 - \frac{1}{2}\xi_2(\ln\Delta + \frac{3}{2}) + \frac{1}{6}(\ln\Delta + \frac{3}{2})^3] \;\; , \qquad (49)$$

$$\theta_t = 48 \; \{\frac{1}{2} \; \frac{1+x^2}{1-x} \; [\frac{9}{32} - \frac{1}{2}\xi_2 + \frac{3}{4}\ln(1-x) - \frac{3}{8}\ln x + \frac{1}{12}\ln^2(1-x)$$
$$+ \frac{1}{12}\ln^2 x - \frac{1}{2}\ln x \ln(1-x)] + \frac{1}{8}(1+x)\ln x \ln(1-x) - \frac{1}{4}(1-x)\ln(1-x)$$
$$+ \frac{1}{32}(5-3x)\ln x - \frac{1}{16}(1-x) - \frac{1}{32}(1+x)\ln^2 x - \frac{1}{8}(1+x)\int\limits_0^{1-x} dy \frac{\ln(1-y)}{y}\} \;\; .$$

The contribution to Σ of the pair production accompanied by the photon emission when both, pair and photons, can be real and virtual is given below (with respect to Ref. [16] we include also the non-singlet mechanism of the pair production):

$$\Sigma^{e^+e^-\gamma} = \frac{1}{4}(\frac{\alpha}{\pi}\mathcal{L})^3 \int\limits_1^{\rho^2} dz \; z^{-2} \int\limits_{x_c}^{1} dx_1 \int\limits_{x_c}^{1} dx_2 \; \theta(x_1 x_2 - x_c)$$
$$\{\frac{1}{3}[R^P(x_1) - \frac{1}{3}R^s(x_1)] \; \delta(1-x_2)\theta(x_1^2\rho^2 - z) + \frac{1}{2} \; P(x_2)R(x_1) \; \theta_1\theta_2\} \; [1 + O(x_c^3)] \;\; ,$$

where

$$R(x) = R^s(x) + \frac{2}{3}P(x) \;\; , \qquad R^s(x) = \frac{1-x}{3x}(4 + 7x + 4x^2) + 2(1+x)\ln x \;\; , \; (50)$$

$$R^P(x) = R^s(x)(\frac{3}{2} + 2\ln(1-x)) + (1+x)(-\ln^2 x - 4\int\limits_0^{1-x} dy \frac{\ln(1-y)}{y})$$
$$+ \frac{1}{3}(-9 - 3x + 8x^2)\ln x + \frac{2}{3}(-\frac{3}{x} - 8 + 8x + 3x^2) \; .$$

The total expression for Σ in Eq. (18) is the sum of the contributions given in Eqs. (19), (20), (34), (43), (48) and (50). The quantity Σ is a function of the parameters x_c, ρ, and Q_1^2.

6 Estimates of neglected terms

Let us now estimate the terms not taken into account here in accordance with the required accuracy:

a)Weak radiative corrections:

$$\Sigma^{\text{w.r.c.}} \sim \frac{\alpha Q_1^2}{\pi M_z^2} \lesssim 10^{-5} \quad . \tag{51}$$

b) Electromagnetic corrections to weak contributions, including interference terms :

$$\Sigma_W^{\text{h.o.}} \sim \delta_{\text{weak}}|_{\theta=\theta_1} \frac{\alpha}{\pi} \mathcal{L} \lesssim 10^{-4} \quad . \tag{52}$$

Here δ_{weak} is given by Eq. (7).

c) Radiative corrections to the annihilation mechanism, including its interference with the scattering mechanism :

$$\Sigma_{\text{st}}^{\text{r.c.}} \sim \theta_1^2 \frac{\alpha}{\pi} \mathcal{L} \lesssim 10^{-4} \quad . \tag{53}$$

Our explicit expressions for Σ^{γ}, without annihilation terms, coincide numerically with the results obtained at the same order in Ref. [17] by using exact matrix elements.

d) The interference between photon emissions by electron and positron:

$$\Sigma_{\text{int}} \sim \theta_1^2 \frac{\alpha}{\pi} \lesssim 10^{-5} \quad . \tag{54}$$

This contribution is connected with terms violating the eikonal form [10] in the expression:

$$A(s,t) = A_0(s,t) e^{i\phi(t)} + O(\frac{\alpha t}{\pi s}) \quad . \tag{55}$$

e) The interference terms between one- and two-photon mechanisms of pair production, including the effect final particles identity:

$$\Sigma_{\text{int}}^{\text{pairs}} \sim (\frac{\alpha}{\pi})^2 \lesssim 10^{-5} \quad . \tag{56}$$

f) The semi-collinear mechanism of pair production gives a contribution which contains a small factor $\mathcal{L}^{-1}$ with respect to the collinear terms, and is numerically small: $\Sigma_{\text{SC}}^{\text{pair}} \lesssim 10^{-4}$. A more accurate estimate should be interesting.

g) The creation of heavy pairs ($\mu\mu, \tau\tau, \pi\pi, ...$) is at least one order of magnitude smaller than the corresponding contribution due to the light particle production and is therefore not essential.

h) Higher-order corrections, including soft and collinear multi-photon contributions, can be safely neglected since they only give contributions of the type $(\alpha\, \mathcal{L}/\pi)^n$ for $n \geq 4$.

7 Results and their discussion

The total cross-section for the ${\rm e}^+{\rm e}^-$ distribution is:

$$\sigma = \frac{4\pi\alpha^2}{Q_1^2}\Sigma \ , \tag{57}$$

where Σ is given by Eq. (18).

Let us define Σ_0^0 to be equal to $\Sigma_0|_{\Pi=0}$ [see Eq. (19)], which corresponds to the Born cross-section obtained by switching out the vacuum polarization contribution $\Pi(t)$ defined in Eq. (10). We obtain:

$$\sigma = \frac{4\pi\alpha^2}{Q_1^2}\Sigma_0^0 \left(1 + \delta_0 + \delta^{\gamma} + \delta^{2\gamma} + \delta^{{\rm e}^+{\rm e}^-} + \delta^{3\gamma} + \delta^{{\rm e}^+{\rm e}^-\gamma}\right) \ , \tag{58}$$

where

$$\Sigma_0^0 = \Sigma_0|_{\Pi=0} = 1 - \rho^{-2} + \Sigma_W + \Sigma_\theta|_{\Pi=0} \simeq 1 - \rho^{-2} \ , \tag{59}$$

and

$$\delta_0 = \frac{\Sigma_0 - \Sigma_0^0}{\Sigma_0^0} \ ; \quad \delta^{\gamma} = \frac{\Sigma^{\gamma}}{\Sigma_0^0} \ ; \quad \delta^{2\gamma} = \frac{\Sigma^{2\gamma}}{\Sigma_0^0} \ ; \quad \cdots \ . \tag{60}$$

In the Tables below we give the values of δ_β as well as their sum as functions of x_c for the values of the parameters $\theta_1 = 1.61^0$, and $\theta_2 = 2.80^0$ defining the Narrow-Narrow (NN) case, and for $\theta_1 = 1.5^0$, and $\theta_2 = 3.15^0$, defining the Wide-Wide (WW) case as in Ref. [18].

Each of these contributions to Σ has a sign which can be changed as a result of the interplay between real and virtual corrections. The cross-section corresponding to the Born diagrams for producing a real particle is always positive, whereas the sign of the radiative corrections depends on the order of perturbation theory for the virtual corrections: at odd orders it is negative, and at even orders it is positive. When the aperture of the counters is small the compensation between real and virtual corrections is not complete. In the limiting case of zero aperture only the virtual contributions remain giving a negative result. As a consequence we see that the radiative corrections for the NN case are larger in absolute value than ones in the WW case. These corrections depend on θ_1, x_c, and ρ. When $x_c \to 1$ the corrections increase in absolute value.

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Tables

x_c	δ_0	δ^{γ}	$\delta^{2\gamma}$	$\delta^{e^+e^-}$	$\delta^{3\gamma}$	$\delta^{e^+e^-\gamma}$	$\sum \delta^i$
NN $\rho = 1.74$							
0.2	0.413E-1	-0.922E-1	0.506E-2	-0.750E-3	-0.561E-03	0.911E-4	-0.471E-1
0.3	0.413E-1	-0.962E-1	0.508E-2	-0.633E-3	-0.514E-03	0.767E-4	-0.509E-1
0.4	0.413E-1	-0.101E+0	0.452E-2	-0.635E-3	-0.483E-03	0.669E-4	-0.562E-1
0.5	0.413E-1	-0.108E+0	0.406E-2	-0.672E-3	-0.453E-03	0.589E-4	-0.637E-1
0.6	0.413E-1	-0.119E+0	0.385E-2	-0.733E-3	-0.414E-03	0.517E-4	-0.749E-1
0.7	0.413E-1	-0.138E+0	0.348E-2	-0.823E-3	-0.350E-03	0.456E-4	-0.943E-1
0.8	0.413E-1	-0.174E+0	0.397E-2	-0.964E-3	-0.249E-03	0.442E-4	-0.130E+0
WW $\rho = 2.10$							
0.2	0.409E-1	-0.728E-1	0.276E-2	-0.579E-3	-0.406E-03	0.598E-4	-0.301E-1
0.3	0.409E-1	-0.771E-1	0.290E-2	-0.491E-3	-0.377E-03	0.506E-4	-0.341E-1
0.4	0.409E-1	-0.826E-1	0.257E-2	-0.502E-3	-0.356E-03	0.442E-4	-0.399E-1
0.5	0.409E-1	-0.902E-1	0.223E-2	-0.543E-3	-0.333E-03	0.387E-4	-0.479E-1
0.6	0.409E-1	-0.104E+0	0.200E-2	-0.606E-3	-0.297E-03	0.338E-4	-0.620E-1
0.7	0.409E-1	-0.127E+0	0.207E-2	-0.697E-3	-0.238E-03	0.304E-4	-0.849E-1
0.8	0.409E-1	-0.167E+0	0.303E-2	-0.838E-3	-0.146E-03	0.320E-4	-0.124E+0

Table 1: The values of δ^i in Eq. (58) for $\sqrt{s} = M_Z = 91.161$ GeV, $\sin^2 \Theta_W = 0.2283$, $\Gamma_Z = 2.4857$ GeV. The 'NN counter' corresponds to $\rho = 1.74$ and the 'WW counter' to $\rho = 2.10$ as a function of x_c.

x_c	0.2	0.3	0.4	0.5	0.6	0.7	0.8
R_{NN}	0.592	0.627	0.708	0.770	0.817	0.763	0.498
R_{WW}	0.653	0.740	0.883	1.020	1.120	0.994	0.500

Table 2: Values of R_{NN} and R_{WW} where R represents the ratio of non-leading with respect to leading contributions and is defined as $R = (\frac{\alpha}{\pi})^2 \frac{\mathcal{L}\phi^{2\gamma}}{\Sigma^{2\gamma}}$ for $\sqrt{s} = M_Z = 91.161$ GeV, $\sin^2 \Theta_W = 0.2283$, $\Gamma_Z = 2.4857$ GeV. The 'NN counter' corresponds to $\rho = 1.74$ and the 'WW counter' to $\rho = 2.10$ as above as a function of x_c.