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A NOTE ON THE STRING ANALOG OF $N=2$ SUPER-SYMMETRIC YANG-MILLS

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Abstract

A connection between the conifold locus of the type II string on the WP^4_{11226} Calabi-Yau manifold and the geometry of the quantum moduli of $N=2$ $SU(2)$ super Yang-Mills is presented. This relation is obtained from the anomalous behaviour of the $SU(2)$ super Yang-Mills special coordinates under S-duality transformation in $Sl(2;Z)/\Gamma_2$.

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1. A string analog of $N=2$ supersymmetric Yang-Mills (SYM) is obtained in reference [1] by means of a "rank three" compactification of the heterotic string on $T^2 \times K_3$. The corresponding $d=4$ $N=2$ theory contains two vector multiplets, one of them to be identified with the dilaton S, and an extra $U(1)$ vector associated with the graviphoton. The basic claim in [1] is the existence of a dual type II string [2, 3, 4, 5] on the Calabi-Yau manifold $W\ P^4_{11226} = M$ [6, 7] such that the quantum moduli of the heterotic string is given by the moduli of the vector multiplets of the type II string on M[2], which receives no quantum corrections. Accepting the previous framework, it should be expected that the geometry of the quantum moduli of $N=2$ $SU(2)$ SYM, solved in [9], can be recovered from the structure of the moduli of vector multiplets on M. In reference [1] some evidence supporting this picture is presented.

The moduli of complex structures for the mirror of M can be parametrized by two complex variables ϕ, ψ, defined by

$$P_{11226} = z_1^{12} + z_2^{12} + z_3^6 + z_4^6 + z_5^2 - 12\psi z_1 z_2 z_3 z_4 z_5 - 2\phi z_1^6 z_2^6 \quad (1)$$

In terms of the coordinates $\bar{x} = \frac{-\phi}{864\psi^6}$, $\bar{y} = \frac{1}{\phi^2}$ introduced in [7], the locus of conifold singularities for (1) is given by

$$(1-\bar{x})^2 - \bar{x}^2 \bar{y} = 0 \quad (2)$$

where j is the Jacobi elliptic function. For fixed $\bar{y} \neq 0$, equation (2) provides two singular points $\bar{x}^{\pm}$. Identifying, at leading order in the coupling, $\bar{y} = e^{-S}$, it was observed in [1] that for $S \to \infty$ the conifold locus merges to the point $\bar{x} = 1$. Using the relation found in [6]

$$\bar{x} = \frac{1728}{j(\tau)} \qquad (j(i) = 1728), \quad (3)$$

the point $\bar{x} = 1$ corresponds to $\tau = i$, which, for τ the elliptic parameter of the internal torus used in the heterotic compactification [1], coincides precisely with the $SU(2)$ point of enhanced gauge symmetry. The picture emerging from this comment being that, for finite but small $\bar{y}$, we should be able to derive the way in which the classical singularity of $N=2$ $SU(2)$ SYM, where the $SU(2)$ symmetry is restored perturbatively, splits into the two singular points of the quantum moduli [9]. It is important to keep in mind that, in the case of $N=2$ $SU(2)$ SYM, the quantum moduli is not the fundamental domain with respect to the full modular (duality) $Sl(2;Z)$ group, while for the string analog, the elliptic parameter τ in (3) is, as a consequence of T-duality, in a $Sl(2;Z)$ fundamental domain.

In this note we will approach the problem of connecting the quantum moduli for $SU(2)$ SYM and the moduli of vector multiplets on M. Our approach will consist in implementing the strong-weak coupling duality transformations of $N=2$ SYM as elements

[2] It was also suggested in reference [8] that the quantum moduli for a heterotic string of rank r should look as the complex moduli of a Calabi-Yau manifold with $h_{21} = r$.

in $Sp(6;Z)$. This will imply a deep connection between the dilaton and the geometry of the quantum moduli of SYM, from which we will try to derive the conifold locus of the Calabi-Yau manifold (1).

2. In [10] we have shown that the obstruction to strong-weak coupling duality invariance in $N=2$ SYM, comes from the fact that[3]

$$a(\gamma(u)) \neq a_D(u) \tag{4}$$

where the transformation γ maps a strong-coupling singular point into the asymptotically free point at infinity. However the effective gauge coupling parameter $\tau = i\frac{4\pi}{g_{eff}^2} + \frac{\theta_{eff}}{2\pi}$ behaves in a dual-like way

$$\tau(\gamma(u)) = -\frac{1}{\tau(u)} \tag{5}$$

For the curve [9] which parametrized the quantum moduli of $N=2$ $SU(2)$ SYM

$$y^2 = (x+\Lambda^2)(x-\Lambda^2)(x-u) \tag{6}$$

with Λ the dynamically generated scale, the transformation γ is given by

$$\gamma(u) = \Lambda^2 \, \frac{u+3\Lambda^2}{u-\Lambda^2} \tag{7}$$

which maps the singular point $u=\Lambda^2$ into ∞. This transformation is an element of $\Gamma_W = Sl(2;Z)/\Gamma_2$, which consists in transformations of the coordinates (x,y) of the curve that can be compensated by a change in the moduli parameter u.

After coupling to gravity, i.e. in non-rigid special geometry, we can improve (4) to [10]

$$a^{gr}(\gamma(u)) = \tilde{f}_{\gamma,\Lambda}(u)\, a_D^{gr}(u) \tag{8}$$

if at the same time we perform the transformation

$$a_0(u) \to \tilde{f}_{\gamma,\Lambda}(u)\, a_0(u) \tag{9}$$

with

$$\tilde{f}_{\gamma,\Lambda}(u) = \frac{\sqrt{2}}{4\pi}\left(\frac{2\Lambda^2}{\Lambda^2-u}\right)^{3/2} \tag{10}$$

Let us explain briefly this result. In the context of rigid special geometry, the 1-form $\lambda(x,u)$, whose periods define the holomorphic sections (a,a_D), satisfies

$$\frac{d\lambda}{du} \sim \lambda_1 \tag{11}$$

[3] From now on we will use the notation of reference [9].

with the proportionality factor determined by imposing the physically correct asymptotic behaviour at the singularities, and where $\lambda_1(x,u)$ is the everywhere non-zero holomorphic 1-form of the curve (6). The 1-form $\lambda^\gamma(x,u)$ which defines $(a(\gamma(u)), a_D(\gamma(u))$, also verifies equation (11) with

$$\frac{d\lambda^\gamma}{du} = \tilde{f}_{\gamma,\Lambda}\lambda_1 \tag{12}$$

In order to pass from (4), where the relation between $a(\gamma(u))$ and $a_D(u)$ is not a symplectic transformation, to (8) which, up to the holomorphic function $\tilde{f}_{\gamma,\Lambda}$, is symplectic, we can replace the derivative in (12) by a covariant derivative relative to the graviphoton (Hodge) $U(1)$-connection [10]. From this, equation (9) immediately follows. However, the gauge transformation $\tilde{f}_{\gamma,\Lambda}$ is singular and generates a topological obstruction to implement strong-weak coupling duality. Thus, the more natural way to interpret (8) and (9) would be to add an extra dilaton field S.

Namely, for the "rank three" heterotic string compactification on $T^2 \times K_3$ we can parametrize the two vector multiplets by the special coordinates (X^0, X^1, X^2), with $S = -iX^1/X^0)$. As it was shown in [11, 12], in order to have all the couplings proportional to S, it is necessary to do the following symplectic change of variables

$$\begin{aligned} \hat{X}^{I\neq 1} = X^I \quad &, \quad \hat{F}_{I\neq 1} = F_I \\ \hat{X}^1 = F_1 \quad &, \quad \hat{F}_1 = -X^1 = -iSX^0 \end{aligned} \tag{13}$$

In these new variables, we can interpret transformation (9) as

$$\hat{X}^0 \to iS\hat{X}^0 = -\hat{F}_1 \tag{14}$$

Performing twice the transformation γ given in (7), we get the identity (up to a sign ambiguity). This implies that under γ satisfying (9), the dilaton changes as

$$S \to S' = \frac{1}{S} \tag{15}$$

At the level of the special coordinates, this equation implies that $\hat{F}_1 \to X^0$, which very likely, together with (14), will be part of a symplectic transformation in $Sp(6;Z)$.

Interpreting now equation (9) as equation (14) boils down to the following formal identification

$$S \sim \tilde{f}_{\gamma,\Lambda}(u) \tag{16}$$

3. Our next task will be to give a natural physical meaning to relation (16). Before doing that it would be worth to come back to our original problem. Once we have added the dilaton field S, we can parametrize the moduli of the two vector multiplets by u, the coordinate of the $N=2$ SYM quantum moduli, and S. Naively this looks like a one parameter family of u-planes labeled by S. The natural physical picture, according with the identification of the conifold locus for $S \to \infty$ with the classical singularity of enhanced $SU(2)$ symmetry, will be to have, for $S \to \infty$, a classical moduli with one singularity at $u=0$ and, for finite S, the quantum moduli with two singularities at $u = \pm\Lambda^2$, where Λ will now depend on the expectation value of the dilaton. However, the previous picture is too crude since it simple fibers the $N=2$ SYM quantum moduli on the dilaton line. This reduces the effect of the dilaton to simply changing the scale Λ, and in this way, the location of the two singularities. Our approach will be to parametrize the moduli space of the two vector multiplets by (u, S) variables, and to try to recover the conifold locus (2) directly from relation (10), which we have derived from the geometry of the quantum moduli of reference [9], by interpreting the transformations (8) and (9) as the symplectic transformation (14) of the heterotic string.

In order to use (16) for defining a locus in the (u, S) variables, we need first to obtain some explicit relation between the dynamical scale Λ and the value of the dilaton field S. Using that the unrenormalized gauge coupling constant g_0 of the effective field theory derived from an heterotic string is proportional to the dilaton expectation value and that changes in g_0 can be absorbed in changing the scale Λ, we will impose, based on the renormalization group equation for $N=2$ $SU(2)$ SYM, the relation

$$\Lambda^2 = e^{-S/2} \tag{17}$$

Combining this with (10), equation (16) becomes

$$\left(\frac{4\pi}{\sqrt{2}} S\right)^{2/3} = \frac{2e^{-S/2}}{e^{-S/2} - u} \tag{18}$$

which can now be used to define a locus $S(u)$ in the (u, S) space. We claim that the so-defined locus in (u, S) variables should coincide with the conifold locus (2).

From (18) we get

$$e^{-S/2} = \frac{u}{1 - 2\left(\frac{\sqrt{2}}{4\pi S}\right)^{2/3}} \tag{19}$$

For $S \to \infty$ we can use, at leading order, the identification $\bar{y} = e^{-S}$ [1]. Solving the conifold locus (2), we obtain

$$\bar{x}^{\pm} = \frac{1}{1 \pm \sqrt{\bar{y}}} \tag{20}$$

which, using (19), can be writen as

$$\bar{x}^{\pm} = \frac{1 - 2\left(\frac{\sqrt{2}}{4\pi S(u)}\right)^{2/3}}{1 \pm u - 2\left(\frac{\sqrt{2}}{4\pi S(u)}\right)^{2/3}} \tag{21}$$

Notice that this equation is equivalent to postulate that the locus (19) is equivalent to the conifold locus (2). However, motivated by this equation we can propose a map from the (u, S) variables into the $(\bar{x}, \bar{y})$ variables parametrizing the complex moduli of (1)

$$\bar{x}(u, S) = \frac{1 - 2\left(\frac{\sqrt{2}}{4\pi S}\right)^{2/3}}{1 + u - 2\left(\frac{\sqrt{2}}{4\pi S}\right)^{2/3}} \tag{22}$$

We need now to check the consistency of (22). First we observe that it reflects correctly the global Z_2 symmetry of the $N=2$ $SU(2)$ SYM gauge theory. The action of this global symmetry in the SYM quantum moduli is given by $u \to -u$, therefore mapping the two finite singular points between themselves. We observe that the change $u \to -u$ in (22) also maps $\bar{x}^+$ into $\bar{x}^-$ and vice-versa.

Second, in the string weak-coupling limit $S \to \infty$, we get

$$\bar{x} = \frac{1}{u+1} + \mathcal{O}(S^{-2/3}) \tag{23}$$

which implies that the asymptotically free regime of the $N=2$ SYM theory, which corresponds to $u \to \infty$, is associated with $\bar{x} = 0$[4]. Using now the relation $\bar{x} = \frac{-\phi}{864\psi^6}$, we obtain that $\frac{1}{u} = \frac{-\phi}{864\psi^6}$ in a neighborhood of the asymptotically free regime $(u \to \infty)$. In order to compare the instanton expansion of $SU(2)$ SYM with the expansion in the (ϕ, ψ) variables of the periods of the holomorphic top form for the Calabi-Yau (1) [6], we will consider Λ and e^{-S} as independent parameters defining the following doble limit[5]

$$lim_{\Lambda, S \to \infty} \Lambda^4 e^{-S} = 1 \tag{24}$$

The instanton expansion of $SU(2)$ SYM goes like $\left(\frac{1}{a(u)}\right)^{4k}$, $k > 0$. Using that $a = \sqrt{2u}$ when $u \to \infty$ [9], in the u variable, we get $\left(\frac{1}{u}\right)^{2k}$. On the other hand, the leading term as $\phi \to \infty$ in the expansion of the periods of the top form behaves as $\left(\frac{\phi}{\psi^6}\right)^k$. The presence in one case of powers of $2k$ and in the other of powers of k can be explained by the fact that, for the Calabi-Yau case, the two singular points of the locus (2) are related by the transformation $\phi \to -\phi$ which is not a symmetry, while the Z_2 transformation $u \to -u$, that corresponds by (19) to $\phi \to -\phi$, is a global symmetry of $N = 2$ $SU(2)$ SYM. Therefore, in order to reproduce the SYM quantum moduli inside the Calabi-Yau moduli space, we should mod by the transformation $\phi \to -\phi$, or equivalently, consider only even powers of $\frac{\phi}{\psi^6}$.

[4]In reference [1] is is also suggested that $\bar{x} = 0$ can be interpreted as a weak coupling region related to the degeneration of the compactification torus.

[5]In reference [1] it has been already proposed to use a suitable "doble scaling" limit in order to recover the $N=2$ $SU(2)$ SYM quantum moduli from the moduli space of complex deformations of the $W\,P^4_{11226}$ Calabi-Yau.

Summarizing, we have shown that by identifying the conifold locus with equation (16) we get, at leading order in S, a consistent physical picture. The important thing for us is that equation (16) was obtained by promoting, introducing a dilaton field, the anomalous strong-weak coupling duality transformations of the $N=2$ SYM field theory, to good symplectic transformations. In other words the quantum moduli in reference [9] describes more than the complex structure of the curve (6), something which is related to the existence of a dynamically generated scale Λ. When we include the dilaton and work in the string framework, what was geometrical (no moduli) information for $SU(2)$ SYM becomes now pure moduli information of the Calabi-Yau manifold (1).

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