CERN-TH/95-138

UWThPh-19-1995

hep-th/9505175

Towards Finite Quantum Field Theory in Non-commutative Geometry

H. Grosse[1]

Institut for Theoretical Physics, University of Vienna,

Boltzmanngasse 5, A-1090 Vienna, Austria

C. Klimčík[2]

Theory Division CERN, CH-1211 Geneva 23, Switzerland

P. Prešnajder

Department of Theoretical Physics, Comenius University

Mlynská dolina, SK-84215 Bratislava, Slovakia

Abstract

We describe the self-interacting scalar field on the truncated sphere and we perform the quantization using the functional (path) integral approach. The theory posseses a full symmetry with respect to the isometries of the sphere. We explicitely show that the model is finite and the UV-regularization automatically takes place.

CERN-TH/95-138

May 1995

[1]Part of Project No. P8916-PHY of the 'Fonds zur Förderung der wissenschaftlichen Forschung in Österreich'.

[2]Partially supported by the grant GAČR 2178

1 Introduction

The basic ideas of the non-commutative geometry were developed in [1, 2], and in the form of the matrix geometry in [3, 4]. The applications to physical models were presented in [2, 5], where the non-commutativity was in some sense minimal: the Minkowski space was not extended by some standard Kaluza-Klein manifold describing internal degrees of freedom but just by two non-commutative points. This led to a new insight on the $SU(2)_L \otimes U(1)_R$ symmetry of the standard model of electro-week interactions. The model was further extended in [6] inserting the Minkowski space by pseudo-Riemannian manifold, and thus including the gravity. Such models, of course, do not lead to UV-regularization, since they do not introduce any space-time short-distance behaviour.

To achieve the UV-regularization one should introduce the non-commutativity into the genuin space-time manifold in the relativistic case, or into the space manifold in the Euclidean version. One of the simplest locally Euclidean manifolds is the sphere S^2. Its non-commutative (fuzzy) analog was described by [7] in the framework of the matrix geometry. More general construction of some non-commutative homogenous spaces was described in [8] using coherent states technique.

The first attempt to construct fields on a truncated sphere were presented in [9] within the matrix formulation. Using more general approach the classical spinor field on truncated S^2 was investigated in detail in [10-11].

In this article article we shall investigate the quantum scalar field Φ on the truncated S^2. We shall explicitely demonstrate that the UV-regularization automatically appears within the context of the non-commutative geometry. We shall introduce only necessary notion of the non-commutative geometry we need in our approach. In Sec. 2 we define the non-commutative sphere and the derivation and integration on it. In Sec. 3 we introduce the scalar self-interacting field Φ on the truncated sphere and the field action. Further, using Feynman (path) integrals we perform the quantization of the model in question. Last Sec. 4 contains a brief discussion and concluding remarks.

2 Non-commutative truncated sphere

A) The infinite dimensional algebra $\mathcal{A}_\infty$ of polynomials generated by $x = (x_1, x_2, x_3) \in R^3$ with the defining relations

$$[x_i, x_j] = 0, \quad \sum_{i=1}^{3} x_i^2 = \rho^2 \tag{1}$$

contains all informations about the standard unit sphere S^2 embedded in R^3. In terms of spherical angles θ and φ one has

$$x_\pm = x_1 \pm i x_2 = \rho e^{\pm i\varphi} \sin\theta \ , \quad x_3 = \rho \cos\theta \ . \tag{2}$$

As a non-commutative analogue of $\mathcal{A}_\infty$ we take the algebra $\mathcal{A}_N$ generated by $\hat{x} = (\hat{x}_1, \hat{x}_2, \hat{x}_3)$ with the defining relations

$$[\hat{x}_i, \hat{x}_j] = i\lambda\varepsilon_{ijk}\hat{x}_k \ , \quad \sum_{i=1}^{3} \hat{x}_i^2 = \rho^2 \ . \tag{3}$$

The real parameter $\lambda > 0$ characterizes the non-commutativity (later on it will be related to N). In terms of $\hat{X}_i = \frac{1}{\lambda}\hat{x}_i$, $i = 1, 2, 3$, eqs. (3) are changed to

$$[\hat{X}_i, \hat{X}_j] = i\varepsilon_{ijk}\hat{X}_k \ , \quad \sum_{i=1}^{3} \hat{X}_i^2 = \rho^2\lambda^{-2} \ , \tag{4}$$

or putting $X_\pm = X_1 \pm iX_2$ we obtain

$$[\hat{X}_3, \hat{X}_\pm] = \hat{X}_\pm \ , \quad [\hat{X}_+, \hat{X}_-] = 2\hat{X}_3 \ , \tag{5}$$

and

$$C = \hat{X}_3^2 + \frac{1}{2}(\hat{X}_+\hat{X}_- + \hat{X}_-\hat{X}_+) = \rho^2\lambda^{-2} \ . \tag{6}$$

We shall realize eqs. (4), or equivalently eqs. (5) and (6), as relations in some suitable irreuducible unitary representations of the $SU(2)$ group. It is useful to perform this construction using Wigner-Jordan realization of the generators $\hat{X}_i$, $i = 1, 2, 3$, in terms of two pairs of annihilation and creation operators $A_\alpha, A_\alpha^*, \alpha = 1, 2$, satisfying

$$[A_\alpha, A_\beta] = [A_\alpha^*, A_\beta^*] = 0 \ , \quad [A_\alpha, A_\beta^*] = \delta_{\alpha,\beta} \ , \tag{7}$$

and acting in the Fock space $\mathcal{F}$ spanned by the normalized vectors

$$|n_1, n_2\rangle = \frac{1}{\sqrt{n_1!n_2!}}(A_1^*)^{n_1}(A_2^*)^{n_2}|0\rangle \ , \tag{8}$$

where $|0\rangle$ is the vacuum defined by $A_1|0\rangle = A_2|0\rangle = 0$. The operators $\hat{X}_\pm$, and $\hat{X}_3$ take the form

$$\hat{X}_+ = 2A_1^*A_2 \ , \ \hat{X}_- = 2A_2^*A_1 \ , \ \hat{X}_3 = \frac{1}{2}(N_1 - N_2) \ , \tag{9}$$

where $N_\alpha = A_\alpha^*A_\alpha$, $\alpha = 1, 2$. Restricting to the $(N+1)$-dimensional subspace

$$\mathcal{F}_N = \{|n_1, n_2\rangle \ \in \mathcal{F}\} \ , \tag{10}$$

we obtain for any given $N = 0, 1, 2, \ldots$, the irreducible unitary representation in which the Casimir operator (6) has the value

$$C = \frac{N}{2}\left(\frac{N}{2}+1\right) \ , \tag{11}$$

i.e. the λ and N are related as

$$\rho\lambda^{-1} = \sqrt{\frac{N}{2}\left(\frac{N}{2}+1\right)} \ . \tag{12}$$

The states $|n_1, n_2\rangle$ are eigenstates of the operator X_3, whereas X_+ and X_- are rising and lowering operators respectively

$$\begin{aligned} X_3 \ |n_1, n_2\rangle &= \frac{n_1 - n_2}{2} \ |n_1, n_2\rangle \ , \\ X_+ \ |n_1, n_2\rangle &= 2\sqrt{(n_1+1)n_2} \ |n_1+1, n_2-1\rangle \ , \\ X_- \ |n_1, n_2\rangle &= 2\sqrt{n_1(n_2+1)} \ |n_1-1, n_2+1\rangle \ . \end{aligned} \tag{13}$$

Since $X_i : \mathcal{F}_N \to \mathcal{F}_N$, we have

$$dim\mathcal{A}_N \leq (N+1)^2 \ . \tag{14}$$

B) As a next step we extend the notions of integration and derivation to the truncated case. The standard integral on S^2

$$I_\infty(F) = \frac{1}{4\pi}\int d\Omega F(x) = \frac{1}{4\pi}\int_{-\pi}^{+\pi} d\varphi \int_0^\pi \sin\theta d\theta F(\theta,\varphi) \tag{15}$$

is uniquely defined if it is fixed for the monomials $F(x) = x_+^l x_-^m x_3^n$. It is obvious that $I_\infty(x_+^l x_-^m x_3^n) = 0$ for $l \neq m$, and that $x_+^l x_-^l x_3^n = \rho^{2l+n}\sin^{2l}\theta\cos^n\theta$ is a polynomial in $\cos\theta = x_3$. An easy calculation gives

$$I_\infty(x_3^{2n+1}) = 0 \ , \quad I_\infty(x_3^{2n}) = \frac{\rho^{2n}}{2n+1} \ ,$$

for $n = 0, 1, 2, \dots$. Putting $\xi = \rho^{-1} x_3 = \cos\theta$ we see that

$$I_\infty(\xi^n) = \frac{1}{2}\int_{-1}^{+1} d\xi\ \xi^n\ . \tag{16}$$

These relations algebraically define the integration in $\mathcal{A}_\infty$.

In the non-commutative case we put

$$I_N(F) = \frac{1}{N+1} Tr[F(\hat{x})] \tag{17}$$

for any polynomial $F(\hat{x}) \in \mathcal{A}_N$ in $\hat{x}_i, i = 1, 2, 3$, where the trace is taken in $\mathcal{F}_N$. Again, the integrals $I(\hat{x}_+^l \hat{x}_-^m \hat{x}_3^n) = 0$ for $l \neq m$ since

$$\hat{x}_+^l \hat{x}_-^m \hat{x}_3^n\ |n_1, n_2\rangle\ \sim\ |n_1 + l - m, n_2 + m - l\rangle\ .$$

Similarly as before, $\hat{x}_+^l \hat{x}_-^l \hat{x}_3^n$ can be expressed using eqs. (5) and (6) as a polynomial in $\hat{x}_3$. The equation

$$\hat{x}_3^n\ |n_1, n_2\rangle = (\lambda \frac{n_1 - n_2}{2})^n\ |n_1, n_2\rangle \tag{18}$$

gives

$$I_N(\hat{x}_3^n) = \sum_{k=0}^{N} \frac{\rho^n}{N+1} \xi_k^n\ , \tag{19}$$

where $\xi_k = \sqrt{\frac{N}{N+2}}(\frac{2k}{N} - 1)$. The formula (19) can be rewritten as a Stieltjes integral with the stair-shape measure $\mu(\xi)$ in the interval (-1,+1) with steps in the points ξ_k

$$I_N(\xi^n) = \int_{-1}^{+1} d\mu(\xi) \xi^n = \sum_{k=0}^{N} \frac{1}{N+1} \xi_k^n\ . \tag{20}$$

Obviously, $I_N(\hat{x}_3^{2n+1}) = 0$, and

$$I_N(\hat{x}_3^{2n}) = \frac{\rho^{2n}}{(\frac{N}{2})^n (\frac{N}{2} + 1)^n (N+1)} \sum_{k=0}^{N} (\frac{2k - N}{2})^{2n}\ .$$

Using the known formula (see e.g. [12], p. 597, eq. (16))

$$\sum_{k=0}^{N}(k+a)^m = \frac{1}{m+1}[B_{m+1}(N+1+a) - B_{m+1}(a)] ,$$

where $B_m(x)$ are Bernoulli polynomials, we obtain

$$I_N(\hat{x}_3^{2n}) = \frac{\rho^{2n}}{2n+1} C(N,n) . \tag{21}$$

Here,

$$C(N,n) = \frac{B_{2n+1}(\frac{N}{2}+1) - B_{2n+1}(-\frac{N}{2})}{(\frac{N}{2})^n(\frac{N}{2}+1)^n(N+1)} \tag{22}$$

represents a non-commutative correction. Since the Bernoulli polynomials are normalized as

$$B_m(x) = x^m \ + \ lower\ powers\ ,$$

we see that

$$C(N,n) = 1 + o(1/N) , \tag{23}$$

i.e. in the limit $N \to \infty$ we recover the commutative result.

The scalar product in $\mathcal{A}_\infty$ can be introduced as

$$(F_1, F_2)_\infty = I_\infty(F_1^* F_2) , \tag{24}$$

and similarly in $\mathcal{A}_N$ we put

$$(F_1, F_2)_N = I_N(F_1^* F_2) . \tag{25}$$

C) The vector fields describing motions on S^2 are linear combinations (with the coeffitiens from $\mathcal{A}_\infty$) of the differential operators acting on any $F \in \mathcal{A}_\infty$ as follows

$$J_i F = \frac{1}{i}\ \varepsilon_{ijk}\ x_j\ \frac{\partial F}{\partial x_k} . \tag{26}$$

In particular,

$$J_i x_j = i\ \varepsilon_{ijk}\ x_k\ . \tag{27}$$

The operators $J_i,\ i = 1,2,3$, satisfy in $\mathcal{A}_\infty$ the $su(2)$ algebra commutation relations

$$[J_i, J_j] = i\varepsilon_{ijk} J_k\ , \tag{28}$$

or for $J_\pm = J_1 \pm iJ_2$ they take the form

$$[J_3, J_\pm] = \pm J_\pm\ ,\quad [J_+, J_-] = 2J_3\ . \tag{29}$$

The operators J_i are self-adjoint with respect to the scalar product (24).

In the non-commutative case the operators J_i act on any element F from the algebra $\mathcal{A}_N$ in the following way

$$J_i F = [X_i\ , F]\ . \tag{30}$$

In particular,

$$J_i \hat{x}_j = i\ \varepsilon_{ijk}\ \hat{x}_k\ . \tag{31}$$

The operators J_i satisfy $su(2)$ algebra commutation relations and are self-adjoint with respect to the scalar product (25).

The functions

$$\Psi_{ll}(\hat{x}) = c_l\ \hat{x}_+^l\ , \tag{32}$$

are the highest weight vectors in $\mathcal{A}_N$ for $l = 0, 1, ..., N$, since

$$J_+ \Psi_{ll}(\hat{x}) = \lambda^l\ [\hat{X}_+, \hat{X}_+^l] = 0\ . \tag{33}$$

For all $l > N$ is $\hat{x}_+^l = 0$ in $\mathcal{A}_N$. The normalization factor c_l is fixed by the condition

$$1 = \parallel \Psi_{ll} \parallel^2 = (\Psi_{ll}, \Psi_{ll})_N = |c_l|^2 I_N(\hat{x}_-^l \hat{x}_+^l)\ ,$$

and is given by the formula ([12], p.618, eq. (36))

$$\rho^{2l} c_l^2 = \frac{(2l+1)!!}{(2l)!!} \frac{(N+1)N^l(N+2)^l(N-l)!}{(N+l+1)!} \ . \tag{34}$$

The second factor on the right hand side represents a non-commutative correction. For $N \to \infty$ it approaches 1. The other normalized functions $\Psi_{lm}, m = 0, \pm 1, ..., \pm l$, in the irreducible representation containing Ψ_{ll} are given as

$$\Psi_{lm} = \sqrt{\frac{(l+m)!}{(l-m)!(2l)!}} \ J_-^{l-m} \Psi_{ll} \ . \tag{35}$$

The normalization factor on the right hand side is the standard one independent of N. The functions Ψ_{lm} are eigenfunctions of the operators J_i^2 and J_3:

$$J_i^2 \Psi_{lm} = l(l+1)\Psi_{lm} \ ,$$
$$J_3 \Psi_{lm} = m \Psi_{lm} \ . \tag{36}$$

We see that $\mathcal{A}_N$ contains all $SU(2)$ irreducible representations with the "orbital momentum" $l = 0, 1, ..., N$. The l-th representation has the dimension $2l+1$, and consequently

$$dim \mathcal{A}_N \geq \sum_{n=0}^{N} (2l+1) = (N+1)^2 \ . \tag{37}$$

Comparing this with eq. (14) we see that $\mathcal{A}_N$ contains no other representations, i.e.

$$\mathcal{A}_N = \bigoplus\nolimits_{l=0}^{N} \mathcal{A}_{(l)} \ , \tag{38}$$

where $\mathcal{A}_{(l)}$ denotes the representation space of the l-th representation spanned by the functions $\Psi_{lm}, m = 0, \pm 1, ..., \pm l$. In particular, $dim \mathcal{A}_N = (N+1)^2$.

3 Scalar field on the truncated sphere

A) The Euclidean field action for a real self-interacting scalar field Φ on a standard sphere S^2 is given as

$$S[\Phi] = \frac{1}{4\pi}\int_{S^2} d\Omega[(J_i\Phi)^2 + \mu^2(\Phi)^2 + V(\Phi)]$$

$$= I_\infty(\Phi J_i^2\Phi + \mu^2(\Phi)^2 + V(\Phi)) \ , \tag{39}$$

where

$$V(\Phi) = \sum_{k=0}^{2K} g_k \Phi^k \ , \tag{40}$$

is a polynomial with $g_{2K} \geq 0$ (and we explicitely indicated the mass term).

The quantum mean value of some polynomial field functional $F[\Phi]$ is defined as the functional integral

$$\langle F[\Phi]\rangle = \frac{\int D\Phi e^{-S[\Phi]} F[\Phi]}{\int D\Phi e^{-S[\Phi]}} \ , \tag{41}$$

where $D\Phi = \prod_x d\Phi(x)$. Alternatively, we can expand the field into spherical functions

$$\Phi(x) = \sum_{l=0}^{\infty}\sum_{m=-l}^{+l} a_{lm} Y_{lm}(x) \tag{42}$$

satisfying

$$J_i^2 Y_{lm} \ = \ l(l+1)Y_{lm} \ .$$

Here the complex coefficients a_{lm} obey

$$a_{l,-m} = (-1)^m a_{lm}^* \ , \tag{43}$$

what guarantees the reality condition $\Phi^*(\hat{x}) = \Phi(\hat{x})$. We can put $D\Phi = \prod_l da_{l0}\prod_{lm} da_{lm}da_{lm}^*$, $l = 0, 1, ..., N$, $m = 1, ..., l$. Both expressions for $D\Phi$

are only formal. The measure in the functional integral can be mathematically rigorously defined (see e.g. [13]) but we shall not follow this direction.

Such problems do not appear in the non-commutative case, where the scalar field $\Phi(\hat{x})$ is an element of the algebra $\mathcal{A}_N$, and consequently it can be expanded as

$$\Phi(\hat{x}) = \sum_{l=0}^{N} \sum_{m=-l}^{+l} a_{lm} \Psi_{lm}(\hat{x}) \ , \tag{44}$$

where $\Psi_{lm}(\hat{x})$ satisfy in $\mathcal{A}_N$ the equation

$$J_i^2 \Psi_{lm} \ = \ l(l+1)\Psi_{lm} \ ,$$

and are orthonormal with respect to the scalar product (25). The coefficients a_{lm} are again restricted by the condition (43).

The action in the non-commutative case is defined as

$$S[\Phi] = I_N(\Phi J_i^2 \Phi + \mu^2 (\Phi)^2 + V(\Phi)) \ , \tag{45}$$

and it is a polynomial in the variables $a_{lm}, l = 0, 1, ..., N, \ m = 0, \pm 1, ..., \pm l$. The measure $D\Phi \ = \ \prod_l da_{l0} \prod_{lm} da_{lm} da^*_{lm} \ , \ l \ = \ 0, 1, ..., N, \ m \ = \ 1, ..., l$, in the quantum mean value (41) is the usual Lebesgue measure, since now the product is finite. The quantum mean values are well defined for any analytic functional $F[\Phi]$.

Under rotations

$$\hat{x}_i \to \hat{x}'_i = \sum_j R_{ij}(\alpha, \beta, \gamma) \hat{x}_j \tag{46}$$

specified by the Euler angles α, β, γ, the field transforms as

$$\Phi(\hat{x}) \to \Phi(\hat{x}') = \sum_{l=0}^{N} \sum_{m=-l}^{+l} a_{lm} \Psi_{lm}(\hat{x}') \ . \tag{47}$$

Using the transformation rule for the functions Ψ_{lm} (see e.g. [15])

$$\Psi_{lm'}(\hat{x}') = \sum_{m'} D^l_{m'm}(\alpha, \beta, \gamma)\Psi_{lm}(\hat{x}) \ , \tag{48}$$

we obtain the transformation rule for the coefficients a_{lm}

$$a_{lm} \to a'_{lm'} = \sum_{m} D^l_{m'm}(\alpha, \beta, \gamma) a_{lm} \ . \tag{49}$$

The last equation is an orthogonal transformation not changing the measure $D\Phi$ (see e.g. [14]).

The Schwinger functions we define as follows

$$S_n(F) = \langle F_n[\Phi] \rangle \ , \tag{50}$$

where

$$F_n[\Phi] = \sum \alpha_{l_1 m_1 \ldots l_n m_n} a_{l_1 m_1} \ldots a_{l_n m_n} \equiv \sum \alpha_{l_1 m_1 \ldots l_n m_n} (\Psi_{l_1 m_1}, \Phi)_N \ldots (\Psi_{l_n m_n}, \Phi)_N \ . \tag{51}$$

The functions (49) satisfy the following Osterwalder-Schrader axioms:

(OS1) *Hermiticity*

$$S^*_n(F) = S_n(\Theta F) \ , \tag{52}$$

where ΘF is an involution

$$\Theta F_n[\Phi] = \sum \alpha^*_{l_1 - m_1 \ldots l_n - m_n} (-1)^{m_1 + \ldots m_n} a_{l_1 m_1} \ldots a_{l_n m_n} \ .$$

(OS2) *Covariance*

$$S_n(F) = S_n(\mathcal{R}F) \ , \tag{53}$$

where $\mathcal{R}F$ is a mapping induced by Eq. (49).

(OS3) *Reflection positivity*

$$\sum_{n,m\in\mathcal{I}} S_{n+m}(\Theta F_n \otimes F_m) \geq 0 \ . \tag{54}$$

(OS4) *Symmetry*

$$S_n(F) = S_n(\pi F) \ , \tag{55}$$

where πF is a functional obtained from F by arbitrary permutation of a_{lm}'s in Eq. (51).

Note: The positivity axiom (53) can be rewritten as $\langle F^* F\rangle \geq 0$, $F = \sum_{n\in\mathcal{I}} F_n$. In fact, the standard formulation of (OS3) axiom requires the specification of the support of the functionals F_n. In our case the axiom holds in the "strong" sense , i.e. without the specification. We expect, however, that in the continuum limit ($N \to \infty$) the issue will emerge. We do not include the last Osterwalder-Schrader axiom - the cluster property, since the compact manifold requires a special treatment (however, it can be recovered in the limit when the radius of the sphere grows to infinity, but this goes beyond the presented scheme).

B) In many practical applications the perturbative results are sufficient. Interpreting the term $V(\Phi)$ as a perturbation, we present below as an illustration the Feynman rules for the model in question. We give the Feynman rules in the (lm)-representation defined by the expansios (42) and (43). The diagrams are constructed from

(i) *External vertices* assigned to any operator a_{lm} appearing in the functional $F[\Phi]$.

(ii) *Internal vertices* given by the expansion of $V(\Phi)$ in terms of $a_{l_1 m_1} \ldots a_{l_k m_k}$.

This gives the following Feynman rules:

(a) *Propagator*

$$2\langle a_{lm} a^*_{l'm'} \rangle = \frac{1}{l(l+1)+\mu^2} \delta_{l'l} \delta_{m'm} \ , \tag{56}$$

where the admissible values of l and m for $\mathcal{A}_\infty$ are $l = 0, 1, 2, ..., \ m = 0, 1, ..., l$, whereas in the case of $\mathcal{A}_N$ they are $l = 0, 1, ..., N, \ m = 0, 1, ..., l$.

(b) *Vertex*

$$V_{l_1 m_1, ..., l_k m_k} = g_k I_\infty (Y_{l_1 m_1} ... Y_{l_k m_k}) \ for \ \mathcal{A}_\infty \ , \tag{57}$$

$$V_{l_1 m_1, ..., l_k m_k} = g_k I_N (Y_{l_1 m_1} ... Y_{l_k m_k}) \ for \ \mathcal{A}_N \ , \tag{58}$$

(c) Finally the summation over all *internal* indices should be performed.

This procedure leads for $\mathcal{A}_\infty$ finite Feynman diagrams except the diagrams containing the tadpole contribution

$$T_\infty \equiv \sum_{lm} \langle a_{lm} a^*_{lm} \rangle \sim \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{l(l+1)+\mu^2} = \infty \ .$$

This divergence is closely related to the divergence of the propagator

$$G(x,y) = \sum_{lm} \frac{1}{l(l+1)+\mu^2} Y_{lm}(x) Y^*_{lm}(y)$$

in the x-representation at points $x = y$. This requires, of course, the regularization of $G(x,y)$, which is, in our case, simply a cut-off in the l-summations. Indeed, for $\mathcal{A}_N$ all diagrams are obviously finite (since all summations are finite). In particular the tadpole contribution reads

$$T_N = \sum_{l=0}^{N} \sum_{m=-l}^{l} \frac{1}{l(l+1)+\mu^2} \sim \ \ln N \ .$$

For practical applications an effective method for the calculation of vertex coefficients $V_{l_1 m_1,\ldots,l_k m_k}$ is needed, both in the standard and non-commutative cases. We shall describe the latter one. Since the multiplication by Ψ_{lm} acts in the algebra $\mathcal{A}_N$ as an irreducible tensor operator, we can apply the Wigner-Eckart theorem. Then the product $\Psi_{l_1 m_1}(\hat{x})\Psi_{l_2 m_2}(\hat{x})$ can be expressed as

$$\Psi_{l_1 m_1}(\hat{x})\Psi_{l_2 m_2}(\hat{x}) = \sum_{l=|l_1-l_2|}^{l_1+l_2} (l_1 m_1, l_2 m_2|lm)\ (l_1 l_2 \parallel l)\ \Psi_{lm}(\hat{x})\ , \qquad (59)$$

where $m = -m_1 + m_2$, $(l_1 m_1, l_2 m_2|lm)$ is a Clebsch-Gordon coefficient, and the symbol $(l_1 l_2 \parallel l)$ denotes the so called reduced matrix element (and depends on the particular algebra in question). Introducing the non-commutative Legendre polynomials $P_l(\xi) = \Psi_{l0}(\hat{x}),\ \xi = \rho^{-1}\hat{x}_3$, the previous equation leads to the coupling rule

$$P_{l_1}(\xi)P_{l_2}(\xi) = \sum_{l=|l_1-l_2|}^{l_1+l_2} (l_1 0, l_2 0|l0)\ (l_1 l_2 \parallel l)\ P_l(\xi)\ . \qquad (60)$$

The repeated application of (59) then allows to calculate the required vertices.

Note: The well known explicit formula for the usual Legendre polynomials allows us to calculate the reduced matrix elements

$$(l_1 l_2 \parallel l) = (l_1 0, l_2 0|l0)$$

enterring the coupling rule in the algebra $\mathcal{A}_\infty$ in terms of a particular Clebsch-Gordon coefficients. Similarly, the explicit formula for the non-commutative Legendre polynomials presented in the Appendix allows to deduce the reduced matrix elements enterring the coupling rule in the algebra $\mathcal{A}_N$.

4 Concluding Remarks

We have demonstrated above that the interacting scalar field on the non-commutative sphere represents a quantum system which has the following properties:

1) The model has a full space symmetry - the full symmetry under isometries (rotations) of the sphere S^2. This is exactly the same symmetry as the interacting scalar field on the standard sphere has.

2) The field has only a finite number of modes. Then the number of degrees of freedom is finite and this leads to the non-perturbative UV-regularization, i.e. all quantum mean values of polynomial field functionals are well defined and finite.

Consequently, all Feynman diagrams in the perturbative expansion are finite, even the diagrams containing the tadpole diagram which are divergent in the model on a standard sphere. Technically, the tadpole is finite due to the cut-off in the number of modes. In our approach the UV cut-off in the number of modes is supplemented with a highly non-trivial vertex modification (compare eqs. (57) and (58)). Moreover, our UV-regularization is non-perturbative and is completely determined by the algebra $\mathcal{A}_N$. It is originated by the short-distance structure of the space, and does not depend on the field action of the model in question. From the presented point of view, it would be desirable to analyze a quantization of the models on a non-commutative sphere S^2 containing spinor, or gauge fields. In the standard case such models have a more complicated structure of divergencies. It is evident, that our approach will lead again to a non-perturbative UV-regularization. The usual divergencies will appear only in the limit $N \to \infty$. It would be very interesting to isolate the large N behaviour non-perturbatively. By this we mean the Wilson-like approach in which the renormalization group flow in the space of Lagrangeans is studied. This can lead to the better understanding of the origin and properties of divergencies in the quantum field theory. Another interesting direction would consist in making connection with the matrix models where, from the technical point of view, very similar integral have been studied. We strongly believe, that qualitatively just the same situation will repeat on the four-dimensional sphere S^4 too. Investigations in all these directions are under current study.

Ackonwledgement

We are grateful to A. Alekseev, L. Álvarez-Gaumé, M. Bauer, A. Connes, V. Černý, T. Damour, J. Fröhlich, J. Ftáčnik, K. Gawędzki, J. Hoppe, B. Jurčo, E. Kiritsis, C. Kounnas, M. Rieffel, R. Stora and D. Sullivan for useful discussions. The part of the research of C.K. has been done at I.H.E.S. at Bures-sur-Yvette and of C.K. and P.P. at the Schrödinger Institute in Vienna. We thank both of them for hospitality.

Appendix

The truncated Legendre polynomials

$$P_l(\xi) = \xi^l a_0^l + \xi^{l-2} a_1^l + \ldots \ , \quad l = 0, 1, \ldots, N \ ,$$

we define as orthonormal polynomials with respect to the scalar product

$$(P_l, P_m)_N = I_N(P_l P_m) = \delta_{lm} \ .$$

Here the non-commutative integral is given as (see eq. (19))

$$I_N(\xi^n) = \sum_{k=0}^{N} \frac{1}{N+1} \xi_k^n \ ,$$

where $\xi_k = \sqrt{\frac{N}{N+2}}(\frac{2k}{N} - 1)$. The polynomials $P_l(\xi)$ can be obtained from the reccurence relation

$$P_{m+1}(\xi) = \frac{1}{a_m}[\xi P_m(\xi) - c_m P_{m-1}(\xi)] \ ,$$

where $c_m = I(\xi P_m P_{m-1})$ and $a_m = \sqrt{I_N(\xi^2 P_m^2) - c_m^2}$.

The truncated spherical functions $\Psi_{lm}(\hat{x})$ satisfy in $\mathcal{A}_N$ equation

$$J_i^2 \Psi_{lm}(\hat{x}) = l(l+1)\Psi_{lm}(\hat{x}) \ .$$

Putting $P_l(\xi) = \Psi_{l0}(\hat{x})$, $\xi = \hat{x}_3$, the last equation reduces to a difference equation for the truncated Legendre polynomials

$$(1-\xi^2)\frac{P_l(\xi+\lambda) - 2P_l(\xi) + P_l(\xi-\lambda)}{\lambda^2}$$

$$+ \ 2\xi \frac{P_l(\xi+\lambda) - P_l(\xi-\lambda)}{2\lambda} \ + \ l(l+1)P_l(\xi) = 0 \ ,$$

where $\lambda = 2/\sqrt{N(N+2)}$. This equation leads to the reccurence relation for the coefficients a_s^l appearing in the Legendre polynomials:

$$a_s^l = -\frac{1}{s(2l-2s+1}\sum_{r=0}^{s-1} a_r^l [\binom{l-2r}{l-2s} \ - \lambda^2 \ \binom{l-2r+1}{l-2s+1}] \ \lambda^{2s-2r-2} \ .$$

In the limit $N \to \infty$ (or equivalently $\lambda \to 0$) all formulas reduce to the standard expressions valid for usual Legendre polynomials.

References

[1] A. Connes, Publ. IHES *62* (1986) 257.

[2] A. Connes, Geometrie Noncommutative (Inter Editions, Paris 1990).

[3] M. Dubois-Violette, C. R. Acad. Sci. Paris *307*, Ser. I (1988) 403.

[4] M. Dubois-Violette, R. Kerner and J. Madore, Journ. Math. Phys. *31* (1990) 316.

[5] R. Coquereaux, G. Esposito-Farese and G. Vailant, Nucl. Phys. *B353* (1991) 689.

[6] A.H. Chamsedine, G. Felder and J. Fröhlich, *Gravity in the non-commutative geometry*, preprint ETH-TH/1992-18.

[7] J. Madore, Journ. Math. Phys. *32* (1991) 332.

[8] H. Grosse and P. Prešnajder, Lett. Math. Phys. *28* (1993) 239.

[9] H. Grosse and J. Madore, Phys. Lett. *B283* (1992) 218.

[10] H. Grosse, C. Klimčík and P. Prešnajder, *Field theory on truncated superspace* - to be published.

[11] H. Grosse, C. Klimčík and P. Prešnajder, *Finite gauge model on truncated sphere* - to be published.

[12] A. P. Prudnikov, Yu. A. Brytshkov and O. I. Maritshev, *Integrals and Series*, Publ. by Nauka, Moscow 1981 (in Russian).

[13] B. Simon, *The* $P(\Phi)_2$ *Euclidean (Quantum) Field Theory*, Princeton University Press, Princeton - New Jersey 1974.

[14] N. Ya. Vilenkin, *Special Functions and the Theory of Group Representations*, Publ. by Nauka, Moscow 1965 (in Russian).