SISSA 64/95/EP
POLFIS-TH 07/95
CERN-TH/95-140
UCLA/95/TEP/19
IFUM 508FT
KUL-TF-95/18
hep-th/yymmnnn
June, 1995

# A Search for Non-Perturbative Dualities of Local $N = 2$ Yang–Mills Theories from Calabi–Yau Threefolds *

M. Billó[1], A. Ceresole[2], R. D'Auria[2], S. Ferrara[3],
P. Fré[1], T. Regge[2], P. Soriani[4] and A. Van Proeyen[5] †

[1] International School for Advanced Studies (ISAS), via Beirut 2-4, I-34100 Trieste
and Istituto Nazionale di Fisica Nucleare (INFN) - Sezione di Trieste, Italy

[2] Dipartimento di Fisica, Politecnico di Torino,
Corso Duca degli Abruzzi 24, I-10129 Torino
and Istituto Nazionale di Fisica Nucleare (INFN) - Sezione di Torino, Italy

[3] CERN Theoretical Division, CH 1211 Geneva, Switzerland
and UCLA Physics Department, Los Angeles CA, USA

[4] Dipartimento di Fisica, Università di Milano, via Celoria 6, I-20133 Milano,
and Istituto Nazionale di Fisica Nucleare (INFN) - Sezione di Milano, Italy

[5] Instituut voor Theoretische Fysica - Katholieke Universiteit Leuven
Celestijnenlaan 200D B–3001 Leuven, Belgium



## Abstract

The generalisation of the rigid special geometry of the vector multiplet quantum moduli space to the case of supergravity is discussed through the notion of a dynamical Calabi–Yau threefold. Duality symmetries of this manifold are connected with the analogous dualities associated with the dynamical Riemann surface of the rigid theory. N=2 rigid gauge theories are reviewed in a framework ready for comparison with the local case. As a byproduct we give in general the full duality group (quantum monodromy) for an arbitrary rigid $SU(r+1)$ gauge theory, extending previous explicit constructions for the $r = 1, 2$ cases. In the coupling to gravity, R–symmetry and monodromy groups of the dynamical Riemann surface, whose structure we discuss in detail, is embedded into the symplectic duality group $\Gamma_D$ associated with the moduli space of the dynamical Calabi–Yau threefold.

† Onderzoeksleider, NFWO, Belgium

* Supported in part by DOE grant DE-FGO3-91ER40662, Task C. and by EEC Science Program SC1*CT92-0789.

# 1 Introduction

Recent progress [1, 2] towards understanding non-perturbative properties of N=2 Yang–Mills theories has been obtained by associating the holomorphic N=2 prepotential [3] to the periods of an auxiliary Riemann surface (of genus $r$ equal to the rank of the gauge group $G = SU(r+1)$), where the monodromy group is directly related to the electric–magnetic duality symmetries of the theory[4, 5].

Non-perturbative monodromies related to monopole point singularities (i.e. points where particles with non-vanishing magnetic charges, monopoles or dyons, become massless) correspond to an infinite sum of instanton contributions to the prepotential in the microscopic G-invariant theory.

Perturbative monodromies, on the other hand, correspond to the unique one-loop perturbative correction [6] to the prepotential in the original G-invariant theory, broken down to $U(1)^r$.

Very recently, these exact results for the low-energy effective N=2 Yang–Mills theory have been extended to include gravity by using several different informations [7]-[18]. Firstly, in a paper by some of us [7], it was pointed out that, in the case of coupling Yang–Mills theories to gravity, the N=2 rigid special geometry encompassing the moduli space of hyperelliptic Riemann surfaces is drastically modified by the gravitational effects, in particular by the presence of the graviphoton. The immediate consequence of this is a change of the electric-magnetic duality group and also of the argument of positivity of the metric of the moduli space of locally supersymmetric Yang–Mills theories. Indeed, the same argument used in the rigid case to introduce an auxiliary Riemann surface to solve the theory, strongly suggests that the auxiliary surface should be, in this case, a Calabi–Yau threefold with third Betti number $b_3 = 2n$ where $n$ is the total number of vectors in the theory.

Thus it would follow immediately that the electric-magnetic duality group $\Gamma_D$ for the gravitational case is related to the monodromy group $\Gamma_M \subset Sp(b_3, \mathbb{Z})$ of the Calabi–Yau threefold . However we observe here a conceptual difference from the rigid theory, that for $r = 1$ was recently pointed out in [19], namely that in the local case the electric-magnetic duality is larger than in the rigid theory because also the symmetries of the Calabi–Yau threefold that are not in the monodromy group will be in $\Gamma_D$. The symmetries of the Calabi–Yau defining equation are $\Gamma_D/\Gamma_M$. This has no analogue in the rigid theories, except for those symmetries of the auxiliary Riemann surface defining polynomial that correspond to a unimodular rescaling factor, such as the R-symmetry [20].

In this paper we will study classes of Calabi–Yau threefolds which are potential candidates to satisfy the important requirement to realize the embedding of the auxiliary Riemann surfaces of the rigid Yang–Mills theories. This entails a suitable embedding of duality symmetries and monodromies of the rigid case.

We will actually derive an explicit representation of the monodromy group of the $SU(n)$ rigid theories by extending some techniques introduced in ref. [21] and later used to study the monodromies of Calabi–Yau manifolds for more than one modulus. (The monodromy for the $SU(n)$ case has been recently analysed in [22], where explicit results were given for the $SU(3)$ theory).

The criterion of searching for the right embedding can be naturally implemented, using a series of different recent results, in the context of string theories. There, it is natural to associate to a given model some dual theory, where the dynamical Calabi–Yau manifold is not just an auxiliary geometrical tool suitable for the analysis of the quantum behaviour, but the target space of the dual theory. This means that, at least in the (abelian) phase where the electric-magnetic duality is manifest, it should be possible to solve the original theory (which is known only in the region of weak coupling) in the strong coupling regime by means of another theory in its semiclassical regime [9, 10]. It is natural to associate gravitationally coupled N=2 Yang–Mills theories to heterotic strings having N=2 supersymmetry in D=4 [7], and to identify their dual theories with type-IIA (or B) theories [4, 8] compactified on the appropriate Calabi–Yau threefolds [8, 7]. In this (in principle) more restricted framework the auxiliary Calabi–Yau threefold previously considered should then be identified with the compactification manifold of the dual type II theory. Thus we have, as proposed in [17], a second-quantized mirror symmetry in the sense that, for vector multiplets, the *classical* moduli space of the Calabi–Yau manifold in the dual theory should give the *quantum* moduli space of the heterotic theory on $K3 \times T_2$. This is made possible by the peculiar role of the dilaton-axion complex field $S$ in string theory. On one side it plays the role of string "coupling constant", on the other side it sits in a vector multiplet in heterotic theories and in a hypermultiplet in type-II theories [23]. This has a two-fold consequence: using N=2 supersymmetry, which forbids [3, 24, 25] the mixing of neutral moduli in vector multiplets with those of hypermultiplets in the low-energy lagrangian, it allows to extend to N=2 string theories powerful non-renormalization theorems of renormalizable N=2 gauge theories [26]. In particular, on the heterotic side, the classical hypermultiplet quaternionic manifold does not receive any quantum correction [11, 12]. For the dual type-II theory, the same is true for the manifold of the vector multiplets [15].

Since the exact moduli space of vector multiplets can be obtained by first-quantized mirror symmetry[27], then it follows that the full heterotic string moduli space of vector multiplets, i.e. the perturbative + instanton corrected prepotential is given by a "classical" computation on the type-II side.

This precisely realizes the fact that the Calabi–Yau threefold moduli space generalizes, in the case of strings, the auxiliary Riemann surface [1, 2]. In this context we notice that the duality group of the Calabi–Yau space, which corresponds to the quantum monodromy of the heterotic strings, also realizes at the N=2 level, the $U$-duality idea of Hull and Townsend [9] and the $S$-$T$ duality of Duff [28], since $S_D$, the dual of $S$, must be one of the Calabi–Yau moduli. Of course the BPS saturated states of type-II theories must be non-perturbative since they must contain electric and magnetically charged states with respect to the $U(1)$ gauge field of the three-form cohomology. These states can be interpreted as particular topological states coming from the compactification of a three-brane soliton. In the N=4 case where, unlike the N=2 case, quantum corrections are absent, there is a unique pair of naturally dual theories, namely heterotic on $T_4$ and type-II on $K3 \times T_2$. The spectrum of BPS states in N=4 heterotic (using string–three-brane duals) has been studied by Sen and Schwarz [29] and the enhanced symmetries of $K3 \times T_2$ by Hull and Townsend [9], Witten[10], Harvey and Strominger[14].

In string theories this duality pairing is also made possible by two other important facts:

*i)* the realization that on Calabi–Yau manifolds one can have a change of Hodge numbers by [15, 16] conifold transitions, i.e. black hole condensation through VEV's of hypermultiplets carrying Ramond-Ramond charges which lower the rank of the gauge group and increase the number of neutral hypermultiplets. This phenomenon is dual to enhanced symmetries or monopole singularities in heterotic theories which may also change the number of massless vector multiplets and hypermultiplets. This also allows to connect a web of heterotic theories to a web of Calabi–Yau manifolds, through non-perturbative black-hole condensation or monopole point singularities.

*ii)* Some Calabi–Yau manifolds can also be obtained [17] by a $\mathbb{Z}_2$ modding of the $K3 \times T_2$ manifold which is in turn the type-II manifold yielding the model dual to the heterotic string compactified on $T_6$ in six dimensions[14]. This is also the approach which allows to make an explicit construction of the dual Calabi–Yau manifold and then can be possibly extended to more general cases by conifold transitions.

Coming back to our original motivation to use a Calabi–Yau threefold to embed the dynamical Riemann surface of rigid theories, an important restriction comes from the fact that the intersection-form of this Calabi–Yau should coincide (up to a symplectic change of basis) with the intersection form of the special homogeneous spaces $SU(1,1)/U(1) \times O(2,r)/O(2) \times O(r)$. This implies that the classical prepotential be of the form

$$\mathcal{F}(S,t_i) = SC_{ij}t^it^j, \qquad d_{Sij} = C_{ij} \neq 0 \tag{1.1}$$

where $S$ is the four dimensional dilaton-axion field, $C_{ij}t^it^j$ is a quadratic real form with signature $(1, r-1)$ and $d_{Sij}$ is the intersection number. This property poses a strong constraint on the Calabi–Yau manifold and a list of such manifolds will be given for $r \leq 2$.

Recently Kachru and Vafa [18] enumerated Calabi–Yau threefolds which are potential duals of heterotic strings with a given number of vector and hypermultiplets (in their abelian phase). Only manifolds for which the two distinct types of multiplets agree with known examples of heterotic string are considered. Moreover in some cases these authors give indications that also terms in the prepotential with $S$ large but $t_i$ finite (such as $\lim_{S\to\infty} \partial_{ijk}\mathcal{F}(t,S)$) agree with the pole structure expected from one loop calculations in heterotic theory [11, 12].

Our analysis is somewhat complementary, in the sense that it does not focus on the number of multiplets but rather on the possibility that a given Calabi–Yau manifold can embed the monodromies of the Riemann surface of the rigid theory, in presence of a dynamical coupling constant (dilaton).

This last fact gives us the condition on the intersection numbers which should possibly restrict the search for "dual" Calabi–Yau threefolds, especially with $r$ large. It is in fact rather obvious that the larger is $r$, the more Calabi–Yau manifolds exist, but the more stringent will probably look the constraints on the intersection matrices. Thus, the potential Calabi–Yau candidates will perhaps not be too many (if any).

The other important embedding is the R–symmetry, related to non–perturbative corrections. We identify such symmetry with the $\mathbb{Z}_p$ symmetry which usually occurs in Calabi–Yau threefolds. Once this identification is made, one can explore how it acts on the vector multiplets of the theory. This will be exhibited in a particular example in section 4. The conjecture that suitable Calabi–Yau manifolds are non–perturbative solutions of $N=2$ local Yang–Mills theories (or heterotic $N=2$ strings), should pass the stringent test that such R–symmetries should find an explanation in terms of the instanton configurations which occur in gravitationally coupled Yang–Mills theories. Relations between space–time instantons and world–sheet instanton sums have been recently discussed for a two parameter Calabi–Yau threefold in [30]. We will just observe that in some cases such R–symmetry can be explained by noticing that the Calabi–Yau manifold contains two dimensional submanifolds realizing a multiple cover of a Riemann surface identical in form with that occurring in the rigid theory.

It is important to stress that our search for dynamical Calabi–Yau manifolds, without imposing a string duality, is neither more nor less general than the counting of Hodge numbers made in [18] by imposing string duality only. The reason is that the identification of Calabi–Yau dual to heterotic strings by just matching the number of vectors and hypermultiplets, without imposing a priori the constraints from the quantum monodromy and the intersection matrices, seems a necessary but not sufficient assumption. On the other hand, our criterion of searching for quantum Calabi–Yau manifolds does not guarantee a priori that the number of neutral hypermultiplets, in the abelian phase of the heterotic theory, is the correct one. The analysis seem to almost completely overlap for $r = 1, 2$ but we expect them to be different and complementary for large $r$. In particular, in [18] it is found the heterotic realization of four out of the five Calabi–Yau manifolds with $h^{1,1} = 2$ ($r = 1$) and of two of the manifolds with $h^{1,1} = 3$ that we have listed in eq. (4.27). In Section 3, we analyze in particular two of the models with $r = 1$ having heterotic counterparts, for which, thanks to the results of [31], the test can be done completely (writing explicitly also the $Sp(6, \mathbb{Z})$ duality matrices) in one case, and almost completely in the other.

It would be extremely interesting to see whether requiring both the quantum consistency and the matching of the number of supermultiplets one could obtain a unique and perhaps universal classification.

This paper is organized as follows. Section 2 is devoted to rigid theories. We show how to define the auxiliary Riemann surfaces that encode the exact solution of the rigid N=2 gauge theories as hypersurfaces in weighted projective spaces. Symmetry of the potential, monodromy, duality and the emerging of the $\mathbb{Z}_{2r+2}$ R-symmetry group are discussed along this line. In section 3 the structure of the monodromy group for rigid $SU(r+1)$ theories is derived as a subgroup of the braid group $B(2r+2)$. In section 4 we consider local N=2 theories. The notion of a dynamical Calabi–Yau manifold and the requirements that such a manifold should satisfy in order to embed correctly the rigid theories are discussed, and examples are given. In particular we stress the role of the intersection numbers (corresponding to the structure constants of the chiral ring) and the embedding of the R-symmetry group. In section 5 the definition of central charge in type-II strings on Calabi–Yau threefolds is discussed; this is a basic point in showing how the Seiberg-Witten construction is realized at the string level under the mapping to the dual heterotic theories. Finally, the Appendix contains additional remarks about the rigid $SU(2)$ theory, mainly concerning the actual role of the symmetries of the potential.

## 2 Non-perturbative solutions of Rigid N=2 gauge theories revisited

Let us summarize the results obtained for pure N=2 gauge theories [1, 4, 5, 22], without hypermultiplets coupling. For the N=2 *microscopic gauge theory* of a group $G$, with Lie algebra $\mathbf{G}$ the rigid special geometry is encoded in a "minimal coupling" quadratic prepotential of the form:

$$\begin{aligned} \mathcal{F}^{(micro)}(Y) &= g^{(K)}_{IJ}\, Y^I\, Y^J \\ g^{(K)}_{IJ} &= \text{Killing metric of the Lie algebra } \mathbf{G} \end{aligned} \tag{2.1}$$

This choice is motivated by renormalizability, positivity of the energy and canonical normalization of the kinetic terms. Consider next the effective lagrangian describing the dynamics of the massless modes. This is an abelian N=2 gauge theory that admits the maximal torus $H \subset G$ as new gauge group and is based on a new rigid special geometry:

$$\begin{aligned} \mathcal{F}^{(eff)}(Y) &= g^{(K)}_{\alpha\beta}\, Y^\alpha\, Y^\beta + \Delta\mathcal{F}^{(eff)}\,(Y^\alpha) \\ Y^\alpha &\in \mathbf{H} \subset \mathbf{G} \end{aligned} \tag{2.2}$$

In general the effective prepotential $\mathcal{F}^{(eff)}(Y)$ has a transcendental dependence on the scalar fields $Y^\alpha$ of the Cartan subalgebra multiplets, due to the correction $\Delta\mathcal{F}^{(eff)}\,(Y^\alpha)$. The main problem is the determination of this correction. Perturbatively one can get information on $\Delta\mathcal{F}^{(eff)}\,(Y^\alpha)$ and discover its logarithmic singularity for large values of the scalar fields $Y^\alpha$. In particular one has a logarithmic correction to the gauge coupling matrix

$$\Delta\overline{\mathcal{N}}_{\alpha\beta} = \frac{\partial^2}{\partial Y^\alpha \partial Y^\beta}\Delta\mathcal{F} \overset{Y\to\infty}{\sim} \sum_\alpha \alpha^\alpha \alpha^\beta \log\frac{(Y\cdot\alpha)^2}{\Lambda^2} \tag{2.3}$$

where $\alpha$ are the root vectors of the gauge Lie algebra and $\Lambda^2$ is the dynamically generated scale. The perturbative monodromy following from

$$\begin{aligned} \mathcal{N}_{\alpha\beta} &\rightarrow [(C + D\mathcal{N})(A + B\mathcal{N})^{-1}]_{\alpha\beta} \\ Sp(2r,\mathbb{R}) &\ni \begin{pmatrix} A & B \\ C & D \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ \sum_\alpha \alpha^\alpha\alpha^\beta & 1 \end{pmatrix} \end{aligned} \tag{2.4}$$

is assumed to be a part of the monodromy group of a genus $r$ Riemann surface having a symplectic action on the periods of the surface. Guessing such a dynamical Riemann surface gives the nonperturbative structure $\Delta\mathcal{F}^{(eff.)}$.

Denoting by $r$ the rank of the original gauge group $G$, one derives the structure of the effective gauge theory of the maximal torus $\mathbf{H}$ from the geometry of an $r$–parameter family $\mathcal{M}_1[r]$ of dynamical genus $r$ Riemann surfaces. The essential steps of the procedure are as follows: naming $u_i$ (i=1,... r) the $r$ gauge invariant moduli of the family, (described as the vanishing locus of an appropriate polynomial) one makes the identifications:

$$u_i \;\rightarrow\; \langle\, d_{\alpha_1\ldots\alpha_{i+1}} Y^{\alpha_1} \,\ldots\, Y^{\alpha_{i+1}} \rangle \quad (i = 1, \ldots r) \tag{2.5}$$

where $Y^\alpha$ are the special coordinates of rigid special geometry and $d_{\alpha_1\ldots\alpha_{i+1}}$ is the restriction to the Cartan subalgebra of the rank $i+1$ symmetric tensor defining the $(i+1)$–th

Casimir operator. The identification (2.5) is only an asymptotic equality for large values of $u_i$ and $Y^\alpha$; at finite values, the relation between the moduli $u_i$ and the special coordinates (namely the elementary fields appearing in the lagrangian) is much more complicated. One considers the derivatives :

$$\Omega_{u_i} \stackrel{\mathrm{def}}{=} \partial_{u_i} \Omega \;=\; \partial_{u_i} \begin{pmatrix} Y^\alpha \\ \frac{\partial \mathcal{F}}{\partial Y^\alpha} \end{pmatrix} \tag{2.6}$$

where $\Omega(u_i)$ is a section of the flat $Sp(2r, \mathbb{R})$ holomorphic vector bundle whose existence is encoded in the definition of rigid special Kähler geometry [7, 8, 25]. On one hand the Kähler metric is given by the following general formula:

$$g_{ij^\star} \;=\; -\,\mathrm{i}\,\overline{\Omega}^T_{\overline{u}_{j^\star}} \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \Omega_{u_i} \tag{2.7}$$

On the other hand, one identifies the symplectic vectors $\Omega_{u_i}$ with the period vectors:

$$\Omega_{u_i} \;=\; \begin{pmatrix} \int_{A^\alpha} \omega^i \\ \int_{B^\alpha} \omega^i \end{pmatrix} \qquad (i = 1, \dots r = \text{ genus}) \tag{2.8}$$

of the $r$ holomorphic 1-forms $\omega^i$ along a canonical homology basis:

$$A_\alpha \cap A_\beta = 0 \qquad B_\alpha \cap B_\beta = 0 \qquad A_\alpha \cap B_\beta = -B_\beta \cap A_\alpha = \delta_{\alpha\beta} \tag{2.9}$$

of a genus $r$ dynamical Riemann surface $\mathcal{M}_1[r]$. The generic moduli space $M_r$ of genus $r$ surfaces is $3r-3$ dimensional. The dynamical Riemann surfaces $\mathcal{M}_1[r]$ fill an $r$-dimensional sublocus $L_R[r]$. The problem is that of characterizing intrinsically this locus. Let

$$i \;:\; L_R[r] \rightarrow M_r \tag{2.10}$$

be the inclusion map of the wanted locus and let

$$H \xrightarrow{\pi} M_r \tag{2.11}$$

be the Hodge bundle on $M_r$, that is the rank $r$ vector bundle whose sections are the holomorphic forms on the Riemann surface $\Sigma_r \in M_r$. As fibre metric on this bundle one can take the imaginary part of the period matrix:

$$\mathrm{Im}\,\mathcal{N}_{\alpha\beta} = \int_{\Sigma_r} \omega^\alpha \wedge \overline{\omega}^{\beta^*} \tag{2.12}$$

where $\omega^\alpha$ is a basis holomorphic one-forms. The locus $L_R[r]$ is defined by the following equation:

$$i^* \partial\overline{\partial} ||\omega||^2 = i^* \mathcal{K} \tag{2.13}$$

where $||\omega||^2 = \int_{\Sigma_r} \omega \wedge \overline{\omega}$ is the norm of any section of the Hodge bundle and $\mathcal{K}$ is the Kähler class of $M_r$.

Using very general techniques of algebraic geometry, the dependence of the periods (2.6) on the moduli parameters can be determined through the solutions of the Picard–Fuchs differential system, once $\mathcal{M}_1[r]$ is explicitly described as the vanishing locus of a holomorphic superpotential $\mathcal{W}(Z, X, Y; u_i)$. In particular one can study the monodromy group $\Gamma_M$ of the differential system and the symmetry group of the potential $\Gamma_{\mathcal{W}}$, that are related to the full group of *electric–magnetic duality rotations* $\Gamma_D$ as follows [32]:

$$\Gamma_{\mathcal{W}} \;=\; \Gamma_D/\Gamma_M \tag{2.14}$$

The elements of $\Gamma_D \supset \Gamma_M$ are given by integer valued symplectic matrices $\gamma \in Sp(2r, \mathbb{Z})$ that act on the symplectic section $\Omega$. Given the geometrical interpretation (2.8) of these sections, the elements $\gamma \in \Gamma_D \subset Sp(2r, \mathbb{Z})$ correspond to changes of the canonical homology basis respecting the intersection matrix (2.9).

To be specific we mention the results obtained for the gauge groups $G = SU(r+1)$. The rank $r = 1$ case, corresponding to $G = SU(2)$, was studied by Seiberg and Witten in their original paper [1]. The extension to the general case, with particular attention devoted to the $SU(3)$ case, was obtained in [4, 5]. In all these cases the dynamical Riemann surface $\mathcal{M}_1[r]$ belongs to the hyperelliptic locus of genus $r$ moduli space, the general form of a hyperelliptic surface being described (in inhomogeneous coordinates) by the following algebraic equation:

$$w^2 \;=\; P_{(2+2r)}(z) \;=\; \prod_{i=1}^{2+2r} (z-\lambda_i) \tag{2.15}$$

where $\lambda_i$ are the $2+2r$ roots of a degree $2+2r$ polynomial. The hyperelliptic locus

$$L_H[r] \subset M_r \quad , \quad \dim L_H[r] = 2r-1 \tag{2.16}$$

is a closed submanifold of codimension $r-2$ in the $3r-3$ dimensional moduli space of genus $r$ Riemann surface[1]. The $2r-1$ hyperelliptic moduli are the $2r+2$ roots of the polynomial appearing in (2.15), minus three of them that can be fixed at arbitrary points by means of fractional linear transformations on the variable $z$. Because of their definition, however, the dynamical Riemann surfaces $\mathcal{M}_1[r]$, must have $r$ rather than $2r-1$ moduli. We conclude that the $r$–parameter family $\mathcal{M}_1[r]$ fills a locus $L_R[r]$ of codimension $r-1$ in the hyperelliptic locus:

$$L_R[r] \subset L_H[r], \quad \mathrm{codim}\, L_R[r] = r-1, \quad \mathrm{dim} L_R[r] = r. \tag{2.17}$$

This fact is expressed by additional conditions imposed on the form of the degree $2+2r$ polynomial of eq.(2.15). In references [4, 5] $P_{(2+2r)}(z)$ was determined to be of the following form:

$$\begin{aligned} P_{(2+2r)}(z) &= P^2_{(r+1)}(z) - P^2_{(1)}(z) \\ &= \Big(P_{(r+1)}(z) + P_{(1)}(z)\Big) \, \Big(P_{(r+1)}(z) - P_{(1)}(z)\Big) \end{aligned} \tag{2.18}$$

where $P_{(r+1)}(z)$ and $P_{(1)}(z)$ are two polynomials respectively of degree $r+1$ and 1. Altogether we have $r+3$ parameters that we can identify with the $r+1$ roots of $P_{(r+1)}(z)$ and with the two coefficients of $P_{(1)}(z)$

$$P_{(r+1)}(z) \;= \prod_{i=1}^{r+1} (z-\lambda_i) \quad , \quad P_{(1)}(z) \;=\; \mu_1\, z + \mu_0 \;\; . \tag{2.19}$$

[1]For genus 1, the moduli space is also 1–dimensional and the hyperelliptic locus is the full moduli space.

Indeed, since the polynomial (2.18) must be effectively of order $2+2r$, the highest order coefficient of $P_{(r+1)}(z)$ can be fixed to 1 and the only independent parameters contained in $P_{(r+1)}(z)$ are the roots. On the other hand, since $P_{(1)}(z)$ contributes only subleading powers, both of its coefficients $\mu_1$ and $\mu_0$ are effective parameters. Then, if we take into account fractional linear transformations, three *gauge fixing* conditions can be imposed on the $r+3$ parameters $\{\lambda_i\}, \{\mu_i\}$. In ref. ([4, 5]) this freedom was used to set:

$$
\begin{aligned}
\sum_{i=1}^{r+1} \lambda_i &= 0 \\
\mu_1 &= 0 \\
\mu_0 &= \Lambda^{r+1}
\end{aligned}
\tag{2.20}
$$

where $\Lambda$ is the dynamically generated scale. With this choice the $r$–parameter family of dynamical Riemann surfaces is described by the equation:

$$
w^2 = \left( z^{r+1} - \sum_{i=1}^{r} u_i(\lambda)\, z^{r-i} \right)^2 - \Lambda^{2r+2} \tag{2.21}
$$

where the coefficients

$$
u_i(\lambda_1, \ldots, \lambda_{r+1}) \qquad\qquad (i = 1, \ldots r) \tag{2.22}
$$

are symmetric functions of the $r+1$ roots constrained by the first of eq.s (2.20) and can be identified with the moduli parameters introduced in eq.(2.5). In the particular case $r=1$, the gauge–fixing (2.20) leads to the following quartic form for the elliptic curve studied in [1]

$$
w^2 = \left(z^2 - u\right)^2 - \Lambda^4 = z^4 - 2\,u\,z^2 + u^2 - \Lambda^4 \tag{2.23}
$$

Of course other gauge fixings give equivalent descriptions of $\mathcal{M}_1[r]$; however, for our next purposes, it is particularly important to choose a gauge fixing of the $SL(2,\mathbb{C})$ symmetry such that the equation $\mathcal{M}_1[r]$ can be recast in the form of a Fermat polynomial in a weighted projective space deformed by the marginal operators of its chiral ring. In this way it is quite easy to study the symmetry group of the potential $\Gamma_{\mathcal{W}}$ identifying the $R$-symmetry group and to derive the explicit form of the Picard-Fuchs equations satisfied by the periods. This is relevant for the embedding of the monodromy and $R$-symmetry groups in $Sp(2r,\mathbb{Z})$. The alternative gauge-fixing that we choose is the following:

$$
\begin{aligned}
\sum_{i=1}^{r+1} \lambda_i &= 0 \\
\mu_1\,\mu_0 + \left( \sum_{i=1}^{r+1} \frac{1}{\lambda_i} \right) \prod_{i=1}^{r+1} \lambda_i^2 &= 0 \\
-\mu_0^2 + \prod_{i=1}^{r+1} \lambda_i^2 &= 1
\end{aligned}
\tag{2.24}
$$

To appreciate the convenience of this choice let us consider the general inhomogeneous form of the equation of the hyperelliptic surface (2.18) and let us (quasi-)homogenize it by setting:

$$
w = \frac{Z}{Y^{r+1}} \quad z = \frac{X}{Y} \quad . \tag{2.25}
$$

With this procedure (2.18) becomes a quasi–homogeneous polynomial constraint:

$$\begin{aligned} 0 &= \mathcal{W}\left(Z,X,Y;\{\lambda\},\{\mu\}\right) \\ &= -Z^2+\left(\prod_{i=1}^{r+1}\left(X-\lambda_i\,Y\right)\right)^2-\left(\mu_1\,X\,Y^r+\mu_0\,Y^{r+1}\right)^2 \end{aligned} \tag{2.26}$$

of degree:

$$\deg\mathcal{W} = 2r+2 \tag{2.27}$$

in a weighted projective space $W\mathbb{CP}^{2;r+1,1,1}$, where the quasi–homogeneous coordinates $Z$, $X$, $Y$ have degrees $r+1$,1 and 1, respectively. Adopting the notations of [33], namely denoting by[2]

$$WCP^n(d;q_1,q_2,\ldots,q_{n+1})_\chi \tag{2.28}$$

the zero locus (with Euler number $\chi$) of a quasi–homogeneous polynomial of degree $d$ in an $n$–dimensional weighted projective space, whose $n+1$ quasi–homogeneous coordinates have weights $q_1$,. . .,$q_{n+1}$:

$$\mathcal{W}\left(\lambda^{q_1}\,X_1,\ldots\,\lambda^{q_{n+1}}\,X_{n+1}\right) = \lambda^d\,\mathcal{W}\left(X_1,\ldots,X_{n+1}\right) \qquad \forall\lambda\in\mathbb{C} \tag{2.29}$$

we obtain the identification:

$$\mathcal{M}_1[r] = WCP^2(2r+2;r+1,1,1)_{2(1-r)} \tag{2.30}$$

that yields, in particular:

$$\mathcal{M}_1[1]=WCP^2(4;2,1,1)_0\ ; \qquad \mathcal{M}_1[2]=WCP^2(6;3,1,1)_2\ . \tag{2.31}$$

for the $SU(2)$ case studied in [1] and for the $SU(3)$ case studied in [4, 5]. Using the alternative gauge fixing (2.24), the quasi–homogeneous Landau–Ginzburg superpotential (2.26), whose vanishing locus defines the dynamical Riemann surface, takes the standard form of *a Fermat superpotential* deformed by the marginal operators of its *chiral ring*:

$$\mathcal{W}\left(Z,X,Y;\{\lambda\},\{\mu\}\right)=-Z^2+X^{2r+2}+Y^{2r+2}+\sum_{i=1}^{2r-1}v_i\left(\lambda\right)\,X^{2r+1-i}\,Y^{i+1} \tag{2.32}$$

The coefficients $v_i \quad (i=1,\ldots 2r-1)$ are the $2r-1$ moduli of a hyperelliptic curve. In our case, however, they are expressed as functions of the $r$ independent roots $\lambda_i$ that remain free after the gauge–fixing (2.24) is imposed. The coefficients $v_i$ have a simple expression as symmetric functions of the $r+1$ roots $\lambda_i$ subject to the constraint that their sum should vanish:

$$\begin{aligned} v_1\left(\lambda\right) &= \sum_i\lambda_i^2\,+\,4\sum_{i<j}\lambda_i\,\lambda_j \\ v_2\left(\lambda\right) &= -2\sum_{i<j}\left(\lambda_i^2\,\lambda_j\,+\,\lambda_i\,\lambda_j^2\right)\,-\,8\sum_{i<j<k}\lambda_i\,\lambda_j\,\lambda_k \\ v_3\left(\lambda\right) &= \sum_{i<j}\lambda_i^2\,\lambda_j^2\,+\,16\sum_{i<j<k}\left(\lambda_i^2\,\lambda_j\,\lambda_k\,+\,\lambda_i\,\lambda_j^2\,\lambda_k\,+\,\lambda_i\,\lambda_j\,\lambda_k^2\right) \end{aligned}$$

[2]Note the difference of notation: $W\mathbb{CP}^{n;q_1,q_2,\ldots,q_{n+1}}$ is the full weighted projective space, in which (2.28) is a hypersurface.

$$
\begin{aligned}
v_4(\lambda) &= -2 \sum_{i<j<k} \left( \lambda_i^2 \lambda_j^2 \lambda_k + \lambda_i^2 \lambda_j \lambda_k^2 + \lambda_i \lambda_j^2 \lambda_k^2 \right) \\
&\quad - 8 \sum_{i<j<k<\ell} \left( \lambda_i \lambda_j \lambda_k \lambda_\ell^2 + \lambda_i \lambda_j \lambda_k^2 \lambda_\ell + \lambda_i \lambda_j^2 \lambda_k \lambda_\ell + \lambda_i^2 \lambda_j \lambda_k \lambda_\ell \right) \\
v_5(\lambda) &= \ldots\ldots
\end{aligned}
\tag{2.33}
$$

In particular for the first two cases $r = 1$ and $r = 2$ we respectively obtain:

$$
\begin{aligned}
\mathcal{M}_1[1] &\hookrightarrow \\
0 &= \mathcal{W}(Z, X, Y; v = 2u) = -Z^2 + X^4 + Y^4 + v(\lambda) X^2 Y^2 \qquad (2.34) \\
& \quad \lambda_1 + \lambda_2 = 0 \\
& \quad \mu_1 = 0 \\
& \quad \mu_0 = \sqrt{\lambda_1^2 \lambda_2^2 - 1} \\
v &= \lambda_1^2 + \lambda_2^2 + 4\lambda_1 \lambda_2 = -2\lambda_1^2 \stackrel{\text{def}}{=} 2u \qquad (2.35)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{M}_1[2] &\hookrightarrow \\
0 &= \mathcal{W}(Z, X, Y; v_1, v_2, v_3) \\
&= -Z^2 + X^6 + Y^6 + v_1 X^4 Y^2 + v_2 X^3 Y^3 + v_3 X^2 Y^4 \\
& \quad \lambda_1 + \lambda_2 + \lambda_3 = 0 \\
& \quad \mu_1 = -\frac{\lambda_1 \lambda_2 \lambda_3}{\sqrt{\lambda_1^2 \lambda_2^2 \lambda_3^2 - 1}} (\lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3) \\
& \quad \mu_0 = \sqrt{\lambda_1^2 \lambda_2^2 \lambda_3^2 - 1} \\
v_1 &= 2(\lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3) \\
v_2 &= -2\lambda_1 \lambda_2 \lambda_3 \\
v_3 &= -\mu_1^2 + (\lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3)^2
\end{aligned}
\tag{2.36}
$$

Alternatively, using as independent parameters the coefficients $u_i(\lambda)$ appearing in eq. (2.21), we can characterize the locus $L_R[r]$ of dynamical Riemann surfaces by means of the following equations on the hyperelliptic moduli $v_i$:

$$
\begin{aligned}
v_k &= -2u_k + \sum_{i+j=k-1} u_i u_j, \qquad k = 1, \ldots, r \\
v_{r+k} &= \sum_{i+j=r+k-1} u_i u_j - \delta_{r-1,k} \mu_1^2
\end{aligned}
\tag{2.37}
$$

Considering now the Hodge filtration of the middle cohomology group $H_{DR}^{(1)}(\mathcal{M}_1[r])$:

$$
\begin{aligned}
& \mathcal{F}^0 \subset \mathcal{F}^1 \\
\mathcal{F}^0 = H^{(1,0)} \quad &; \quad \mathcal{F}^1 \equiv H_{DR}^{(1)} = H^{(1,0)} + H^{(0,1)}
\end{aligned}
\tag{2.38}
$$

the Griffiths residue map ([34, 35]) provides an association between elements of $\mathcal{F}^k$ and polynomials $P^I_{k|(2r+2)}(X)$ of the chiral ring $\mathcal{R}(\mathcal{W}) \stackrel{\text{def}}{=} \mathbb{C}[X]/\partial\mathcal{W}$ of degree $k|(2r+2) \equiv$

$(k+1)(2r+2)-r-3$ according to the following pattern:

$$\begin{array}{llll} \text{cohom.} & \text{deg} & \text{polynom.} & \\ & & & \\ \mathcal{F}^0 & r-1 & P^i_{0|(2r+2)} & i=1,\dots,r \\ \mathcal{F}^1 & 3r+1 & P^{i^*}_{1|(2r+2)} & i^*=1,\dots,r \end{array} \tag{2.39}$$

Explicitly, the periods of eq. (2.8) are represented by:

$$\begin{aligned} \int_C \omega^i &= \int_C \frac{X^{r-i}Y^{i-1}}{\mathcal{W}}\,\omega \\ \int_C \omega^{i^*} &= \int_C \frac{X^{r+i}Y^{2r-i+1}}{\mathcal{W}^2}\,\omega \end{aligned} \tag{2.40}$$

where $C$ denotes any of the homology cycles and $\omega = Z\,\mathrm{d}X \wedge \mathrm{d}Y + \text{cycl.}$ Using standard reduction techniques [36, 32] one can obtain the first-order Picard-Fuchs differential system

$$\left(\frac{\partial}{\partial v^I}\mathbb{1} - A_I(v)\right) V = 0 \qquad I = 1,\dots 2r-1 \tag{2.41}$$

satisfied by the $2r$-component vector:

$$V = \begin{pmatrix} \int_C \omega^i \\ \int_C \omega^{i^*} \end{pmatrix} \tag{2.42}$$

in the $2r-1$-dimensional moduli space of elliptic surfaces. Using the explicit embedding of the locus $L_R[r] \subset L_H[r]$ described by equations (2.37), we obtain the Picard-Fuchs differential system of rigid special geometry by a trivial pull-back of eq. (2.41):

$$\left(\frac{\partial}{\partial u^i}\mathbb{1} - A_I(v)\frac{\partial v^I}{\partial u^i}\right) V = 0. \tag{2.43}$$

The explicit solution of the Picard–Fuchs equations for $r = 1, 2$ has been given respectively in [7, 22]. The solution of the Picard–Fuchs equations for generic $r$ determines in principle the period of the surface and the monodromy group. We do not attempt to solve (2.43) for generic $r$, but rather we will determine in the next section the monodromy group by relying on the defining polynomial of the surface only.

In the remaining part of this section we discuss the symmetry group of $\mathcal{M}_1(r)$, which, together with the monodromy group $\Gamma_M$ defines the duality group $\Gamma_D$ according to equation (2.14). This symmetry group can be defined by considering those linear transformations $\mathbf{X} \to M_A\mathbf{X}$ of the quasi–homogeneous coordinate vector $\mathbf{X} = (X, Y, Z)$ such that

$$\mathcal{W}(M_A\mathbf{X}; v) = f_A(v)\mathcal{W}(\mathbf{X}; \phi_A(v)) \tag{2.44}$$

where $\phi_A(v)$ is a (generally non–linear) transformation of the moduli and $f_A(v)$ is a compensating overall rescaling of the superpotential that depends both on the moduli $v$ and on the chosen transformation $A$. An important subgroup $\Gamma^0_{\mathcal{W}} \subset \Gamma_{\mathcal{W}}$ is given by the set of those transformations $H \in \Gamma_{\mathcal{W}}$ that have a compensating rescaling factor equal to unity:

$$H \in \Gamma^0_{\mathcal{W}} \subset \Gamma_{\mathcal{W}} \leftrightarrow f_H(u) = 1 \tag{2.45}$$

The zero–locus of the potential is obviously invariant under the full group $\Gamma_{\mathcal{W}}$ yet in the case of *rigid special geometry* only $\Gamma^0_{\mathcal{W}}$ turns out to be an isometry group of the special Kählerian metric, in contrast with the case of *Special Geometry* where the whole $\Gamma_{\mathcal{W}}$ generates isometries. This very crucial point will be appreciated in the sequel.

The hyperelliptic superpotential (2.32) admits a $\Gamma^0_{\mathcal{W}}$ symmetry group which is isomorphic to the dihedral group $D_{2r+2}$, defined by the following relations on two generators $A, B$:

$$A^{2r+2} = \mathbb{1} \quad ; \quad B^2 = \mathbb{1} \quad ; \quad (AB)^2 = \mathbb{1}. \tag{2.46}$$

The action of the generators on the moduli is the following. Let $\alpha^{2r+2} = 1$ be a $(2r+2)^{\mathrm{th}}$ root of the unit and let the moduli $v_i$ be arranged into a column vector $\mathbf{v}$. Then we have:

$$\begin{aligned} \mathbf{v}' = A\mathbf{v}, \qquad & A = \begin{pmatrix} \alpha^2 & 0 & \dots & 0 \\ 0 & \alpha^3 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \alpha^{2r} \end{pmatrix} \\ \mathbf{v}'' = B\mathbf{v}, \qquad & B = \begin{pmatrix} 0 & 0 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & \dots & 0 \\ 1 & 0 & \dots & 0 \end{pmatrix} \end{aligned} \tag{2.47}$$

For the transformations $A$ and $B$ the compensating transformations on the homogeneous coordinates $M_A$ and $M_B$ are

$$M_A = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad ; \quad M_B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.48}$$

Consequently the differential Picard–Fuchs system for the period (2.42) of the generic hyperelliptic surface has a $\Gamma^0_{\mathcal{W}} = D_{2r+2}$ symmetry as defined above and the generators $A$ and $B$ act by means of suitable $Sp(2r, \mathbb{Z})$ matrices on the period vector (2.8). However the equations (2.43) are invariant only under the cyclic subgroup $\mathbb{Z}_{2r+2} \in D_{2r+2}$ generated by $A$. Hence the potential $\widetilde{\mathcal{W}}(u) = \mathcal{W}(v(u))$ of the $r$-dimensional locus $L_R[r]$ of dynamical Riemann surfaces and the Picard-Fuchs first order system admits only the duality symmetry $\Gamma^0_{\widetilde{\mathcal{W}}} = \mathbb{Z}_{2r+2}$.

The physical interpretation of this group is R-symmetry. Indeed, recalling eq. (2.5) we see that when each of the elementary fields $Y^\alpha$ appearing in the lagrangian is rescaled as $Y^\beta \to \alpha Y^\beta$, then the first $u_i$ moduli are rescaled with the powers of $\alpha$ predicted by equation (2.5). According to the analysis of reference [20] this is precisely the requested R-symmetry for the topological twist. All the scalar components of the vector multiplets have the same R-charge ($q_R = 2$) under a $U_R(1)$ symmetry of the classical action, which is broken to a discrete subgroup in the quantum theory. Henceforth the integer symplectic matrix that realizes $A$ yields the R-symmetry matrix of rigid special geometry for $SU(r+1)$ gauge theories. An important problem is the derivation of the corresponding R-symmetry matrix in $Sp(2r+4, \mathbb{Z})$, when the gauge theory is made locally supersymmetric by coupling it to supergravity including also the dilaton-axion vector multiplet suggested by string theory.

# 3 The monodromy group of $SU(r+1)$ rigid gauge theory

In this section we give a concise account of the explicit construction of the monodromy group $\Gamma_M(r) \in Sp(2r;\mathbb{Z})$ for the subclass of hyperelliptic surfaces given by eqs. (2.18) or (2.21) associated with the $SU(r+1)$ rigid gauge theory. In our approach the required monodromy group is selected as a particular subgroup of the monodromy group of a generic hyperelliptic surface of genus $r$

$$\Sigma_r \in L_H[r] : \{w^2 = P_{(2r+2)}(z) = z^{2r+2} + c_1\ z^{2r+1} + \cdots + c_{2r+1}\ z + c_{2r+2} = \Pi_{i=1}^{2r+2}(z-\lambda_i)\}\ , \tag{3.1}$$

where only $2r-1$ of the $c_k$ are independent moduli. The method presented here yields a complete solution for any $SU(r+1)$ and uses some tools that were introduced in [21]. Our results can be compared with those recently obtained in [22], where the particular case of $SU(3)$ has been thoroughly discussed and where the corresponding periods of the theory have been explicitly obtained[3].

Our basic observation is the following: the monodromy group for $\Sigma_r$ is given by a $2r$-dimensional representation of $B(2r+2)$, the braid group acting on $2r+2$ strands, on the homology basis of $\Sigma_r$ . Indeed, it is sufficient to recall that:
i)The monodromy group of a p-fold $\mathcal{M}$ is given by the representation on the homology basis of the p-fold of the fundamental group $\pi_1$ of the complement of the bifurcation set of $\mathcal{M}$.
ii) For the case $\mathcal{M} = \Sigma_r$, where $\Sigma_r$ is described by the polynomial (3.1), denoting by $Q^{(2r-2)}$ the bifurcation set of eq. (3.1), and by $C$ the base point, we have

$$\pi_1(\mathbf{CP}^{(2r-1)} - Q^{(2r-2)}; C) \equiv B(2r+2)\ , \tag{3.2}$$

since $B(2r+2)$ is the fundamental group of the space of polynomials of degree $2r+2$ with no multiple roots. Indeed, the bifurcation set of a polynomial is given by the submanifold in the moduli space $\{c_1, \ldots, c_{2r-1}\}$ where two or more roots $\lambda_i$ coincide.

The generators $t_i$ of $B(2r+2)$ correspond to the exchange of the $i$-th and the $i+1$-th strand and satisfy the relations

$$\begin{aligned} t_i t_{i+i} t_i &= t_{i+1} t_i t_{i+1} \\ t_i t_j &= t_j t_i \qquad\qquad |i-j| \geq 2\ . \end{aligned} \tag{3.3}$$

In particular, to each generator $t_i \in \pi_1 \equiv B(2r+2)$ there corresponds a loop in the moduli space which exchanges the roots $\lambda_i, \lambda_{i+1}$ of the polynomial and a vanishing cycle of $\Sigma_r$. For a generic hyperelliptic surface any two roots can be exchanged by a suitable word in the generators $t_i$.

Let us now consider the particular subclass of hyperelliptic surfaces $\mathcal{M}_1[r] \in L_R[r]$ described in the previous section, by eqs. (2.18),(2.21). Corresponding to the factorization in eqs. (2.18),(2.21) we have a natural splitting of the $2r+2$ roots of $P_{(2r+2)}$ into two sets

$$\{\lambda_1, \ldots, \lambda_{r+1}\} \quad \text{and} \quad \{\lambda_{r+2}, \ldots, \lambda_{2r+2}\}\ . \tag{3.4}$$

[3]We refer the reader to that paper for a more complete discussion of the geometrical theory of monodromy and its relevance for $SU(n)$ $N=2$ gauge theories. See also [37] for the monodromy in $SO(2r+1)$ gauge theories

It is obvious that for the particular surface $\mathcal{M}_1[r]$ the fundamental group $\pi_1$ mentioned above will be generated by those elements $t_i \in B(2r+2)$ which respect the splitting (3.4), that is

$$\{t_1, \dots, t_r, t_{r+1}^2, t_{r+2}, \dots, t_{2r+2}, T = (t_1 t_2 \cdots t_{2r+1})^{r+1}\} \tag{3.5}$$

where $T$ corresponds to the exchange of the two sets of roots ($t_1 \cdots t_{2r+1}$ corresponds to the cyclic permutation $\{\lambda_1, \lambda_2, \dots, \lambda_{2r+2}\} \to \{\lambda_2, \lambda_3, \dots, \lambda_{2r+2}, \lambda_1\}$). We conclude that the fundamental group of the hyperelliptic surface $\mathcal{M}_1[r]$ is generated by the elements (3.5). The required monodromy group $\Gamma_M[r]$ is therefore given by the representation on the homology group $H_1(\mathcal{M}_1[r], \mathbb{Z})$ of the generators (3.5). At this point the strategy for computing the explicit monodromy of $\mathcal{M}_1[r]$ is clear: one first obtains the monodromy group of $\Sigma_r$ as a representation $M(t_i)$ of the $B(2r+2)$ generators on the homology basis of $\mathcal{M}_1[r]$. Then the monodromy group of $\mathcal{M}_1[r]$ is given by the subgroup generated by

$$\Big\{M(t_1), \dots, M(t_r), M^2(t_{r+1}), M(t_{r+2}), \dots, M(t_{2r+1}), M(T)\Big\} \tag{3.6}$$

Let us then construct the monodromy group of $\Sigma_r$ as a representation of $B(2r+2)$ on $H_1(\Sigma_r; \mathbb{Z})$. We first choose a basis of cycles $(A^I, B_I)$ on the cut $z$-plane such that

$$A^I \cap A^J = B_I \cap B_J = 0, \;\; A^I \cap B_J = -B_J \cap A^I = \delta^I_J \quad (I, J = 1, \dots, r) \tag{3.7}$$

so that the homology intersection form $C$ takes the canonical form

$$C = \begin{pmatrix} 0 & \mathbb{1}_r \\ -\mathbb{1}_r & 0 \end{pmatrix} \tag{3.8}$$

Actually we may take the cycles $A^I$ to encircle the couple of roots $(\lambda_{2I}, \lambda_{2I+1})$, while the $B_I$ cycles encircle the set of roots $(\lambda_1, \lambda_2, \dots, \lambda_{2I})$. To a generic element $t \in B(2r+2)$ we may associate the corresponding vanishing cycle $L$ of $H_1(\Sigma_r, \mathbb{Z})$, say

$$L = (n^e_I, n^I_m) \tag{3.9}$$

where $(n^e_I, n^I_m)$ are the components of $L$ with respect to the basis $(A^I, B_I)$. Using the Picard–Lefschetz formula [22]

$$\delta \to \delta - (\delta \cup L)L \tag{3.10}$$

which represents the transformation induced on the homology by the vanishing cycle $L$ corresponding to the element $t \in B(2r+2)$, it is easy to see that in the given basis the corresponding monodromy matrix $M(L)$ is given by:

$$M(L) = \mathbb{1} + L \otimes (CL) \equiv \begin{pmatrix} \mathbb{1} + \vec{n}_e \otimes \vec{n}_m & -\vec{n}_e \otimes \vec{n}_e \\ \vec{n}_m \otimes \vec{n}_m & \mathbb{1} - \vec{n}_m \otimes \vec{n}_e \end{pmatrix} . \tag{3.11}$$

Denoting by $L^{(i)}$ the homology element associated to $t_i$ $(i = 1, \dots, 2r+2)$, their explicit form is found by imposing the braid relations (3.3) on $M(L^{(i)})$, which yield the constraints

$$\begin{aligned} L^{(i)T} C L^{(j)} &= 0 \qquad\qquad |i-j| \geq 2 \\ L^{(i)T} C L^{(i+1)} &= 1 \; . \end{aligned} \tag{3.12}$$

The solution can be written as follows

$$\begin{aligned}
L^{(2j-1)} &= (\vec{e}_j - \vec{e}_{j-1}; \vec{0}) \\
L^{(2j)} &= (\vec{0}; -\vec{e}_j) \qquad\qquad j = 1, \ldots, r \\
L^{(2r+1)} &= (-\vec{e}_r; \vec{0}) \\
L^{(2r+2)} &= (\vec{0}; \vec{e}_1 + \cdots + \vec{e}_r)
\end{aligned} \tag{3.13}$$

where $\vec{e}_i$ is an orthonormal basis in $\mathbb{R}^r$ ($\vec{e}_0 = 0$). Notice that the electric charges of the odd-numbered $L$ and the magnetic charges of the even-numbered $L$ are given in terms of the roots and fundamental weights of $SU(r+1)$ [22]. The restriction of the braid group generated by $M(L^{(i)})$ to the subgroup given in (3.6) (with $(M(L^{(i)}) \equiv M(t_i))$ gives the monodromy group of the hyperelliptic curves $\mathcal{M}_1(r)$ .

We stress that our construction selects uniquely the possible entries of $L^{(i)} = (\vec{n}_e^{(i)}, \vec{n}_m^{(i)})$, corresponding to the values of the electric and magnetic charges of any $SU(r+1)$ gauge theory.

# 4 Coupling to Supergravity and the dynamical Calabi-Yau manifold

When we couple vector multiplets to supergravity, in the scalar sector *rigid special geometry* [7] is replaced by its local version, namely by *special geometry* [3, 24, 38, 39, 25, 40]. For the coupling of the microscopic N=2 gauge theory we have two possibilities. 1) The most natural generalization of the minimal coupling (2.1) is given by the gravitational minimal coupling where the number of vector multiplets $n = \dim_{\mathbf{R}}\mathcal{G}$ remains the same as in the rigid theory and the scalar manifold is given by:

$$\mathcal{SK}_{local}(n) = MK(n) \stackrel{\text{def}}{=} \frac{SU(1,n)}{U(1) \times SU(n)} \tag{4.1}$$

while the prepotential is:

$$\begin{aligned}
F(X) &= (X^0)^2 - g_{IJ}^{(K)}\, X^I\, X^J \\
g_{IJ}^{(K)} &= \text{positive definite Killing metric of the Lie algebra } \mathbf{G} \\
X^\Lambda &= (X^0, X^I) = \text{local special coordinates}
\end{aligned} \tag{4.2}$$

Even if such a choice is a consistent one from the supergravity point of view, it is however not compatible with string theory, since the multiplet containing the dilaton–axion is missing. Furthermore, one can show [20] that there is no off–shell defined R–symmetry that can lead to an off-shell defined ghost number [41] in the topological twisted theory, and that in the topologically twisted theory, the moduli–space of gravitational instantons has dimension $3 \times \tau$ rather than $4 \times \tau$ ($\tau$ = Hirzebruch signature), as needed to obtain non vanishing topological correlators of operators associated with 0–cycles and 2–cycles of the four–manifold.

These problems disappear if we consider instead the generalization of the minimal coupling selected by string theory, which, besides the $n = \dim_{\mathbf{R}}\mathcal{G}$ vector multiplets of the rigid theory, requires also an additional vector multiplet $\left(A^S_\mu, \lambda^S_A, \lambda^{S\star A}, S, \overline{S}\right)$ containing the dilaton–axion field:

$$
\begin{aligned}
S &= \mathcal{A} + \mathrm{i}\exp[D] \\
\partial_{[\mu}\, B_{\nu\rho]} &= \varepsilon_{\mu\nu\rho\sigma}\partial^\sigma \mathcal{A}
\end{aligned}
\tag{4.3}
$$

whose vacuum expectation value provides the effective gauge coupling constant and theta–angle:

$$
\langle\, S\,\rangle \;=\; \frac{\theta}{2\pi} + \mathrm{i}\frac{1}{g^2}
\tag{4.4}
$$

The tree level effective action is based on the following homogeneous special manifold:

$$
\mathcal{SK}_{local}(n+1) \;=\; ST(n) \stackrel{\mathrm{def}}{=} \frac{SU(1,1)}{U(1)} \otimes \frac{O(2,n)}{O(2)\times O(n)}
\tag{4.5}
$$

that, according to a theorem proved sometime ago [42], is also the only special manifold admitting a direct product structure. If we use the coordinate frame of Calabi–Visentini ([43]) for the submanifold $\frac{O(2,n)}{O(2)\times O(n)}$:

$$
\begin{array}{l}
X^\Lambda \stackrel{\mathrm{def}}{=} \left(X^0, X^1, X^I\right) \\
X^\Lambda X^\Sigma \eta_{\Lambda\Sigma} = 0
\end{array}
\quad : \quad
\left\{
\begin{array}{l}
X^0 = \frac{1}{2}\left(1 + g^{(K)}_{IJ} Y^I Y^J\right) \\[2ex]
X^1 = \frac{\mathrm{i}}{2}\left(1 - g^{(K)}_{IJ} Y^I Y^J\right) \\[2ex]
X^I = Y^I \qquad \{I = 1, \ldots, n = \dim_{\mathbb{R}} \mathcal{G}\}
\end{array}
\right.
\tag{4.6}
$$

where $g^{(K)}_{IJ}$ is the Killing metric of the Lie algebra $\mathbf{G}$ and $\eta_{\Lambda\Sigma} = \mathrm{diag}\left(+,+,-g^{(K)}_{IJ}\right)$, the Kähler potential being (the full Kaḧler potential still contains a term $-\log\; i(\overline{S} - S)$),

$$
\mathcal{K}(Y, \overline{Y}) = -\log\left[X^\Lambda \overline{X}^\Sigma \eta_{\Lambda\Sigma}\right] = -\log\left[\frac{1}{2}\left(1 - 2 g^{(K)}_{IJ} Y^I\, \overline{Y}^J + |g^{(K)}_{IJ} Y^I Y^J|^2\right)\right],
\tag{4.7}
$$

then the appropriate $4+2n$–dimensional symplectic section determining special geometry is provided by[8]:

$$
\Omega(S, Y) \;=\; \begin{pmatrix} X^\Lambda \\ F_\Sigma \;=\; S\,\eta_{\Sigma\Delta}\, X^\Delta \end{pmatrix}
\tag{4.8}
$$

When the theory is classical and purely abelian, with matter fields carrying no electric and magnetic charges, the supergravity based on the $ST(n)$ special manifold admits a continuous group of duality transformations á la Gaillard-Zumino [44]:

$$
SL(2, \mathbb{R}) \otimes O(2, n),
\tag{4.9}
$$

the symplectic embedding into $Sp(4+2n, \mathbb{R})$ being as follows.

$$
\begin{array}{lcl}
A \in O(2,n) & \hookrightarrow & \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & \eta A \eta^{-1} \end{pmatrix} \in Sp(2n+4, \mathbb{R}) \\[3ex]
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(2n+4, \mathbb{R}) & \hookrightarrow & \begin{pmatrix} a\mathbb{1} & b\eta^{-1} \\ c\eta & d\mathbb{1} \end{pmatrix} \in Sp(2n+4, \mathbb{R})\ ,
\end{array}
\tag{4.10}
$$

where $A^T \eta A = \eta$.

On the other hand, consider the abelian phase of a spontaneously broken Yang–Mills theory coupled to supergravity. If one takes into account the massive charged modes, the duality group $\Gamma_D^{loc}$ is a discrete group. The reason is that the lattice of electric and magnetic charges of the BPS saturated states must be preserved by the duality rotations. This happens even in those cases where the local special geometry of the moduli space does not receive quantum corrections and remains the same as that of $ST(n)$ [described by eq. (4.6)]. In these cases the duality group $\Gamma_D^{loc}$ is a discrete subgroup of (4.9):

$$\Gamma_D^{loc} \subset SL(2,\mathbb{Z}) \otimes O(2,n;\mathbb{Z}) \,, \tag{4.11}$$

the embedding into $Sp(2n+4,\mathbb{Z})$ being the restriction to the integers of the embedding (4.10).

In general, however, the local geometry of the moduli space $ST(n)$ is modified by perturbative and non perturbative effects. Therefore, considering the effective N=2 lagrangian describing the dynamics of the massless modes, that admits the $r$–dimensional maximal torus $H \subset G$ as gauge group, we are faced with the problem of finding the $r+1$–dimensional special manifold $\widehat{ST}(r)$ that encodes the complete structure of this lagrangian and the exact quantum duality group $\Gamma_D^{local}$.

We note that $\widehat{ST}(r)$ is a quantum deformation of the manifold $ST(r)$: for large values of the moduli, namely in a asymptotic region, to be appropriately defined, where the quantum theory approaches its classical limit, the manifold $\widehat{ST}(r)$ should reduce to $ST(r)$. This manifold is the truncation to the Cartan–subalgebra fields of the manifold $ST(r + \# \text{ of roots} = n)$, that corresponds to the *gravitationally coupled microscopic gauge theory.* At the same time, the quantum duality group of the rigid theory $\Gamma_D^{rig.}$ should be embedded in the quantum duality group of the local theory

$$Sp(2r,\mathbb{Z}) \;\supset\; \Gamma_D^{rig.} \;\hookrightarrow\; \Gamma_D^{loc.} \;\subset\; Sp(4+2r,\mathbb{Z}) \tag{4.12}$$

In special coordinates

$$S = \frac{\widehat{X}^1}{\widehat{X}^0} \quad t^i = \frac{\widehat{X}^i}{\widehat{X}^0} \quad i = 2,\dots,r+1 \tag{4.13}$$

this means that the prepotential of the *quantum local special geometry* is of the following form:

$$\begin{aligned} \mathcal{F}^{loc}(S,t) \;=\; \left(\widehat{X}^0\right)^{-2} F^{loc}\left(\widehat{X}\right) \;&=\; \frac{1}{r!}\, S\, t^i\, t^j \eta_{ij} + \Delta\mathcal{F}^{loc}(S,t) \\ \lim_{t^i \to t^i_0,\; S \to S_0} \Delta\mathcal{F}^{loc}(S,t) \;&=\; 0 \end{aligned} \tag{4.14}$$

the asymptotic region corresponding to a neighbourhood of $S, t^i = S_0, t^i_0$ where $S_0, t^i_0$ are appropriate values, possibly infinite. Eq.(4.14) is the local supersymmetry counterpart of eq.(4.2) that applies instead to the case of rigid supersymmetry.

The reason why we have put a hat on the $X^\Lambda$ is that they cannot be directly identified with the $X^\Lambda$ introduced in eq.(4.6). Indeed, in the symplectic basis defined by eqs. (4.6),(4.8), namely in the basis where, according to the embedding (4.10), the $O(2,n)$ symmetry and, hence, the gauge symmetry $G \subset O(n) \subset O(2,n)$ are linearly realized, the special geometry of the manifold $ST(n)$ admits no description in terms of a prepotential

$F(X) = X^\Lambda F_\Lambda/2$. This is due to the constraint $0 = X^\Lambda X^\Sigma \eta_{\Lambda\Sigma}$ [8]. Hence although the Calabi–Visentini coordinates $Y^I$ are identified with the special coordinates of rigid special geometry, yet the $X^\Lambda$ appearing in (4.6) and (4.8) are not independent special coordinates for local special geometry. To obtain a prepotential one needs to perform a symplectic rotation to a new basis:

$$
\begin{aligned}
\begin{pmatrix} \widehat{X}^\Lambda \\ \partial_\Sigma F(\widehat{X}) \end{pmatrix} &= M \begin{pmatrix} X^\Lambda \\ S\,\eta_{\Sigma\Delta}\,X^\Delta \end{pmatrix} \\
M &= \begin{pmatrix} 1 & 0 & -\mathbb{1} & 0 & 0 & \mathbf{0} \\ 0 & 0 & \mathbf{0} & 1 & 0 & \mathbb{1} \\ \mathbf{0} & -\mathbb{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & \mathbf{0} & \frac{1}{2} & 0 & -\frac{1}{2}\mathbb{1} \\ -\frac{1}{2} & 0 & -\frac{1}{2}\mathbb{1} & 0 & 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbb{1} & \mathbf{0} \end{pmatrix} \in Sp(4+2n,\mathbb{R})
\end{aligned}
\tag{4.15}
$$

leading to a new symplectic embedding:

$$
\begin{aligned}
A \in O(2,n) \qquad &\hookrightarrow \quad M \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & \eta A \eta^{-1} \end{pmatrix} M^{-1} \in Sp(2n+4,\mathbb{R}) \\
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(2n+4,\mathbb{R}) \quad &\hookrightarrow \quad M \begin{pmatrix} a\mathbb{1} & b\eta^{-1} \\ c\eta & d\mathbb{1} \end{pmatrix} M^{-1} \in Sp(2n+4,\mathbb{R})
\end{aligned}
\tag{4.16}
$$

After this change of basis the symmetric constant tensor $\eta_{ij}$ appearing in (4.14) is not the positive definite $g^{(K)}_{\alpha\beta}$, appearing in eq.(4.2), namely the reduction to the Cartan–subalgebra of the Killing metric $g^{(K)}_{IJ}$. It is rather a form with Lorentzian signature $(-,+,+,\ldots,+)$. Now, the basic idea to obtain the explicit form of the gravitationally coupled effective lagrangian is to identify the special Kähler manifold $\widehat{ST}(r)$ with the complex–structure moduli space of an $r+1$–parameter family of dynamical Calabi–Yau three–folds $\mathcal{M}_3[r]$.

This is the obvious generalization of the procedure adopted in the rigid case. In the same way as the rigid special manifold $\mathcal{SK}^{rig}[r]$ is the moduli–space of an $r$–parameter family of genus $r$ dynamical Riemann–surfaces $\mathcal{M}_1[r]$, the local special manifold $\widehat{ST}(r)$ is the moduli–space of a family of Calabi–Yau threefolds. The relation between local special geometry and the variations of Hodge–structures of Calabi–Yau threefolds is well known [39] but we have of course to impose further requirements on $\mathcal{M}_3[r]$ in order for its moduli space to represent the gravitational coupling of an already given rigid effective theory.

Any N=2 globally supersymmetric field theory can be made locally supersymmetric by coupling to N=2 supergravity. This is always possible because of the off–shell structure of N=2 supersymmetry. However the procedure is generally one–to–many as a consequence of the interplay between the auxiliary fields belonging to the matter multiplets and those pertaining to the gravitational multiplet. Once the latter are introduced we have an additional freedom in framing the interaction and various results can be obtained that would be the same if we had only the matter auxiliary fields to play with. Correspondingly the infinite Planck–mass limit

$$M_P = \frac{1}{\kappa} \to \infty \tag{4.17}$$

of a locally supersymmetric theory is not the same thing as a globally supersymmetric theory: this is a quite familiar phenomenon in all the phenomenological applications of supersymmetry.Therefore, in order to state which locally supersymmetric theory can be regarded as the coupling of which rigid theory, one needs some criteria.

In the case of a rigid gauge theory one uses its renormalizability to study the singularities and monodromies produced at the perturbative level and then guesses the complementary singularities introduced by non perturbative effects. This procedure is not available if we start from the gravitational coupling of the microscopic gauge theory since this theory is no longer renormalizable. Obviously one can calculate perturbative effects in string theory and then implement them in the effective lagrangian. This is one possible route and corresponds to the gravitational counterpart of the procedure followed in the rigid case [11]. The now more difficult task of guessing the complementary singularities remains and this amounts to guessing a *dynamical Calabi–Yau* with the appropriate monodromies. This argument shows that one can anyhow by–pass the string step and go directly to the central question: namely *which is the Calabi–Yau three–fold with the appropriate monodromies?* Appropriate monodromies are those that include the monodromies of the rigid theory. More specifically we should have:

$$\begin{array}{ccccccc} Sp(2r+4,\mathbb{Z}) & \supset & \Gamma_D^{local} & \supset & \Gamma_D^{rigid} & & \\ \Gamma_D^{local} & \supset & \Gamma_M^{local} & \supset & \Gamma_M^{rigid} & \subset & \Gamma_D^{rigid} \end{array} \tag{4.18}$$

Recalling the fundamental relation (2.14) between the group of electric–magnetic duality rotations and the monodromy group, we also have:

$$\Gamma_{\mathcal{W}}^{rig} \subset \Gamma_{\mathcal{W}}^{local} \tag{4.19}$$

In the previous section we have studied the general form of $\Gamma_{\mathcal{W}}^{rig}$ for $SU(r+1)$ gauge theories showing that it is $\mathbb{Z}_{2r+2}$ and that it coincides with the R–symmetry group. It follows that the symmetry group of the gravitationally coupled theory, namely of the dynamical Calabi–Yau threefold, should conveniently embed the rigid R–symmetry group. This is the same request formulated in [20] in order to be able to define the topological twist of the quantum theory.

These are the basic criteria that allow to identify the corresponding matter coupled supergravity as the locally supersymmetric version of the already determined globally supersymmetric effective gauge theory.

Let us summarize the results of our discussion. The family of dynamical Calabi–Yau manifolds $\mathcal{M}_3[r]$ must satisfy the following conditions:

- $\mathcal{M}_3[r]$ must be an $r+1$–parameter family of algebraic three–folds in a (product of ) weighted–projective spaces described by the vanishing $\mathcal{W}_i = 0$ $(i = 1,\dots,p)$ of the $p$ addends of a Landau–Ginzburg superpotential:

$$\mathcal{W}(X^1,\dots X^m;\psi_1,\dots,\psi_r) \;=\; \sum_{i=1}^{p} \mathcal{W}_i(X^1,\dots X^m;\psi_1,\dots,\psi_{r+1}) \tag{4.20}$$

  depending on the $r+1$–parameters $\psi_1,\dots,\psi_{r+1}$ and on the $m = 3+p+1$ quasi–homogeneous coordinates of the ambient space.

- The first Chern class of the hypersurface family must obviously vanish

$$c_1\left(\mathcal{M}_3[r]\right) \;=\; 0 \tag{4.21}$$

- The family $\mathcal{M}_3[r]$ must contain some multiple cover of the family $\mathcal{M}_1[r]$ of genus $r$ Riemann surfaces. This guarantees the embedding of the rigid R–symmetry group $\mathbb{Z}_{2r+2}$ into the symmetry group $\Gamma_{\mathcal{W}}$ of the Calabi–Yau potential.

- Writing the degree $\nu$ superpotential (4.20) as the deformation of a reference superpotential $\mathcal{W}_0(X)$

$$\mathcal{W}(X;\psi) \;=\; \mathcal{W}_0(X) + \sum_{I=0}^{r} \psi_I \, P^I_{1|\nu} \tag{4.22}$$

  The chiral ring :

$$\mathcal{R}_{\mathcal{W}_0} \;=\; \frac{\mathbb{C}[X]}{\partial \mathcal{W}_0} \tag{4.23}$$

  of the degree $\nu$ reference superpotential (4.20) should contain a subring of dimension $2 + 2\times(r+1)$ spanned by polynomials $P^J(X)$ of the following degrees:

$$\begin{array}{lll}
\text{degree} & \text{polyn.} & \text{index range} \\
 & & \\
0\times\nu & \mathcal{P}_{0|\nu} = 1 & \\
 & & \\
1\times\nu & \mathcal{P}^0_{1|\nu}(X) & \\
1\times\nu & \mathcal{P}^\alpha_{1|\nu}(X) & \alpha = 1,\dots r \\
 & & \\
2\times\nu & \mathcal{P}^0_{2|\nu}(X) & \\
2\times\nu & \mathcal{P}^\alpha_{3|\nu}(X) & \alpha = 1,\dots r \\
 & & \\
3\times\nu & \mathcal{P}^{top}_{3|\nu}(X) & .
\end{array} \tag{4.24}$$

  such that they satisfy the relations

$$\begin{array}{ccc}
 & \mathcal{P}^0_{1|\nu} \cdot \mathcal{P}^0_{1|\nu} \;\sim\; 0 & \\
\mathcal{P}^0_{1|\nu} \cdot \mathcal{P}^\alpha_{1|\nu} \;\sim\; \mathcal{P}^\alpha_{2|\nu} & & \mathcal{P}^\alpha_{1|\nu} \cdot \mathcal{P}^\beta_{1|\nu} \;\sim\; g^{(K)}_{\alpha\beta}\, \mathcal{P}^0_{2|\nu} \\
 & \mathcal{P}^0_{1|\nu} \cdot \mathcal{P}^\alpha_{1|\nu} \cdot \mathcal{P}^\beta_{1|\nu} \;\sim\; g^{(K)}_{\alpha\beta}\, \mathcal{P}^{top}_{3|\nu} & .
\end{array} \tag{4.25}$$

  This condition guarantees that in the asymptotic region where the classical limit of the moduli space is attained, the geometry of $\widehat{ST}[r]$ does indeed converge to that of $ST[r]$. The fusion coefficients of the chiral ring displayed in eq. (4.25) coincide with the anomalous magnetic moments of the $ST[r]$ manifold in its asymptotic region.

An obvious approach to the construction of suitable dynamical Calabi–Yau threefolds for rank $r$ locally supersymmetric gauge theories is that of identifying these manifolds with the mirrors of Calabi–Yau threefolds with $h^{(1,1)} = r+1$:

$$\mathcal{M}_3[r] \;=\; \widetilde{M}_3\left(h^{1,1} = r+1\,;\; h^{2,1} = x\right) \tag{4.26}$$

Next one looks at the duality–monodromy groups and at the structure of their deformation ring to see whether the other requests are satisfied. This programme corresponds to a viable possibility if the class of manifolds with given $h^{(1,1)} = r + 1$ is known and small. Such a situation occurs, under additional reasonable assumptions, for low values of $r$, in particular for $r = 1$ and $r = 2$. Restricting one's attention to those threefolds that are described as the vanishing locus of a single polynomial constraint in weighted $WCP^4$, the class of $h^{(1,1)} = 2, 3$ threefolds is known [31] and displayed below

| | Hypersurface | $h^{1,1}$ | $h^{2,1}$ |
|---|---|---|---|
| #1 | $WCP^4(8;2,2,2,1,1)$ | 2 | 86 |
| #2 | $WCP^4(12;6,2,2,1,1)$ | 2 | 128 |
| #3 | $WCP^4(12;4,3,2,2,1)$ | 2 | 74 |
| #4 | $WCP^4(14;7,2,2,2,1)$ | 2 | 122 |
| #5 | $WCP^4(18;9,6,1,1,1)$ | 2 | 272 |
| #6 | $WCP^4(12;6,3,1,1,1)$ | 3 | 165 |
| #7 | $WCP^4(12;3,3,3,2,1)$ | 3 | 69 |
| #8 | $WCP^4(15;5,3,3,3,1)$ | 3 | 75 |
| #9 | $WCP^4(18;9,3,3,2,1)$ | 3 | 99 |
| #10 | $WCP^4(24;12,8,2,1,1)$ | 3 | 243 |

(4.27)

Hence, under these assumptions, for the gravitational coupling of an $r = 1$ gauge theory, (i.e. for the $G = SU(2)$ case) we have five possibilities distinguished by five different values of the second Hodge number $h^{(2,1)}$. Since this number counts the Kähler classes of the mirror manifold under consideration it has no relevance as long as we deal with locally supersymmetric pure gauge theories. So we are allowed to inquiry which of these manifolds satisfy the additional embedding criteria outlined above.

Consider, for instance the second model in table (4.27). Its mirror manifold with $h^{(1,1)} = 128$, $h^{(2,1)} = 2$ is described as the vanishing locus of the following weighted projective polynomial

$$\widetilde{W} = Z_1^{12} + Z_2^{12} + Z_3^6 + Z_4^6 + Z_5^2 - 12\psi Z_1 Z_2 Z_3 Z_4 Z_5 - 2\phi Z_1^6 Z_2^6 \tag{4.28}$$

This two moduli potential admits the $\Gamma_{\mathcal{W}} = \mathbb{Z}_{12}$ symmetry given by:

$$\begin{pmatrix} \psi \\ \phi \end{pmatrix} \rightarrow \begin{pmatrix} \alpha^{11} & 0 \\ 0 & \alpha^6 \end{pmatrix} \begin{pmatrix} \psi \\ \phi \end{pmatrix} \tag{4.29}$$

where $\alpha$ denotes a $12^{\text{th}}$ root of the unity. Clearly $\mathbb{Z}_{12}$ contains a subgroup $\mathbb{Z}_4$ acting as

$$\begin{pmatrix} \psi \\ \phi \end{pmatrix} \rightarrow \begin{pmatrix} \alpha'^3 & 0 \\ 0 & \alpha'^2 \end{pmatrix} \begin{pmatrix} \psi \\ \phi \end{pmatrix} \tag{4.30}$$

with $\alpha'^4 = 1$. This $\mathbb{Z}_4$ group should be the R-symmetry group of the rigid SU(2) theory which, therefore, should be embedded in the gravitational symplectic group $Sp(6,\mathbb{Z})$ with generators $A = (A_{12})^3$ where $A_{12}$ is the matrix generating the $\mathbb{Z}_{12}$ $\Gamma_{\mathcal{W}}$ group in $Sp(6,\mathbb{Z})$. Such a triple covering of the of the rigid theory R-symmetry inside the gravitational one (and quite possibly also of the monodromy group) appears to be the result of a triple covering (apart from exceptional points) of a dynamical Riemann[4] surface $\mathcal{M}_1[1]$ inside this particular $M_3[1]$. To see this it suffices to set $Z_3 = Z_4 = 0$, $Z_1^3 = X$, $Z_2^3 = Y$, $Z_5 = Z$ in eq. (4.28) and compare with eq. (2.34). What is only a plausible conjecture for model #2 can instead be proved for model #1 of table (4.27) thanks to the explicit results contained in [31]. Indeed the mirror manifold of $WCP^4(8;2,2,2,1,1)$ has been studied in detail in [31] and it is described as the vanishing locus of the following octic superpotential[5]:

$$\mathcal{W} = X_1^8 + X_2^8 + X_3^4 + X_4^4 + X_5^4 - 8\psi\, X_1X_2X_3X_4X_5 - 2\phi\, X_1^4X_2^4 \qquad (4.31)$$

Also this manifold embeds (apart for exceptional points) a multiple covering of the rigid theory elliptic surface $M_1[1]$, which, this time, is double rather than triple. For its realization it suffices to set, in eq. (4.31):

$$X_4 = X_5 = 0 \qquad X_1^2 = X \qquad X_2^2 = Y \qquad X_3^2 = Z \qquad (4.32)$$

The potential (4.31) has a $\Gamma_{\mathcal{W}} = \mathbb{Z}_8$ symmetry group whose action on the moduli $\psi, \phi$ is the following:

$$\mathcal{A} : \{\psi\,,\,\phi\} \longrightarrow \{\,\alpha\,\psi\,,\,-\phi\} \qquad \alpha^8 = 1 \qquad (4.33)$$

Clearly the transmutation of the rigid $\mathbb{Z}_4$ R-symmetry into $\mathbb{Z}_8$ is due to the double covering, just as in the other possible case $WCP^4(12;6,2,2,1,1)$ of gravitational coupling, its transmutation into $\mathbb{Z}_{12}$ was due to the triple covering. In the present case, however, using the results of [31], this statement can be verified explicitly. The integer symplectic matrix that represents the $\mathbb{Z}_8$ generator on the periods has been calculated in [31] and has the following form:

$$Sp(6,\mathbb{Z}) \ni \mathcal{A} = \begin{pmatrix} -1 & 0 & 1 & -2 & 2 & 0 \\ -2 & 1 & 0 & -2 & 4 & 4 \\ 0 & 1 & -1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & -1 \end{pmatrix} \qquad (4.34)$$

It is obtained by a change of basis which makes it integer symplectic from the matrix

[4]It is important to stress that we do not mean that such Riemann surface should be identified with the rigid theory solution, but as a mathematical explanation why the R–symmetry is $\mathbb{Z}_8$ rather than $\mathbb{Z}_4$. We expect that the more profound argument should be found in the microscopic original theory in terms of space–time instanton sums.

[5]Note that this example is connected through a conifold transition [16] to the Calabi–Yau manifold described by a quintic equation in $\mathbb{C}P_4$ ($h_{11} = 1$, $h_{21} = 101$).

given in [31]. Its second power

$$\mathcal{R}_4 = \mathcal{A}^2 = \begin{pmatrix} -3 & 1 & -2 & -2 & 0 & 4 \\ -2 & 1 & -2 & 0 & 4 & 4 \\ 0 & 0 & 1 & 0 & 4 & 0 \\ 0 & 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & -1 & 1 & -1 & 0 \\ -1 & 0 & 1 & -2 & 2 & 1 \end{pmatrix} \tag{4.35}$$

is the generator of the $\mathbb{Z}_4$ R–symmetry of the original theory. If we calculate its eigenvalues we find:

$$\text{eigenvalues of } \mathcal{R}_4 = \{-1, i, -i, -1, i, -i\} \tag{4.36}$$

As we see, in agreement with the properties of R–symmetry discussed in [20], (apart from an overall change of sign) there is a pair of complex conjugate eigenvalues $\pm i$ corresponding to the graviphoton and gravidilaton directions and a unit eigenvalue corresponding to the physical vector multiplet of $SU(2)$. Indeed going to the basis of the eigenvectors of $\mathcal{R}_2 = \mathcal{R}_4^2$ we find

$$\widetilde{\mathcal{R}}_2 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.37}$$

This is the matrix that realizes the $\mathbb{Z}_2$ R–symmetry in a Calabi–Visentini basis for the classical manifold $ST(1) = SU(1,1)/U(1) \times O(2,1)/O(2)$. Hence, as we see, the $\mathbb{Z}_2$ R–symmetry of the $SU(2)$ theory is indeed transplanted into the gravitationally coupled theory and can be reduced to the canonical form it takes as discrete subgroup of the $O(2)$ group in the corresponding classical moduli manifold, by means of a change of basis. This change of basis, however, is not symplectic and in the same basis the monodromy matrices are not symplectic integer valued. The quantum basis where both the R–symmetry and the monodromies are symplectic integers is determined via the Picard–Fuchs equations and gives for $\mathcal{R}_2$ the expression

$$\mathcal{R}_2 = \mathcal{A}^4 = \begin{pmatrix} 3 & -2 & 4 & 0 & 0 & -4 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 4 & 0 & 0 \\ 0 & 0 & -2 & 3 & 0 & 0 \\ 0 & 0 & 1 & -2 & -1 & 0 \\ 2 & -1 & 0 & 4 & 0 & -3 \end{pmatrix} \tag{4.38}$$

The matrix $\mathcal{R}_2$ realizes, in the gravitational coupled theory, the symmetry:

$$u \rightarrow -u \tag{4.39}$$

of the rigid theory discussed below in eq.(A.3).

Next we verify that the deformation ring has the correct structure. The Griffith residue mapping associates the Hodge filtration of the middle cohomology group:

$$\begin{aligned} &\mathcal{F}^0 \subset \mathcal{F}^1 \subset \mathcal{F}^2 \subset \mathcal{F}^3 \\ F^k &= H^{(3,0)} + H^{(2,1)} + \ldots + H^{(3-k,k)} \\ H^{(3)}_{DR} &= H^{(3,0)} + H^{(2,1)} + H^{(1,2)} + H^{(0,3)} \end{aligned} \tag{4.40}$$

with polynomials

$$\mathcal{P}^{\alpha}_{k|8}(X_1, X_2, X_3, X_4, X_5) \in \frac{\mathbb{C}[X]}{\partial \mathcal{W}^0} \tag{4.41}$$

of degrees 0, 8, 16, 24, according to the following pattern:

$$\begin{array}{lll} \text{cohom.} & \text{deg} & \text{polynom.} \\ & & \\ \mathcal{F}^0 & 0 & \mathcal{P}_{0|8} \; = \; 1 \\ \mathcal{F}^1 & 8 & \mathcal{P}^{\alpha}_{1|8} \\ \mathcal{F}^2 & 16 & \mathcal{P}^{\alpha}_{2|8} \\ \mathcal{F}^3 & 24 & \mathcal{P}_{3|8} \; = \; X_1^6\, X_2^6\, X_3^6\, X_4^2\, X_5^2 \stackrel{\mathrm{def}}{=} \mathcal{P}^{top} \; . \end{array} \tag{4.42}$$

where the deformations should satisfy the following algebra:

$$\begin{array}{l} \mathcal{P}^0_{1|8} \cdot \mathcal{P}^0_{1|8} \sim 0 \\ \mathcal{P}^0_{1|8} \cdot \mathcal{P}^0_{2|8} \sim \mathcal{P}^{top} \\ \mathcal{P}^1_{1|8} \cdot \mathcal{P}^1_{2|8} \sim \mathcal{P}^{top} \\ \mathcal{P}^0_{1|8} \cdot \mathcal{P}^1_{1|8} \cdot \mathcal{P}^1_{1|8} \sim \mathcal{P}^{top} \; . \end{array} \tag{4.43}$$

The deformation $\mathcal{P}^1_{1|8}$ corresponds to the matter multiplet while the deformation $\mathcal{P}^0_{1|8}$ corresponds to the additional dilaton–axion multiplet and the algebra (4.43) guarantees that the classical limit of the moduli–space (obtained for large complex structures $\psi_I \to \infty$ ) is given by the coset manifold

$$ST(1) \; = \; \frac{SU(1,1)}{U(1)} \otimes \frac{O(2,1)}{O(2)} \tag{4.44}$$

as requested by tree–level string theory. From the explicit identification of the deformation polynomials:

$$\begin{array}{rcl} \mathcal{P}_{0|8} & = & 1 \\ \mathcal{P}^0_{1|8} & = & X_1^4\, X_2^4 \\ \mathcal{P}^1_{1|8} & = & X_1\, X_2\, X_3\, X_4\, X_5 \\ \mathcal{P}^0_{2|8} & = & X_1^2\, X_2^2\, X_3^2\, X_4^2\, X_5^2 \\ \mathcal{P}^1_{2|8} & = & X_1^3\, X_2^3\, X_3\, X_4\, X_5 \\ \mathcal{P}_{3|8} & = & X_1^4\, X_2^4\, X_3^2\, X_4^2\, X_5^2 \end{array} \tag{4.45}$$

we immediately obtain that (4.43) is a viable candidate for the description of the moduli space of a locally supersymmetric N=2 theory.

The list of Calabi–Yau threefolds obtained in [18] as examples of dual heterotic/type-II models where the matching of vector and hypermultiplet numbers is realized, overlaps with the models selected by our embedding criterion, as we have already emphasized in the introduction. Moreover the check of the large $S$–limit of the anomalous magnetic couplings $W_{ijk}$ made by these authors agrees with our previous discussion.

## 5 Central charges and BPS states from three-form cohomology

The purpose of this section is to describe BPS states and central-charge formulas directly from the three-form cohomology of Calabi–Yau threefolds. Strictly speaking it is only in type II B models that the vector multiplets are associated with three–forms since in type II A models they are rather associated with the two–forms. Yet, by using mirror symmetry we can always interchange the moduli–space of Kähler structures of one Calabi–Yau manifold (two–form case) with the moduli–space of complex structures (three–form case) of its mirror. Hence for definiteness we always refer to the type II B case and to the three–form cohomology.

It then will appear evident that, under the assumption that Calabi–Yau classical moduli space of three-form cohomology (in type-II theories) describes the quantum moduli space of vector multiplets in heterotic strings (second-quantized mirror symmetry [17]), the Calabi–Yau lattice of saturated states will correspond to the lattice of monopoles and dyons of the heterotic quantum theory. In particular conifold points on Calabi–Yau, as shown in [15, 16] will correspond to monopole point (non-perturbative) singularities in N=2 heterotic strings.

In order to carry out this program we will make use of the cohomology decomposition of the self-dual five form $\mathcal{F}$ (which exists in type-II strings), adopting the results of [45], and the recent analysis of conifold points, corresponding to vanishing three-cycles, as points at which some hypermultiplets, carrying Ramond-Ramond magnetic and electric charges, become massless.

The five-form of type-IIB theory is selfdual: $\mathcal{F} = {}^*\mathcal{F}$, so that it satisfies both Bianchi identities and equations of motions:

$$\begin{aligned} \mathrm{d}\mathcal{F} &= 0 \\ \mathrm{d}^*\mathcal{F} &= 0. \end{aligned} \tag{5.1}$$

When the ten-dimensional space-time is compactified to $M_4 \times \mathcal{M}_3$, where $M_4$ is four-dimensional space-time and $\mathcal{M}_3$ is a Calabi–Yau threefold, the Poincaré dual of the five-form $\mathcal{F}$, which is a five-cycle, can be decomposed along a basis $S_2^i \times C_3^\Lambda$, where $S_2^i$ are two-cycles of $M_4$ ($i = 1, \ldots, b^2(M_4)$) and $C_3^\Lambda$ are three-cycles of $\mathcal{M}_3$. Choosing a symplectic basis $(A_\Lambda, B^\Lambda)$, $(\Lambda = 0, 1, \ldots h^{2,1})$ for the three-cycles and introducing the dual basis $(\alpha_\Lambda, \beta^\Lambda)$ of harmonic three-forms we can write:

$$\mathcal{F} = \mathcal{F}^\Lambda \alpha_\Lambda + \mathcal{G}_\Lambda \beta^\Lambda \tag{5.2}$$

where

$$\mathcal{G}_\Lambda = \mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma} \widetilde{\mathcal{F}}^\Sigma + \mathrm{Re}\,\mathcal{N}_{\Lambda\Sigma} \mathcal{F}^\Sigma. \tag{5.3}$$

$\mathcal{F}^\Lambda = \mathcal{F}^\Lambda_{\mu\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$ and $\mathcal{G}^\Lambda = \mathcal{G}^\Lambda_{\mu\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$ are respectively the electric and magnetic field strengths of the gauge vectors emerging from $\mathcal{F}$ in the compactification. When monopole and dyon states are present the topology of space-time is modified, as it is well-known. There are non-contractible spheres $S_2^i$ that surround the $\mathrm{i}^{\mathrm{th}}$ singularity of the gauge field corresponding to each monopole (or dyon) state and the integrals of $\mathcal{F}^\Lambda$ or $\mathcal{G}^\Lambda$ on such spheres yield the value of electric ($n^e_\Lambda$) or magnetic ($n^\Lambda_m$) charges for the state wrapped by

$S_2$. For any such sphere we can write:

$$\begin{aligned} \int_{S_2\times A_\Lambda} \mathcal{F} = \int_{S_2} \mathcal{G}_\Lambda &= n^e_\Lambda \\ \int_{S_2\times B^\Lambda} \mathcal{F} = \int_{S_2} \mathcal{F}^\Lambda &= n^\Lambda_m \end{aligned} \tag{5.4}$$

These are the integral charges, with respect to the integral cohomology basis for $H^3$.

However the physical charges, related to the "central charge" and the $h^{2,1}$ complex charges (electric and magnetic) are those associated with the graviphoton $T^-_{\mu\nu}$ and with the others vectors in the theory, as they appear in the transformation laws of the gauginos[6]:

$$\delta\lambda^{\Lambda A} = \widehat{\mathcal{F}}^{\Lambda -}_{\mu\nu}\gamma^{\mu\nu}\epsilon^{AB}\varepsilon_B \qquad\qquad (A,B=1,2). \tag{5.5}$$

The graviphoton $T^-_{\mu\nu}$ appears in the gravitino transformation:

$$\delta\psi_{\mu A} = T^-_{\rho\sigma}\gamma^{\rho\sigma}\gamma_\mu\epsilon_{AB}\varepsilon^B + \ldots; \tag{5.6}$$

Note that

$$T^-_{\mu\nu} = T_\Lambda \mathcal{F}^{-\Lambda}_{\mu\nu} = (F_\Lambda\mathcal{F}^{-\Lambda}_{\mu\nu} - X^\Lambda \mathcal{G}^-_{\Lambda\mu\nu})\mathrm{e}^{\frac{\mathcal{K}}{2}} \tag{5.7}$$

where we used the graviphoton projector

$$T_\Lambda = (F_\Lambda - \overline{\mathcal{N}}_{\Lambda\Sigma}X^\Sigma)\,\mathrm{e}^{\frac{\mathcal{K}}{2}} \tag{5.8}$$

and $\widehat{\mathcal{F}}^{-\Lambda}T_\Lambda = 0$, i.e.

$$\widehat{\mathcal{F}}^{-\Lambda} = \mathcal{F}^{-\Lambda} - \mathrm{i}T^-\overline{X}^\Lambda\mathrm{e}^{\frac{\mathcal{K}}{2}} \qquad\qquad \left(T_\Lambda\overline{X}^\Lambda\mathrm{e}^{\frac{\mathcal{K}}{2}} = -\mathrm{i}\right) \tag{5.9}$$

By the definition of $T^-_{\mu\nu}$ and the definition of the holomorphic three-form on $\mathcal{M}_3$

$$\Omega(\phi) = X^\Lambda(\phi)\alpha_\Lambda - F_\Lambda(\phi)\beta^\Lambda \tag{5.10}$$

(we work in arbitrary coordinates $\phi$, so that $X^\Lambda = X^\Lambda(\phi)$) it follows (using also $\overline{F}_\Lambda\mathcal{F}^{-\Lambda} - \overline{X}^\Lambda\mathcal{G}^-_\Lambda = 0$) that

$$\int_{S_2\times\mathcal{M}_3}\mathcal{F}\wedge\Omega = \int_{S_2}\mathrm{e}^{-\frac{\mathcal{K}}{2}}T^- = (X^\Lambda n^e_\Lambda - F_\Lambda n^\Lambda_m) \equiv Z(\phi) \tag{5.11}$$

which is precisely the (holomorphic) central charge.

The other charges, for the $h^{2,1}$ electric and magnetic field strengths, are

$$\int_{S_2\times C_3}\mathcal{F}\wedge D_i\Omega = \int_{S_2}\mathrm{Im}\mathcal{N}_{\Lambda\Sigma}\widehat{\mathcal{F}}^\Sigma D_iX^\Lambda = D_iZ \equiv q_i(\phi) \tag{5.12}$$

In the above discussion we used the decomposition of the five-form $\mathcal{F}$[45]:

$$\mathcal{F} = \mathcal{F}^\Lambda\alpha_\Lambda - \mathcal{G}_\Lambda\beta^\Lambda = \mathrm{e}^{\frac{\mathcal{K}}{2}}(T^-\overline{\Omega} + T^+\Omega + \mathcal{F}^{-i}D_i\Omega + \mathcal{F}^{+i^*}D_{i^*}\overline{\Omega}) \tag{5.13}$$

[6]We write here the transformations for left-handed fermions; by $\mathcal{F}^-$ we intend the antiselfdual part of $\mathcal{F}$; we use the notation in which the index $i = 1,\ldots h^{2,1}$ carried by the $h^{2,1}$ gauginos $\lambda^{iA}$ is extended to the range $\Lambda = 0,1,\ldots,h^{2,1}$ by writing $\lambda^{\Lambda A} = f^\Lambda_i\lambda^{iA}$, $i = 1,\ldots,h^{2,1}$ where $f^\Lambda_i = D_i(\mathrm{e}^{\frac{\mathcal{K}}{2}}X^\Lambda)$ (for notations see [8]).

recalling also the relation between $\widehat{\mathcal{F}}^{-\Sigma}$ and $\mathcal{F}^{-\Sigma}$ given by eq. (5.9). Now it is easy to prove that

$$(\mathcal{N}_{\Lambda\Delta} - \overline{\mathcal{N}}_{\Lambda\Delta})(\delta^{\Lambda}_{\Sigma} - \mathrm{i}T_{\Sigma}\overline{X}^{\Lambda})\mathcal{F}^{-\Sigma}D_{i^*}\overline{X}^{\Delta} \tag{5.14}$$

is the correct ansatz for the additional field strengths, orthogonal to the graviphoton, since, due to

$$(\mathcal{N}_{\Lambda\Delta} - \overline{\mathcal{N}}_{\Lambda\Delta})\overline{X}^{\Lambda}D_{i^*}\overline{X}^{\Delta} = 0 \tag{5.15}$$

the components of (5.14) along $T^-_{\mu\nu}$ is zero. We have then

$$\mathcal{F}^{-i} = \mathrm{e}^{\mathcal{K}/2}g^{ij^*}D_{j^*}\overline{X}^{\Lambda}(\mathcal{N}_{\Lambda\Delta} - \overline{\mathcal{N}}_{\Lambda\Delta})\widehat{\mathcal{F}}^{-\Delta} \tag{5.16}$$

Therefore the difference between the quantized electric and magnetic charges $n^e_{\Lambda}, n^{\Lambda}_m$ and the moduli dependent charges $(Z, q_i)$ is that the first are fluxes on the real de–Rham cohomology group $H^3_{DR}$ while in the second case the fluxes are on complex Hodge filtration of this latter: $H^3_{DR} = H^{(3,0)} + H^{(2,1)} + H^{(1,2)} + H^{(0,3)}$. The central charge lies in $H^{(3,0)}$, while the other charges lie in $H^{(2,1)}$.

Note that from:

$$\int_{S_2} T^- \; = \; Z \tag{5.17}$$

it follows that $\mathrm{Re}\, Z, \mathrm{Im}\, Z$ are the electric and magnetic charge of the graviphoton field. The conifold points are *poles* in the Yukawa couplings (Consequence of the Picard–Fuchs equations) and correspond to the logarithmic singularities around a non–perturbative monopole point of the prepotential in heterotic string considered in the previous section. Note also that the Yukawa couplings have no monodromies since they are tensors under duality rotations. Actually the tensors $W_{ijk} = \partial_i\partial_j\partial_k\mathcal{F}$, that are commonly named Yukawa couplings because of their physical interpretation when the Calabi–Yau manifold is used to compactify the heterotic string to an N=1 theory, in type-II N=2 compactification have the physical interpretation of anomalous magnetic moments of the gauginos $\lambda^i$.

Using the holomorphic expression of the central charge (5.11) we may write a general formula for the behaviour of the prepotential $F(\phi)$, near the singular locus $Z(\phi) = 0$, [11], i.e.

$$F(\phi) \sim \frac{\mathrm{i}c}{\pi} Z^2(\phi) \log Z(\phi), \tag{5.18}$$

where $c$ is a constant.

The singularities of the prepotential $\mathcal{F}$, that are interpreted as monopole point singularities, have their origin in the mass singularities of these magnetic moments, that in turn are a consequence of the Picard–Fuchs equations. When hypermultiplets are contained in the theory, they can contribute an anomalous magnetic moment term to $W_{ijk}$.

In the example of the quintic hypersurface [46] in $\mathbb{CP}^4$, for which $h^{2,1} = 1$,[15]

$$\partial^3\mathcal{F} \overset{z\sim 0}{\approx} \frac{1}{Z} \qquad \left(\text{in general } F_{ijk} \sim \frac{n_i\, n_j\, n_k}{n^{\Lambda}X_{\Lambda}}\right) \tag{5.19}$$

which has the physical meaning of a hypermultiplet of mass $Z$ contributing to the anomalous magnetic moment of the gaugino which is partner of the unique R–R vector field other than the graviphoton.

Now, on the heterotic side, the meaning of the symplectic section $\Omega = \Big( X^\Lambda \, , \, F_\Sigma \Big)$ is precisely the same as here, in the sense that $\Omega$ is related to the gauge coupling matrix $\mathcal{N}_{\Lambda\Sigma}$ of the heterotic vectors by the same formulae. However the explicit expression of $\Omega$ looks pretty different in this case. For example on $K_3 \times T_2$ the dependence on the vector multiplet moduli is:

$$F_\Lambda \; = \mathcal{S}\, X^\Lambda \qquad (\mathcal{S} \text{ is the heterotic dilaton}) \tag{5.20}$$

at the classical level and:

$$F_\Lambda \; \sim \mathcal{S}\, X_\Lambda \, + \, \frac{\mathrm{i}\beta}{\pi} n_\Lambda \, (X \cdot n) \log(X \cdot n)/X_0 \tag{5.21}$$

after perturbative quantum correction, where $\beta$ is a model-dependent constant proportional to the field theory $\beta$-function and $X \cdot n = 0$ is a perturbative singularity.

# 6 Final Remarks

In this paper, following a previous conjecture [7, 8], we have provided a search for Calabi–Yau manifolds embedding the R-symmetry and the quantum monodromy of the Riemann surfaces encompassing the non-perturbative dynamics of rigid Yang–Mills theories. In the framework of string theory this search finds a natural setting in dual pairing of N=2 superstring theories [17, 18], i.e.heterotic strings on $K3 \times T_2$ and type-II strings on Calabi–Yau threefolds, where the number of neutral massless hypermultiplets $N_H$ of heterotic string and the vector multiplets in the abelian phase $N_V$ match the Hodge numbers of the dual pair according to the formula (for type II B, for instance):

$$N_V = h^{(2,1)} \quad , \quad N_H = h^{(1,1)} + 1 \tag{6.1}$$

A recent construction [17] of a dual pair, based on the analysis of the soliton string worldsheet (in the context of N=2 orbifolds of dual N=4 compactifications of type-II and heterotic strings) and the classifications of many other candidate pairs [18] including stringy analogue of Seiberg–Witten monopole points, gives a further strong evidence that dynamical Calabi–Yau manifolds, considered in this paper purely from the point of view of extending the quantum monodromy of the rigid theories, are the natural candidates for describing the non-perturbative regime of strongly coupled N=2 superstring theories, in four dimensions.

**Acknowledgements**

We would like to thank I. Antoniadis, P. Candelas, R. Catenacci, R. Finkelstein, C. Gomez, J. Harvey, A. Klemm, W. Lerche, K. S. Narain, C. Reina, A. Strominger, T. Taylor, C. Vafa, R. Varadarajan for enlightening discussions.

# Appendix. Some further consideration on the $r = 1$ case

Let us rewrite the potential for the $SU(2)$ dynamical Riemann surface as follows:

$$0 = \mathcal{W}(X,Y,Z;u) = -Z^2 + \frac{1}{4}\left(X^4+Y^4\right) + \frac{u}{2}X^2Y^2 \tag{A.1}$$

One realizes that this potential has a $\Gamma_{\mathcal{W}} = D_3$ symmetry group [47, 19] defined by the following generators and relations

$$\widehat{A}^2 = \mathbb{1} \quad , \quad C^3 = \mathbb{1} \quad , \quad (C\widehat{A})^2 = \mathbb{1} \tag{A.2}$$

with the following action on the homogeneous coordinates and the modulus $u$. (We forget about the action on the $Z$ coordinate, which is immaterial, contributing with a quadratic term to the polynomial).

$$\begin{array}{lllll} \widehat{A}: & M_{\widehat{A}} = \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix} ; & \phi_{\widehat{A}}(u) = -u \, ; & f_{\widehat{A}}(u) = 1 \\ C: & M_C = \frac{1}{\sqrt{2}}\begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix} ; & \phi_C(u) = \frac{u-3}{u+1} \, ; & f_C(u) = \frac{1+u}{2} \end{array} \tag{A.3}$$

Eq. (A.3) is given in reference [19], where the authors posed themselves the question why only the $\mathbb{Z}_2$ cyclic group generated by $\widehat{A}$ is actually realized as an isometry group of the rigid special Kählerian metric. The answer is contained in the general discussion of section 3:

$$\mathbb{Z}_2 = \Gamma^{rig}_{\mathcal{W}} \subset \Gamma_{\mathcal{W}} = D_3 \tag{A.4}$$

Namely it is only $\mathbb{Z}_2$ that preserves the potential with a unit rescaling factor. The natural question at this point is what is the relation of this $\mathbb{Z}_2 \subset D_3$ with the dihedral $D_4$ symmetry expected for $r = 1$. The answer is simple: the $\mathbb{Z}_4$ action in $D_4$ becomes a $\mathbb{Z}_2$ action on the $u$ variable, $u \to \alpha^2 u$, $(\alpha^4 = 1)$.

**The rigid special Kähler metric for $SU(2)$**

As it has been shown in [7] the Picard–Fuchs equation associated, in the $SU(2)$ case, to the symplectic section:

$$\Omega_u = \partial_u \Omega = \partial_u \begin{pmatrix} Y \\ \frac{\partial \mathcal{F}}{\partial Y} \end{pmatrix} = \begin{pmatrix} \int_A \omega \\ \int_B \omega \end{pmatrix} \tag{A.5}$$

is

$$(\partial_u \mathbb{1} - A_u)\, V = 0 \, , \tag{A.6}$$

where $V$ is defined in (2.42), and the $2\times 2$ matrix connection $A_u$ is given by:

$$A_u = \begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{-1/2}{1-u^2} & \frac{2u}{1-u^2} \end{pmatrix} \tag{A.7}$$

with solutions

$$\begin{cases} \partial_u Y \equiv f^{(1)}(u) = F(\frac{1}{2},\frac{1}{2},1;\frac{1+u}{2}) \\ \partial_u \frac{\partial \mathcal{F}}{\partial Y} \equiv f^{(2)}(u) = \mathrm{i}F(\frac{1}{2},\frac{1}{2},1;\frac{1-u}{2}). \end{cases} \tag{A.8}$$

The duality group of electric–magnetic rotations is, in this case [1]:

$$\Gamma_D \stackrel{def}{=} G_\theta \subset PSL(2,\mathbb{Z}) \tag{A.9}$$

defined as the subgroup of the elliptic modular group $\Gamma = PSL(2,\mathbb{Z})$ generated by the two matrices acting on the section $\Omega_u$:

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad T_1 = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \tag{A.10}$$

where $S$ is the $R$-symmetry generator and $T_1$ is the monodromy matrix associated to the singular point $u = 1$ of the Picard–Fuchs system (A.6). Relying on eq.(2.7), we easily derive the relation between isometries $u_i \to \phi_i(u)$ of the rigid special Kähler metric and symplectic transformations [20]: there exist $M_\phi \in Sp(2r,\mathbb{R})$ such that

$$\begin{aligned} \Omega\left(\phi(u)\right) &= e^{i\theta_\phi} M_\phi \, \Omega\left(u\right) \\ \Omega_{u_i}\left(\phi(u)\right) \tfrac{\partial \phi^i}{\partial u_j} &= e^{i\theta_\phi} M_\phi \, \Omega_{u_j}\left(u\right) \end{aligned} \tag{A.11}$$

In our case we have that the $R$-symmetry isometry $u \to -u$ induces the transformation

$$-\,\Omega_u(-u) = \mathrm{i} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} f^{(1)}(u) \\ f^{(2)}(u) \end{pmatrix} = \mathrm{i}\, S\, \Omega_u(u) \tag{A.12}$$

while the monodromy transformation around $u = 1$ gives

$$\Omega_u\left((u-1)\, e^{2\pi i}\right) = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} f^{(1)}(u-1) \\ f^{(2)}(u-1) \end{pmatrix} = T_2\, \Omega_u(u-1)\ . \tag{A.13}$$

Having recalled the explicit form of the isometry–duality group let us now study the structure of the rigid special metric. To this effect let us introduce the ratio of the two solutions to eq. (A.6),

$$\overline{\mathcal{N}}(u) = \frac{f^{(2)}(u)}{f^{(1)}(u)}. \tag{A.14}$$

Such a ratio is identified with the matrix $\overline{\mathcal{N}}$ appearing in the vector field kinetic terms:

$$\mathcal{L}^{vector}_{kin} = \frac{1}{2\mathrm{i}} [\overline{\mathcal{N}}(u)\, F^-_{\mu\nu}\, F^-_{\mu\nu} - \mathcal{N}(\overline{u})\, F^+_{\mu\nu}\, F^+_{\mu\nu}] \tag{A.15}$$

If we look at the *inverse function* $u(\overline{\mathcal{N}})$, this latter is a modular form of the group $\Gamma(2)$ that has the following behaviour:

$$\begin{array}{ccc} \forall\, \gamma \in \Gamma(2) & u\left(\gamma \cdot \overline{\mathcal{N}}\right) = & u\left(\overline{\mathcal{N}}\right) \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \Gamma/\Gamma(2) = D_3 & u\left(-\frac{1}{\overline{\mathcal{N}}}\right) = & -\,u\left(\overline{\mathcal{N}}\right) \\ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma/\Gamma(2) = D_3 & u\left(\overline{\mathcal{N}}+1\right) = & \frac{u(\overline{\mathcal{N}})-3}{u(\overline{\mathcal{N}})+1} \end{array} \tag{A.16}$$

Actually we have:

$$
\begin{aligned}
u\left(\overline{\mathcal{N}}\right) &= 1 - 2\,\kappa^2\left(\overline{\mathcal{N}}\right) \quad ; \quad \kappa^2\left(\overline{\mathcal{N}}\right) = \left[\frac{\theta_2(0|\overline{\mathcal{N}})}{\theta_3(0|\overline{\mathcal{N}})}\right]^4 \\
f^1\left(u(\overline{\mathcal{N}})\right) &= F\left(\frac{1}{2},\frac{1}{2},1;\frac{1+u(\overline{\mathcal{N}})}{2}\right) = \left[\theta_3(0|\overline{\mathcal{N}})\right]^2
\end{aligned}
\tag{A.17}
$$

where $\kappa^2(\tau)$ is the elliptic modulus and $\theta_i(z|\tau) \quad (i = 1,\dots,4)$ are the elliptic $\theta$–functions [48]. Recalling eq.(2.8) we can now write the explicit form of the rigid special Kähler metric in the variable $u$:

$$
ds^2 = g_{u\overline{u}}\,|du|^2 \;; \qquad g_{u\overline{u}} = 2\,\mathrm{Im}\,\overline{\mathcal{N}}(u)\,|f^{(1)}(u)|^2 \tag{A.18}
$$

Calculating the Levi–Civita connection and Riemann tensor of this metric we obtain:

$$
\begin{aligned}
\Gamma^u_{uu} &= -\,g^{u\overline{u}}\,\partial_u\,g_{u\overline{u}} = -\frac{1}{2\mathrm{i}}\,\frac{\partial\overline{\mathcal{N}}/\partial u}{\mathrm{Im}\,\overline{\mathcal{N}}(u)} - \partial_u \log f^{(1)}(u) \\
R^u_{u\overline{u}u} &= \partial_{\overline{u}}\,\Gamma^u_{uu} = \frac{1}{4}\,\frac{1}{(\mathrm{Im}\,\overline{\mathcal{N}}(u))^2}\,|\partial\overline{\mathcal{N}}/\partial u|^2 \\
R_{\overline{u}u\overline{u}u} &= g_{u\overline{u}}R^u_{u\overline{u}u} = \frac{1}{2}\,\frac{1}{\mathrm{Im}\,\overline{\mathcal{N}}(u)}\,|\partial\overline{\mathcal{N}}/\partial u|^2\,|f^{(1)}(u)|^2
\end{aligned}
\tag{A.19}
$$

so that we can verify that the above metric is indeed *rigid special Kählerian*, namely that it satisfies the constraint:

$$
R_{\overline{u}u\overline{u}u} - C_{uuu}\,\overline{C}_{\overline{u}\overline{u}\overline{u}}\,g^{u\overline{u}} = 0 \tag{A.20}
$$

by calculating the Yukawa coupling or anomalous magnetic moment tensor:

$$
C_{uuu} = \partial_u\overline{\mathcal{N}}\,\left(f^{(1)}(u)\right)^2 \tag{A.21}
$$

As one can notice from its explicit form (A.18), the Kähler metric of the rigid N=2 gauge theory of rank $r = 1$ is not the Poincaré metric in the variable $\overline{\mathcal{N}}$, as one might naively expect from the fact that $\overline{\mathcal{N}} = \tau$ is the standard modulus of a torus and that $G_\theta \subset PSL(2,\mathbb{Z})$ linear fractional transformations are isometries. Indeed using (A.18) and (A.17) we can write:

$$
ds^2 = 8\,\mathrm{Im}\,\overline{\mathcal{N}}\,|\theta_3(0|\overline{\mathcal{N}})|^4\,|\partial_{\overline{\mathcal{N}}}\,\kappa^2(\overline{\mathcal{N}})|^2|d\overline{\mathcal{N}}\,|^2 \tag{A.22}
$$

that is to be contrasted with the expression for the Poincaré metric:

$$
ds^2 = g^{Poin}_{\mathcal{N}\overline{\mathcal{N}}}\,|d\overline{\mathcal{N}}\,|^2 = \frac{1}{4}\,\frac{1}{(\mathrm{Im}\,\overline{\mathcal{N}})^2}\,|d\overline{\mathcal{N}}\,|^2 \tag{A.23}
$$

From eq.(A.19) however it is amusing to note that the Ricci form of the rigid metric is precisely the Poincaré metric.

$$
R^{Ricci}{}_{\mathcal{N}\overline{\mathcal{N}}} = g^{Poin}{}_{\mathcal{N}\overline{\mathcal{N}}} \tag{A.24}
$$

This is a consequence of the general equation (2.13) in the case of one modulus where the period matrix $\mathcal{N}$ can be used as a parameter.

**The rigid special coordinates**

Having recalled the solution of the Picard–Fuchs equation, the relation between the special coordinate of rigid special geometry and the invariant variable $u$ is obtained by means of a simple integration:

$$Y(u) \;=\; \int_{u_0}^{u} dt f^{(1)}(t) \;=\; (1+u)\, F\left(\frac{1}{2},\frac{1}{2},2;\frac{1+u}{2}\right) - (1+u_0)F\left(\frac{1}{2},\frac{1}{2},2;\frac{1+u_0}{2}\right) \quad \text{(A.25)}$$

In the special coordinate basis the anomalous magnetic moment tensor is given by:

$$C_{YYY} \;=\; C_{uuu}\left(\frac{\partial u}{\partial Y}\right)^3 \;=\; -\frac{\mathrm{i}}{\pi}\frac{1}{1-u^2}\left(\frac{\partial u}{\partial Y}\right)^3 \quad \text{(A.26)}$$

The second of equations (A.26) follows from the comparison between equation (A.21) and the Picard–Fuchs equation (A.6) satisfied by the periods that yields:

$$C_{uuu} \;=\; -\frac{\mathrm{i}}{\pi}\,\frac{1}{1-u^2} \quad \text{(A.27)}$$

In the large $u$ limit the asymptotic behaviour of the special coordinate is:

$$Y(u) \approx 2\sqrt{u} + \ldots \qquad \text{for } u \to \infty \quad \text{(A.28)}$$

so that we get:

$$C_{YYY}(u) \;=\; \frac{\partial^3 \mathcal{F}}{\partial Y^3}(u) \;\approx\; \frac{\mathrm{i}}{\pi}\, u^{-1/2} + \ldots \qquad \text{for } u \to \infty \quad \text{(A.29)}$$

and by triple integration one obtains the asymptotic behaviour of the prepotential $\mathcal{F}(Y)$ of rigid special geometry:

$$\mathcal{F}(Y) \;\approx\; \mathrm{const}\, Y^2 \log Y^2 \; + \ldots \qquad \text{for } Y \to \infty \quad \text{(A.30)}$$

Formula (A.30) contains the leading classical form of $\mathcal{F}(Y)$ plus the first perturbative correction calculated with standard techniques of quantum field–theory. Eq.(A.30) was the starting point of the analysis of Seiberg and Witten who from the perturbative singularity structure inferred the monodromy group and then conjectured the dynamical Riemann surface. The same procedure has been followed to conjecture the dynamical Riemann surfaces of the higher rank gauge theories. The nonperturbative solution is given by

$$\mathcal{F}(Y) \;=\; \frac{i}{2\pi} Y^2 \log\frac{Y^2}{\Lambda^2} + Y^2 \sum_{n=1}^{\infty} C_n \left(\frac{\Lambda^2}{Y^2}\right)^{2n} \quad \text{(A.31)}$$

The infinite series in (A.31) corresponds to the sum over instanton corrections of all instanton–number.

The important thing to note is that the special coordinates $Y^\alpha(u)$ of rigid special geometry approach for large values of $u$ the Calabi–Visentini coordinates of the manifold $O(2,n)\ /O(2)\times\ O(n)$ discussed in section 4. As stressed there, the $Y^\alpha$ are not special coordinates for local special geometry.

## References


[1] N. Seiberg and E. Witten , Nucl. Phys. B426 (1994) 19.

[2] N. Seiberg and E. Witten, Nucl. Phys. B431 (1994) 484.

[3] B. de Wit, P.G. Lauwers, R. Philippe, Su S.-Q. and A. Van Proeyen, Phys. Lett. 134B (1984) 37; S. J. Gates, Nucl. Phys. B238 (1984) 349; B. de Wit and A. Van Proeyen, Nucl. Phys. B245 (1984) 89.

[4] A. Klemm, W. Lerche, S. Theisen and S. Yankielovicz, Phys. Lett. B344 (1995) 169; "On the Monodromies of N=2 Supersymmetric Yang–Mills Theory", Proceedings of the Workshop on Physics from the Planck Scale to Electromagnetic Scale, Warsaw 1994 and of the 28th International Symposium on Particle Theory, Wendisch-Rietz, preprint CERN-TH.7538/94, hep-th/9412158.

[5] P. Argyres and A. Faraggi, Phys. Rev. Lett. 73 (1995) 3931.

[6] P. di Vecchia, R. Musto, F. Nicodemi and R. Pettorino, Nucl. Phys. B252 (1985) 635; N. Seiberg, Phys. Lett. B206 (1988) 75.

[7] A. Ceresole, R. D'Auria and S. Ferrara, Phys. Lett. 339B (1994) 71.

[8] A. Ceresole, R. D'Auria, S. Ferrara and A. Van Proeyen, "Duality Transformations in Supersymmetric Yang–Mills Theories coupled to Supergravity", preprint CERN-TH. 7547/94, hep-th 9502072 (1995), Nucl. Phys. B, to be published; "On Electromagnetic Duality in Locally Supersymmetric N=2 Yang–Mills Theory", preprint CERN-TH.08/94, hep-th/9412200, Proceedings of the Workshop on Physics from the Planck Scale to Electromagnetic Scale, Warsaw 1994.

[9] C. M. Hull and P.K. Townsend, hep-th 9410167, Nucl. Phys. B438 (1995) 109.

[10] E. Witten, "String Theory Dynamics in Various Dimensions", IASSNS-HEP-95-18, hep-th 9503124.

[11] I. Antoniadis, S. Ferrara, E. Gava, K. S. Narain and T. R. Taylor, "Perturbative Prepotential and Monodromies in N=2 Heterotic Strings", hep-th 9504034 (1995).

[12] B. de Wit, V. Kaplunovsky, J. Louis and D. Lüst, "Perturbative Couplings of Vector Multiplets in N=2 Heterotic String Vacua", preprint hep-th 9504006.

[13] C. M. Hull and P.K. Townsend, "Enhanced Gauge Symmetries in Superstring Theories", QMW-95-10, hep-th 9505073.

[14] J. A. Harvey and A. Strominger, "The Heterotic String is a Soliton", preprint hep-th/9504047.

[15] A. Strominger, "Massless black holes and conifolds in string theories", hep-th 9504090;

[16] B. Greene, D. Morrison and A. Strominger, "Black hole condensation and the unification of string vacua", hep-th 9504145.

[17] S. Ferrara, J. A. Harvey, A. Strominger and C. Vafa, "Second-quantized Mirror Symmetry", hep-th 9505162.

[18] S. Kachru and C. Vafa, "Exact Results for N=2 Compactifications of Heterotic Strings", HUTP-95/A016, hep-th 9505105.

[19] C. Gomez and E. Lopez,"From Quantum Monodromy to Duality", preprint hep-th 9505135.

[20] M. Billó, R. D'Auria, S. Ferrara, P. Frè, P. Soriani and A. Van Proeyen, "R-symmetry and the topological twist of N=2 effective supergravities of heterotic strings", preprint hep-th 9505123.

[21] A. Ceresole, R. D'Auria and T. Regge, Nucl. Phys. B414 (1994) 517.

[22] A. Klemm, W. Lerche and S. Theisen, "Nonperturbative Effective Actions of N=2 Supersymmetric Gauge Theories", preprint CERN-TH/95-104, LMU-TPW 95-7, hep-th 9505150.

[23] S. Cecotti, S. Ferrara, L. Girardello, Int. Jour. Mod. Phys. A4 (1989) 2475.

[24] E. Cremmer, C. Kounnas, A. Van Proeyen, J.P. Derendinger, S. Ferrara, B. de Wit and L. Girardello, Nucl. Phys. B250 (1985) 385; B. de Wit, P. G. Lauwers and A. Van Proeyen, Nucl. Phys. B255 (1985) 569; E. Cremmer and A. Van Proeyen, Class. Quantum Grav. 2 (1985) 445.

[25] L. Castellani, R. D'Auria and S. Ferrara, Phys. Lett. 241B (1990) 57; Class. Quantum Grav. 7 (1990) 1767; R. D'Auria, S. Ferrara and P. Fré, Nucl. Phys. B359 (1991) 705.

[26] K. Stelle, in Proceedings of Nuffield Quantum Gravity Workshop (1981), M. Duff and C. Isham eds., Cambridge U.P. (London); R. Barbieri, S. Ferrara, C. Savoy, F. Palumbo and L. Maiani, Phys. Lett. 115B (1982) 212.

[27] *Essays on Mirror Manifolds*, ed. S.T. Yau (Int. Press, Honk Kong, 1992).

[28] M. J. Duff, preprint NI-94-033, CTP-TAMU-49/94, hep-th/9501030.

[29] A. Sen, Nucl. Phys B388 (1992) 457 and Phys. Lett. B303 (1993) 22; A Sen and J.H. Schwarz, Nucl. Phys. B411 (1994) 35; Phys. Lett. B312 (1993) 105.

[30] C. Gomez and E. Lopez, preprint hep-th/9506024.

[31] P. Candelas, X. De la Ossa, A. Font, S. Katz, d. R. Morrison, Nucl. Phys. B416 (1994) 481

[32] W. Lerche, D. Smit and N. Warner, Nucl. Phys. B372 (1992) 87.

[33] J.K. Kim, C.J. Park and Y. Yoon, Phys. Rev. D40 (1989) 3378.

[34] P. Griffiths, Ann. Math. 90 (1969) 460.

[35] P. Fré and P. Soriani, "The N=2 Wonderland: from Calabi–Yau manifolds to topological field theories", World Scientific, Singapore, 1995 (in press).

[36] A. Cadavid and S. Ferrara, Phys. Lett. B267 (1991) 193.

[37] U. H. Danielsson and B. Sundborg, preprint USITP-95-06, UUITP-4/95, hep-th 9504102.

[38] S. Ferrara and A. Strominger, "N=2 spacetime supersymmetry and Calabi–Yau moduli space", in Proceedings of College Station Workshop (1989) 245.

[39] A. Strominger, Comm. Math. Phys. 133 (1990) 163.

[40] P. Candelas and X. C. de la Ossa, Nucl. Phys. B355 (1991) 455.

[41] D. Anselmi and P. Frè, Nucl. Phys. B404 (1993) 288; Nucl. Phys. B416 (1994) 255.

[42] S. Ferrara and A. Van Proeyen, Class. Quantum Grav. 6 (1989) 124.

[43] E. Calabi and E. Visentini, Ann. of Math. 71 (1960) 472.

[44] M. K. Gaillard and B. Zumino, Nucl. Phys. B193 (1981) 221.

[45] S. Ferrara, M. Bodner and C. Cadavid, Phys. Lett. B247 (1990) 25.

[46] P. Candelas, X.C. de la Ossa, P.S. Green and L. Parkes, Phys. Lett B258 (1991) 118; Nucl. Phys. B359 (1991) 21.

[47] A. Giveon and D. J. Smit, Nucl. Phys. B348 (1991) 168.

[48] A. Erdelyi, F. Obershettinger, W. Magnus and F.G. Tricomi, Higher Transcendental Functions, McGraw Hill, New York (1953).