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Nonperturbative Effective Actions of N=2 Supersymmetric Gauge Theories

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Abstract

We elaborate on our previous work on $N=2$ supersymmetric Yang-Mills theory. In particular, we show how to explicitly determine the low energy quantum effective action for $G = SU(3)$ from the underlying hyperelliptic Riemann surface, and calculate the leading instanton corrections. This is done by solving Picard-Fuchs equations and asymptotically evaluating period integrals. We find that the dynamics of the $SU(3)$ theory is governed by an Appell system of type F_4, and compute the exact quantum gauge coupling explicitly in terms of Appell functions.

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1. Introduction

Seiberg and Witten [1,2] have investigated $N=2$ supersymmetric gauge theories, with gauge group $G = SU(2)$, and solved for their exact non-perturbative low energy effective action. Their construction has been generalized to arbitrary $SU(n)$ gauge groups in [3,4]. A detailed analysis from the viewpoint of large n was presented in [5]. The relation to special geometry was discussed in [6]. More recently, the extension to $SO(2n+1)$ was presented in [7], as well as an analysis of the non-local behavior at the cusp points in moduli space [8]. In addition, $SU(n)$ gauge theories with extra matter were considered in [9]. There is a considerable overlap between various of these papers.

The purpose of the present paper is to elaborate on our previous work [3] on $SU(n)$ $N=2$ Yang-Mills theory, with particular focus on $G = SU(3)$; some of the material has already been presented in a short preview [10].

For arbitrary gauge group G, $N=2$ supersymmetric gauge theories without matter hypermultiplets are characterized by having flat directions for the Higgs vacuum expectation values, along which the gauge group is generically broken to the Cartan sub-algebra. Thus, the effective theories contain $r = \mathrm{rank}(G)$ abelian $N=2$ vector supermultiplets, which can be decomposed into r $N=1$ chiral multiplets A_i plus r $N=1$ $U(1)$ vector multiplets $W_{\alpha i}$ (we will denote the scalar components of A_i by a_i). $N=2$ supersymmetry implies that the leading piece (with up to two derivatives) of the low energy effective lagrangian depends only on a single holomorphic function, $\mathcal{F}(A)$:

$$\mathcal{L} \;=\; \frac{1}{4\pi}\mathrm{Im}\Big[\int d^4\theta\,\big(\sum \frac{\partial\mathcal{F}(A)}{\partial A_i}\overline{A}_i\big) \;+\; \int d^2\theta\,\frac{1}{2}\big(\sum \frac{\partial^2\mathcal{F}(A)}{\partial A_i\partial A_j}W_{\alpha i}W_{\alpha j}\big)\Big]\;.$$

The prepotential $\mathcal{F}$ describes the geometry of the quantum moduli space $\mathcal{M}_\Lambda$, whose metric τ gives the complexified gauge coupling constant: $\tau(a) = \partial_a^2\mathcal{F}(a) = \frac{1}{\pi}\theta_{\rm eff}(a) + 8\pi i(g_{\rm eff}(a))^{-2}$.

An important point is that $\mathcal{M}_\Lambda$ has singularities where the local effective action description breaks down. This is because certain BPS monopoles become massless for the corresponding vacuum expectation values. For example, for $G = SU(2)$ [1,2], there are singularities at $u = \pm\Lambda^2$, where Λ is the dynamically generated scale of the theory, and where u is a gauge invariant coordinate of $\mathcal{M}_\Lambda$ (for large u, $u \sim \frac{1}{2}\langle a^2\rangle$). These singularities correspond to a monopole and a dyon becoming massless, respectively.

(There is also a singularity in the semi-classical region at $u = \infty$, but there are no massless states associated with it.)

Loops in $\mathcal{M}_\Lambda$ around these singularities induce non-trivial monodromy, which acts on the section

$$\pi \;=\; \begin{pmatrix} a_D \\ a \end{pmatrix} \;, \qquad a_D \;\equiv\; \frac{\partial}{\partial a}\mathcal{F}(a)$$

via matrix multiplication. A crucial insight of Seiberg and Witten was to use the global monodromy properties of π to essentially fix it and thus, via integration, to find the prepotential $\mathcal{F}(a)$. In practice, this was done by viewing π as a vector of period integrals related to an auxiliary elliptic curve, and τ as period matrix of this curve.

In section 2, we will review some ideas of Seiberg and Witten about $G = SU(2)$ Yang-Mills theory, with emphasis on the techniques that we are going to use later. In section 3, we will first describe some properties of the classical $SU(n)$ gauge theories as well as their relation to simple singularities, and then discuss their monodromy properties. In section 4 we will consider the exact $G = SU(n)$ quantum theories, which are defined in terms of certain hyperelliptic curves. In particular, we will present some details about the moduli space $\mathcal{M}_\Lambda$ and its monodromies. We will also emphasize the relationship between BPS states and the singular homology of level surfaces. In section 5 we will derive the Picard-Fuchs equations for $G = SU(3)$, whose solutions give an alternative representation of the period integrals.

In section 6 we will use these solutions to compute the series expansion of the exact quantum effective action, in both semi-classical and dual magnetic semi-classical coordinate patches of $\mathcal{M}_\Lambda$. The explicit expressions for the non-perturbative corrections represent the main results of the present paper, and may be viewed as predictions that may –in principle– be checked by some other sort of computation.

Finally, we will present some conclusions in section 7, and in two Appendices we give some details about the computation of the period integrals, and present explicit expressions for the periods.

2. A quick tour through G=SU(2)

In this section we review and detail some of the results for gauge group $SU(2)$ [1,2]. The main purpose is to familiarize the reader with the techniques that will be used later in a more involved context.

By hypothesis, the quantum moduli space $\mathcal{M}_\Lambda$ of $SU(2)$ Yang-Mills theory coincides with the moduli space of the elliptic curve

$$y^2 = W_{A_1}^2 - \Lambda^4 = (x^2 - u)^2 - \Lambda^4 \ . \tag{2.1}$$

This curve is equivalent to the curve given in the second paper of Seiberg and Witten [2], in that it has the same j-function. The prime interest is in the periods,

$$\begin{pmatrix} \varpi_D \\ \varpi \end{pmatrix} \ = \ \frac{\partial}{\partial u}\,\pi(u) \ \equiv \frac{\partial}{\partial u}\begin{pmatrix} a_D \\ a \end{pmatrix} \ \sim \ \begin{pmatrix} \oint_\beta \\ \oint_\alpha \end{pmatrix} \cdot \frac{dx}{y(x,u)} \ , \tag{2.2}$$

since the prepotential can be obtained directly from them by integration: $\mathcal{F} = \int_a a_D(a)$. The periods (2.2) are largely fixed by their monodromy properties around the singularities of $\mathcal{M}_\Lambda$, which just reflect the monodromy properties of the basis homology cycles α and β. Specifically, denoting the the four zero's of $p(x) = y^2(x)$ by $e_1^+ = -\sqrt{u+\Lambda^2}$, $e_1^- = -\sqrt{u-\Lambda^2}$, $e_2^- = \sqrt{u-\Lambda^2}$ and $e_2^+ = \sqrt{u+\Lambda^2}$, we define the basis for the homology cycles as in Fig.1.

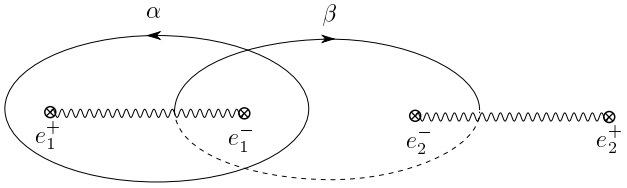


Fig.1: *The definition of the cuts and cycles for the elliptic curve (2.1) in the x-plane. This picture correspond to the choice of the basepoint $u_0 > \Lambda^2$ real.*

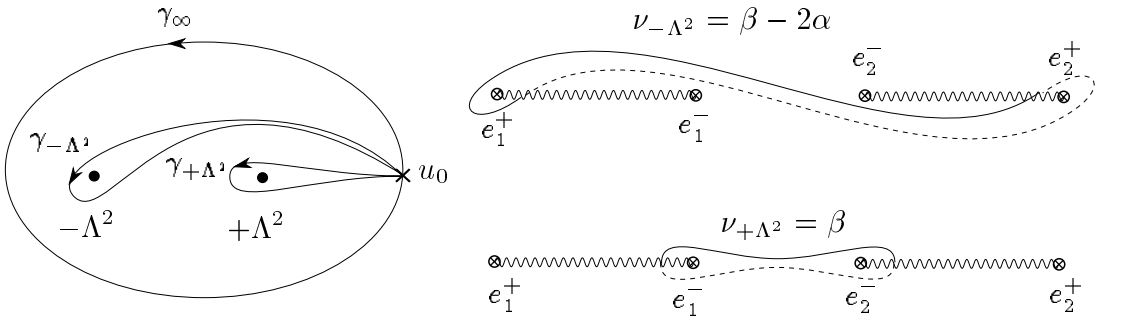
The singularities in the quantum moduli space are given by the the zeros of the discriminant of (2.1), $\Delta_\Lambda = (2\Lambda)^8(u^2 - \Lambda^4)$, and describe the following degenerations of the elliptic curve:

i_+) $u \to +\Lambda^2$, for which $(e_1^- \to e_2^-)$, i.e., the cycle $\nu_{+\Lambda^2} = \beta$ degenerates,

i_-) $u \to -\Lambda^2$, for which $(e_1^+ \to e_2^+)$, i.e., the cycle $\nu_{-\Lambda^2} = \beta - 2\,\alpha$ degenerates,

ii) $\frac{\Lambda^2}{u} \to 0$, for which $(e_1^+ \to e_1^-)$ *and* $(e_2^+ \to e_2^-)$. As two pairs of zero's coincide simultaneously, one has a "non-stable" degeneration. With some care (see below) one can conclude: $\nu_\infty = 2\,\alpha$.

Is is now easy to see that, for example, a loop $\gamma_{+\Lambda^2}$ around the singularity at $u = \Lambda^2$ makes e_1^- and e_2^- rotate around each other, so that the cycle α gets transformed into $\alpha - \beta$, c.f., Fig.2 (one can investigate the monodromy around $u = -\Lambda^2$ in an analogous way). Note that the zeros exchange along a certain path that shrinks as $e_1^- \to e_2^-$. Such paths are called vanishing cycles, and play an important rôle because they directly determine the monodromies; this will be explained in section 4.2.



Paths in the $u-$plane Corresponding vanishing cycles in the $x-$plane

Fig.2: *Monodromy paths and cycles that vanishes as one moves towards the degeneration point.*

To obtain the monodromy around $\frac{\Lambda^2}{u} \to 0$, one can compactify the $u-$plane to $\mathbb{P}^1$ with homogeneous coordinates $(u : \Lambda^2)$, and get the monodromy at infinity from the consistency condition $M_\infty = M_{+\Lambda^2} M_{-\Lambda^2}$ (Fig.2). One may also compute the monodromy at $\frac{\Lambda^2}{u} \to 0$ directly. To obtain a stable situation with a single vanishing cycle, one can employ a projective transformation $x \mapsto \frac{a\,x+b}{c\,x+d}$ that may be fixed by requiring: $(e_1^+ \to \infty, e_1^- \to -1, e_2^- \to 1)$, plus, for example:

$$e_2^+ \to \widetilde{u} = \frac{u\Lambda^2}{\sqrt{u^2 - \Lambda^4}} \ . \tag{2.3}$$

For $\frac{u}{\Lambda^2} \to \infty$, $\widetilde{u} \to 1$ i.e. the α-cycle vanishes, but since $(\widetilde{u} - 1) \sim \frac{\Lambda^2}{2u^2} + O(\frac{\Lambda^8}{u^4})$ this correspond only to half a loop in the $u-$plane i.e. the vanishing cycle for the curve (2.1) is $\nu_\infty = 2\,\alpha$. One also needs to take into account a factor -1 contributed by the form $\frac{dx}{y}$, as well as the fact that the monodromy obtained in this way corresponds to the degeneration point being encircled counter clockwise, ie., the monodromy will be given by ${M_\infty}^{-1}$.

In summary, one obtains the following monodromies:

$$M_\infty = M_{+\Lambda^2} M_{-\Lambda^2} = \begin{pmatrix} -1 & 4 \\ 0 & -1 \end{pmatrix}, \ M_{+\Lambda^2} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \ M_{-\Lambda^2} = \begin{pmatrix} -1 & 4 \\ -1 & 3 \end{pmatrix} \ , \tag{2.4}$$

which generate $\Gamma_0(4) \subset SL(2\mathbb{Z})$. These monodromies are consistent with the one-loop β-function of the weakly coupled $SU(2)$ theory, with the β-function of the magnetic dual $U(1)$ theory coupled to a massless monopole of charges $(g, q) = (1, 0)$, and with the β-function corresponding to a massless dyon of charge $(1, -2)$, respectively [2].

In order to obtain the effective action explicitly, one needs to evaluate the periods (2.2). Instead of directly computing the integrals, one may use the fact that the periods form a system of solutions of the Picard-Fuchs equation associated with (2.1). One then has to evaluate the integrals only in leading order, just to determine the correct linear combinations of the solutions. More precisely, the PF equations satisfied by the periods $(\varpi_D(u), \varpi(u)) \equiv (\partial_u a_D, \partial_u a)$ are given in terms of the second order differential operator $\mathcal{L} = (\Lambda^4 - u^2)\partial_u^2 - 2u\partial_u - \frac{1}{4}$. In terms of the dimensionless and $\mathbf{Z}_8$ invariant variable $\alpha = \frac{u^2}{\Lambda^4}$, this turns into $(\theta_\alpha = \alpha\partial_\alpha)$

$$\mathcal{L} = \theta_\alpha(\theta_\alpha - \frac{1}{2}) - \alpha(\theta_\alpha + \frac{1}{4})^2 \ , \tag{2.5}$$

which constitutes the hypergeometric system $F(\frac{1}{4},\frac{1}{4};\frac{1}{2};\alpha)$. It is also possible to derive a second order differential equation for the section $\pi \equiv (a_D, a)$ directly. In fact, one easily verifies that $\mathcal{L}\partial_u = \partial_u \widetilde{\mathcal{L}}$ with †

$$\widetilde{\mathcal{L}} = \theta_\alpha(\theta_\alpha - \frac{1}{2}) - \alpha(\theta_\alpha - \frac{1}{4})^2 \ , \tag{2.6}$$

and this form the hypergeometric system $F(-\frac{1}{4},-\frac{1}{4};\frac{1}{2},\alpha)$. One may also verify directly that $\oint \lambda$ with $\lambda = \frac{i\sqrt{2}}{4\pi} 2x^2 \frac{dx}{y}$ satisfies this equation.

The solutions of $\widetilde{\mathcal{L}}\,\pi = 0$ in terms of hypergeometric functions, and their analytic continuation over the complex plane, are of course well known. For $|u| > |\Lambda|$ a system of solutions to the Picard-Fuchs equations is given by w_0 and w_1 with

$$w_0(u) = \frac{\sqrt{u}}{\Lambda} \sum c(n) (\frac{\Lambda^4}{u^2})^n \ , \qquad c(n) = \frac{(\frac{1}{4})_n (-\frac{1}{4})_n}{(1)_n^2}$$

and

$$w_1(u) = w_0(u) \log(\frac{\Lambda^4}{u^2}) + \frac{\sqrt{u}}{\Lambda} \sum d(n) (\frac{\Lambda^4}{u^2})^n \ ,$$

where

$$d(n) = c(n)\big(2(\psi(1) - \psi(n+1)) + \psi(n+\frac{1}{4}) - \psi(\frac{1}{4}) + \psi(n-\frac{1}{4}) - \psi(-\frac{1}{4})\big)$$

and where $(a)_m \equiv \Gamma(a+m)/\Gamma(a)$ is the Pochhammer symbol. Matching the asymptotic expansions of the period integrals one finds

$$a(u) = \frac{\Lambda}{\sqrt{2}} w_0(u) \ , \qquad a_D(u) = -\frac{i\Lambda}{\sqrt{2}\pi}(w_1(u) + (4 - 6\log(2)) w_0(u)),$$

which transform under counter-clockwise continuation of u along γ_∞ (c.f., Fig.1) precisely as in (2.4). These expansions correspond to particular linear combinations of hypergeometric functions, the most concise form of which are

$$\begin{aligned} a_D(\alpha) \ &= \oint_\beta \lambda = \frac{i}{4}\Lambda(\alpha-1)\, {}_2F_1\Big(\frac{3}{4},\frac{3}{4},2;1-\alpha\Big) \\ a(\alpha) \ &= \oint_\alpha \lambda = \frac{1}{1+i}\Lambda(1-\alpha)^{1/4}\, {}_2F_1\Big(-\frac{1}{4},\frac{3}{4},1;\frac{1}{1-a}\Big) \ . \end{aligned} \tag{2.7}$$

† The fact that the operator $\mathcal{L}\partial_u$ has the alternative factorization, $\partial_u\widetilde{\mathcal{L}}$, means that (a_D, a) transform irreducibly under monodromy. In the massive case this will no longer be case, and the three solutions (a_D, a, const) will mix under monodromy.

From these expressions, the prepotential in the semi-classical regime (near infinity in the moduli space) can readily be obtained to any given order. Inverting $a(u)$ as series for large a/Λ yields for the first few terms $\frac{u(a)}{\Lambda^2} = 2\left(\frac{a}{\Lambda}\right)^2 + \frac{1}{16}\left(\frac{\Lambda}{a}\right)^2 + \frac{5}{4096}\left(\frac{\Lambda}{a}\right)^6 + O(\left(\frac{\Lambda}{a}\right)^{10})$. After inserting this into $a_D(u)$, one obtains $\mathcal{F}$ by integration w.r.t. a as follows:

$$\mathcal{F} = \frac{i\,a^2}{2\pi}\left(2\log\frac{a^2}{\Lambda^2} - 6 + 8\log 2 - \sum_{k=1}^{\infty}\mathcal{F}_k\left(\frac{\Lambda}{a}\right)^{4k}\right) .$$

Specifically, the first few terms of the instanton expansion are:

k	1	2	3	4	5	6	7	8
$\mathcal{F}_k$	$\frac{1}{2^5}$	$\frac{5}{2^{14}}$	$\frac{3}{2^{18}}$	$\frac{1469}{2^{31}}$	$\frac{4471}{2^{34}\cdot 5}$	$\frac{40397}{2^{43}}$	$\frac{441325}{2^{47}\cdot 7}$	$\frac{866589165}{2^{64}}$

One can treat the dual semi-classical regime is an analogous way. Near the point $u = \Lambda^2$ where the monopole becomes massless, we introduce $z = (u - \Lambda^2)/(2\Lambda^2)$ and rewrite the Picard-Fuchs operator as

$$\mathcal{L} = z(\theta_z - \frac{1}{2})^2 + \theta_z(\theta_z - 1) \tag{2.8}$$

At $z = 0$, the indices are 0 and 1, and we have again one power series

$$w_0(z) = \Lambda^2 \sum c(n) z^{n+1}, \qquad c(n) = (-1)^n \frac{(\frac{1}{2})_n^2}{(1)_n (2)_n} \tag{2.9}$$

and a logarithmic solution

$$w_1(z) = w_0(z)\log(z) + \sum d(n) z^{n+1} - 4 \ , \tag{2.10}$$

with

$$d(n) = c(n)\Big(2(\psi(n+\frac{1}{2}) - \psi(\frac{1}{2})) + \psi(1) - \psi(n+1) + \psi(2) - \psi(n+2)\Big) .$$

For small z one can easily evaluate the lowest order expansion for the integrals (2.7) and thereby determine the analytic continuation of the solutions from the weak coupling to the strong coupling domain:

$$\begin{aligned} a_D &= 2\int_{e_1^-}^{e_2^-} \lambda \;=\; i\Lambda(z + \ldots) \;=\; i\Lambda w_0(z) \\ a &= 2\int_{e_1^+}^{e_1^-} \lambda \;=\; \frac{\Lambda}{2\pi}(4 + z(1 + 4\log(2)) - z\log(z) + \ldots) \\ &= -\frac{\Lambda}{2\pi}(w_1(z) - (1 + \log(2)) w_0(z)) . \end{aligned}$$

This exhibits the monodromy of (2.4) along the path $\gamma_{+\Lambda^2}$. Inverting $a_D(z)$ yields $z(a_D) = -2\widetilde{a}_D + \frac{1}{4}\widetilde{a}_D^2 + \frac{1}{32}\widetilde{a}_D^3 + \mathcal{O}(\widetilde{a}_D^4)$, with $\widetilde{a}_D \equiv ia_D/\Lambda$. After inserting this into $a(z)$ we integrate w.r.t. a_D and obtain the dual prepotential $\mathcal{F}_D$ as follows:

$$\mathcal{F}_D = \frac{i\Lambda^2}{2\pi}\left(\widetilde{a}_D^2 \log\Big[-\frac{i}{4}\sqrt{\widetilde{a}_D}\Big] + \sum_{k=1}^{\infty} \mathcal{F}_{D\,k}\,\widetilde{a}_D^k\right),$$

where the lowest threshold corrections $\mathcal{F}_{D\,k}$ are

k	1	2	3	4	5	6	7	8
$\mathcal{F}_{D\,k}$	4	$-\frac{3}{4}$	$\frac{1}{2^4}$	$\frac{5}{2^9}$	$\frac{11}{2^{12}}$	$\frac{63}{2^{16}}$	$\frac{527}{2^{18}\cdot 5}$	$\frac{3129}{2^{24}}$

Now slightly changing direction, remember that in the first paper of Seiberg and Witten [1], an elliptic curve different from the one in the second paper [2] was considered. This "isogenous" curve has the form

$$y^2 \;=\; (x-\Lambda^2)(x+\Lambda^2)(x-\widetilde{u})\;, \tag{2.11}$$

and leads to the monodromy group $\Gamma(2)$. The motivation for introducing the $\Gamma_0(4)$ curve (2.1) was to have a more convenient electrical charge normalization, and according to [2], the difference between the curves (2.11) and (2.1) just accounts for this. This poses, however, a paradox: by comparing the Weierstraß normal forms, one finds that the parameters u in (2.1) and $\widetilde{u}$ in (2.11) are related as follows:[†]

$$u \;=\; \frac{\widetilde{u}}{\sqrt{\widetilde{u}^2-\Lambda^4}}\Lambda^2\;. \tag{2.12}$$

This means, however, that the semi-classical regions near infinity and near the finite points are exchanged for the two curves, which also means that the electric and magnetic sectors are exchanged. How can this be reconciled with the statement that the two curves just differ the normalization of the electric charge ?

The point is that (2.12) represents a ($\mathbb{Z}_2$-valued) duality transformation. That is, even though π and $\mathcal{F}(a), \mathcal{F}_D(a_D)$ transform in a complicated way under

$$I: \qquad \alpha \;\longrightarrow\; \widetilde{\alpha} \equiv \frac{\alpha}{\alpha-1}\;, \qquad\qquad \alpha \equiv \frac{u^2}{\Lambda^4}\;, \tag{2.13}$$

† This is exactly the $SL(2)$ transformation (2.3).

the physical gauge and dual gauge couplings, τ and τ_D, behave in a simple way. More precisely, we find that one can compactly write the periods $\partial_u \pi$ as follows:

$$\begin{aligned} \varpi_D(\alpha) &= \frac{i\sqrt{2}}{4\pi}\oint_\beta \frac{dx}{y} = (\frac{i}{2\Lambda})_2F_1\Big(\frac{1}{4},\frac{1}{4},1;1-\alpha\Big) \\ \varpi(\alpha) &= \frac{i\sqrt{2}}{4\pi}\oint_\alpha \frac{dx}{y} = \frac{1}{2\Lambda(1-i)}(1-\alpha)^{-1/4}{}_2F_1\Big(\frac{1}{4},\frac{1}{4},1;\frac{1}{1-\alpha}\Big)\ . \end{aligned} \tag{2.14}$$

In this particular representation, the arguments of the hypergeometric functions simply exchange under the transformation (2.13), and one finds for expansions around $(1-\alpha) \in \mathbb{R}^-$ that

$$\begin{aligned} \tau(\alpha) \equiv \frac{\varpi_D(\alpha)}{\varpi(\alpha)} &= -2\frac{\varpi(\widetilde{\alpha})}{\varpi_D(\widetilde{\alpha})} \equiv 2\,\tau_D(\widetilde{\alpha}) \\ &= \widetilde{\tau}_D(\widetilde{\alpha})\ . \end{aligned} \tag{2.15}$$

Here, $\widetilde{\tau}_D(\widetilde{\alpha})$ is the dual coupling corresponding to the $\Gamma(2)$ curve (2.11). (For expanding around the fixed point, $\alpha = 0$, one has to take into account that because of $(1-\alpha) \notin \mathbb{R}^-$ one picks up an extra overall phase, ie., $\tau(u \sim 0) = (i+1) + \mathcal{O}(u) = -2i\tau_D(\widetilde{u} \sim 0)$.) We thus see that rescaling the electric charges, performing the isogeny map I (exchanging the curves) and exchanging the electric and magnetic sectors is the identity map. In fact, (2.13) can be viewed as the effect of the transformation $I:\ \tau \to -\frac{2}{\tau}$ on the $\Gamma_0(4)$ modular function u. This is not to be confused with the $SL(2,\mathbb{Z})$ S-duality transformation $S:\ \tau \to -\frac{1}{\tau}$, under which the period matrix is invariant:

$$\tau(\alpha) = \tau_D(\widehat{\alpha})\ ,\quad \widehat{\alpha} = \frac{\left(3\sqrt{\alpha-1}+\sqrt{\alpha}\right)^2}{8\left(\sqrt{\alpha(\alpha-1)}+\alpha-1\right)}\ . \tag{2.16}$$

Note that since (2.15), (2.16) act non-trivially on the physical moduli space, these transformations are not a symmetries of the theory, but rather relate the different semi-classical regimes to each other.

3. Semi-classical Yang-Mills theory

3.1. Classical moduli spaces and Simple Singularities

Before we jump in the discussion of the exact quantum theory, we first explain some properties of the classical and semi-classical (perturbative) theory. Most of these

features, especially the relation to simple singularities, directly generalize to groups other than $G = SU(n)$.

The classical potential for the scalar superfield component ϕ is given by

$$V(\phi) \;=\; \frac{1}{g^2}[\phi, \phi^\dagger]^2 \;. \tag{3.1}$$

It leads to a continuous family of inequivalent ground states, which constitutes the classical moduli space $\mathcal{M}_0$. In order to characterize $\mathcal{M}_0$, note that one can always rotate ϕ into the Cartan sub-algebra,

$$\phi = \sum_{k=1}^{n-1} a_k H_k \tag{3.2}$$

with $H_k = E_{k,k} - E_{k+1,k+1}$, $(E_{k,l})_{i,j} = \delta_{ik}\delta_{jl}$. For generic eigenvalues of ϕ, the $SU(n)$ gauge symmetry is broken to the maximal torus $U(1)^{n-1}$, whereas if some eigenvalues coincide, some larger, non-abelian group $H \subseteq G$ remains unbroken. Precisely which gauge bosons are massless for a given background $\vec{a} = \{a_k\}$, can easily be read off from the central charge formula,

$$Z_q(a) \;=\; \vec{q}\cdot\vec{a}\;, \qquad \text{with} \quad m^2(q) = 2|Z_q|^2\;, \tag{3.3}$$

where we take for the charge vectors $\vec{q}$ the roots $\alpha \in \Lambda_R(G)$ in Dynkin basis.

The Cartan sub-algebra variables a_k are not gauge invariant and in particular not invariant under discrete Weyl transformations. Therefore, one introduces other variables for parametrizing the classical moduli space, which are given by Weyl invariant Casimirs $u_k(a)$. These variables parametrize the Cartan sub-algebra modulo the Weyl group, ie, $\{u_k\} \cong \mathbb{C}^{n-1}/W_{A_{n-1}}$, and transform under the the anomaly free global $\mathbb{Z}_{2n}$ subgroup of $U(1)_R$ as $u_k \to e^{i\pi k/n}u_k$. More precisely, in order to go to the Casimir variables $u_k(a)$, we first change basis according to

$$Z_{\alpha_{(i,j)}}(a) \;\equiv\; e_i - e_j\;, \quad i \neq j\;, i,j = 1,n, \tag{3.4}$$

with $\sum e_i = 0$. These variables are then related to the Casimir variables u_k by a Miura transformation:

$$\prod_{i=1}^{n}\big(x - e_i(a)\big) \;=\; x^n - \sum_{l=0}^{n-2} u_{l+2}(a)\, x^{n-2-l} \;\equiv\; \mathcal{W}_{A_{n-1}}(x,u)\;. \tag{3.5}$$

Here, $\mathcal{W}_{A_{n-1}}(x,u)$ is nothing but the *simple singularity* [11] associated with $SU(n)$, where

$$\begin{aligned} u_k(a) &= (-1)^{k+1} \sum_{j_1 \neq \ldots \neq j_k} e_{j_1} e_{j_2} \ldots e_{j_k}(a) \\ &\equiv \frac{1}{k} \mathrm{Tr} \langle \phi^k \rangle + \text{products of lower order Casimirs} \end{aligned} \tag{3.6}$$

are the symmetric polynomials. These are manifestly invariant under the Weyl group $W_{A_{n-1}} \cong S(n)$, which acts by permutation of the e_i. Specifically, in terms of the original variables a_k, one has for the bottom and top Casimirs:

$$u_2(a) = \tfrac{1}{2n} \sum_{\substack{\text{positive} \\ \text{roots } \alpha}} (Z_\alpha)^2 = \tfrac{1}{2}\, \vec{a}^{\,t} \cdot C \cdot \vec{a} \qquad u_n(a) = (-1)^{n+1} \prod_{\substack{\text{fund. rep} \\ \text{weights } \lambda}} Z_\lambda(a) \ , \tag{3.7}$$

where C is the Cartan matrix of $SU(n)$. In addition, let us note for later reference that for $G = SU(3)$:

$$\begin{aligned} u(a_1,a_2) &\equiv u_2 = {a_1}^2 + {a_2}^2 - a_1 a_2 \\ v(a_1,a_2) &\equiv u_3 = a_1 a_2 (a_1 - a_2) \\ a_1(u,v) &\equiv e_1(u,v) = \xi_+ + \xi_- \\ a_2(u,v) &\equiv -e_2(u,v) = e^{-2\pi i/6} \xi_+ + e^{2\pi i/6} \xi_- \ , \quad \text{where} \\ \xi_\pm(u,v) &\equiv 2^{-1/3} \sqrt[3]{v \pm \sqrt{v^2 - \tfrac{4}{27} u^3}} \ , \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} Z_1 &\equiv Z_{(2,-1)} = 2a_1 - a_2 = e_1 - e_3 \\ Z_2 &\equiv Z_{(-1,2)} = 2a_2 - a_1 = e_3 - e_2 \\ Z_3 &\equiv Z_{(1,1)} = a_1 + a_2 = e_1 - e_2 \ . \end{aligned} \tag{3.9}$$

From the above we know that whenever $e_i = e_j$ for some i and j, there are, classically, extra massless non-abelian gauge bosons, since $Z_\alpha = 0$ for some root α. For such backgrounds the effective action becomes singular. The classical moduli space is thus given by the space of Weyl invariant deformations modulo such singular regions: $\mathcal{M}_0 = \{u_k\} \backslash \Sigma_0$. Here, $\Sigma_0 \equiv \{u_k : \Delta_0(u_k) = 0\}$ is the zero locus of the "classical" discriminant

$$\Delta_0(u) = \prod_{i<j}^{n} (e_i(u) - e_j(u))^2 \equiv \prod_{\substack{\text{positive} \\ \text{roots } \alpha}} (Z_\alpha)^2(u) \ , \tag{3.10}$$

of the simple singularity (3.5). Specifically, one has up to normalization:[†]

$$\begin{aligned} SU(2): \quad \Delta_0 &= u \\ SU(3): \quad \Delta_0 &= 4u^3 - 27v^2 \\ SU(4): \quad \Delta_0 &= 27\,v^4 - 4\,u^3\,v^2 + 16\,u^4\,w - 144\,u\,v^2\,w + 128\,u^2\,w^2 + 256\,w^3 \end{aligned} \tag{3.11}$$

We schematically depicted these singular loci $\Delta_0(u) = 0$ in Fig.3.

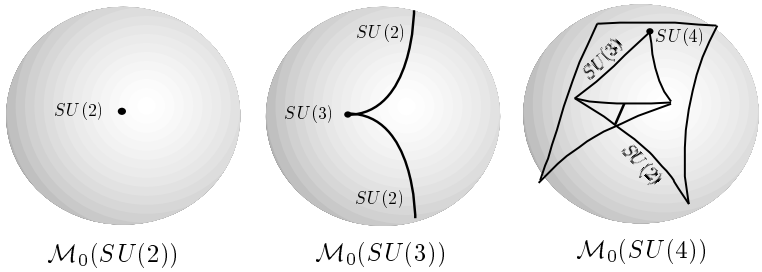


Fig.3: *Singular loci Σ_0 in the classical moduli spaces $\mathcal{M}_0$ of pure $SU(n)$ $N=2$ Yang-Mills theory. They are nothing but the bifurcation sets of the type A_{n-1} simple singularities, and reflect all possible symmetry breaking patterns in a gauge invariant way (for $SU(3)$ and $SU(4)$ we show only the real parts). The picture for $SU(4)$ is known in singularity theory as the "swallowtail".*

The discriminant loci Σ_0 are generally given by intersecting hypersurfaces of complex codimension one. On each such surface one has $Z_\alpha = 0$ for some pair of roots $\pm\alpha$, so that there is an unbroken $SU(2)$, and the Weyl group action $r_\alpha : Z_\alpha \to -Z_\alpha$ is singular. On intersections of these hypersurfaces one has, correspondingly, larger unbroken gauge groups. All planes together intersect in just one point, namely in the origin, where the gauge group $SU(n)$ is fully restored. Thus, all possible classical symmetry breaking patterns are encoded in the discriminants of $\mathcal{W}_{A_{n-1}}(x,u)$.

Finally, note that for a general singularity with n variables, the discriminant locus Σ_0 coincides with what is called the level bifurcation set of a singularity $\mathcal{W}(x_i,u)$, because on it the level surface

$$V_u \;=\; \left\{\, x_i \,:\, \mathcal{W}(x_i,u) = 0,\ ||x|| \le \epsilon \right\} \tag{3.12}$$

† We will often denote u_2, u_3, u_4 by u, v, w.

becomes singular [11]. More specifically, V_u becomes singular in that certain homology cycles $\nu \in H_{n-1}(V_u, \mathbb{Z})$ shrink to zero. Such cycles ν are called vanishing cycles. For the case at hand, the level surface determined by $\mathcal{W}_{A_{n-1}}(x,u) = 0$ is zero dimensional and given by the set of points $V_u = \{e_i,\, i = 1, \ldots, n\}$. This space is singular if any two of the $e_i(u)$ coincide, and indeed, the vanishing cycles are given by the differences, $\nu_{i,j} = e_i - e_j$, ie., by the central charges Z_α. They generate the root lattice: $H_0(V_u, \mathbb{Z}) \cong \Lambda_R$.

What we learn is that the classically massless non-abelian gauge bosons are directly related to vanishing cycles of level surfaces. This reflects an apparent general property of BPS states and, as will be explained below, generalizes in particular to massless magnetic monopoles and dyons in the exact quantum theory, where the relevant "level" surfaces are given by special Riemann surfaces.

*3.2. Classical and semi-classical monodromy**

Since the map $u_k \to a_k$ is multi-valued and since u_k are Weyl group invariant, closed paths in $\mathcal{M}_0$ space will in general be closed in $\{a_k\}$-space only up to Weyl transformations. This means that the classical part of the monodromy group is given by the corresponding Weyl group, which is a well-known fact in the theory of simple singularities [11]. More precisely, the singular locus Σ_0 has various branches that are the images of the lines $Z_\alpha = 0$ in a-space, and encircling such a branch will induce a classical monodromy given by a Weyl reflection corresponding to the root α. In addition, we have monodromy acting on the dual magnetic variables,

$$a_{Di} \;\equiv\; \frac{\partial}{\partial a_i}\mathcal{F}(a)\;, \tag{3.13}$$

which is dual to the monodromy acting on the a_i. Hence the total classical monodromy is

$$\begin{pmatrix} \vec{a}_D \\ \vec{a} \end{pmatrix} \;\longrightarrow\; P^{(r)}\cdot\begin{pmatrix} \vec{a}_D \\ \vec{a} \end{pmatrix}\;, \qquad \text{where} \quad P^{(r)} \;=\; \begin{pmatrix} (r^{-1})^t & 0 \\ 0 & r \end{pmatrix}\;, \tag{3.14}$$

with $r \in W_G$. Which specific Weyl transformation actually occurs depends of course on the specific closed path in $\mathcal{M}_0$. It is clear that all possible classical monodromies

* While preparing the manuscript, we received the preprint [7] with results that overlap with some results of this section.

can be generated by loops associated with the fundamental Weyl reflections. These have the following matrix representations:

$$r_i \;=\; \mathbb{1} - \alpha_i \otimes \lambda_i \;, \quad i = 1, ..., n-1 \;, \tag{3.15}$$

where α_i, λ_i are the simple roots and fundamental weights in Dynkin basis.

In addition to the classical monodromy (3.14), there is a semi-classical contribution ("θ-shift") from the logarithmic piece of the one-loop effective action. Specifically, the general formula for the one-loop correction to the gauge coupling constant is $\delta(\frac{1}{g^2})_{\alpha\beta} \sim \mathrm{Tr}(T_\alpha T_\beta \log[m^2(a)])$, where $m^2(a)$ is the vev-dependent mass matrix of the non-abelian gauge bosons, and T_α denotes a generator in the adjoint representation. From this one obtains the one-loop correction to the prepotential as follows:

$$\mathcal{F}_{\text{1-loop}}(a(u)) \;=\; \frac{i}{4\pi} \sum_{\substack{\text{positive} \\ \text{roots } \alpha}} (Z_\alpha)^2 \log\left[(Z_\alpha)^2/\Lambda^2\right] \;. \tag{3.16}$$

The scale parameter Λ is arbitrary and reflects the breakdown of conformal invariance at the quantum level. Note that the gauge coupling constant, $\tau_{ij} = \partial_{a_i}\partial_{a_j}\mathcal{F}$, blows up logarithmically precisely when $\Delta_0 = 0$, ie., whenever there are massless charged fields in the theory that lead to an IR divergence. Note also that even though $\mathcal{F}_{\text{1-loop}}$ is manifestly Weyl group invariant, it does not have a simple form in terms of the Casimir variables u_k.

The effect of $\mathcal{F}_{\text{1-loop}}$ is to modify the classical monodromy by a perturbative quantum piece. That is, the monodromy around a singular line in $\mathcal{M}_0$ that is the image of some singular line $Z_\alpha = 0$ in weight space, will induce a $2\pi i$ phase contribution from $\log(Z_\alpha)^2$ in $\partial_a \mathcal{F}_{\text{1-loop}}(a(u))$. Though there exist in general more complicated paths, it suffices [7] to consider the following "fundamental" monodromies that generate the semi-classical monodromy group:

$$M^{(r_i)} \;=\; P^{(r_i)} \cdot {T_{\alpha_i}}^{-1} \tag{3.17}$$

with theta-shifts given by †

$$T_{\alpha_i} \;=\; \begin{pmatrix} \mathbb{1} & \alpha_i \otimes \alpha_i \\ 0 & \mathbb{1} \end{pmatrix} \;. \tag{3.18}$$

† Note that the off-diagonal terms can be modified at wish by a change of homology basis, ie., $a_{Di} \to a_{Di} + h_{ij}a_j$. A different basis was used in the discussion of the semi-classical monodromy in [3,10].

For our discussion of the Picard-Fuchs equations, we will be particularly interested in the monodromy associated with a large loop (around "infinity") in a given u_l-plane, the other u_k being held fixed. Such a monodromy is given by a certain product of the generators (3.17), depending on which sub-set of branches of Σ_0 is actually encircled by the large loop. Though the precise monodromy transformation depends on the chosen basepoint u_0, the conjugacy class of the Weyl transformation just depends on the given u_l-plane. That is, for large loops $u_l \to e^{2\pi i t} u_l$, where $t \in [0,1]$, the classical Weyl monodromy is of order l. In particular, for large loops in the u_2-plane, the monodromy at infinity is given by a single Weyl reflection, while for large loops in the top Casimir plane, u_n, the monodromy is given by a Coxeter element.

One can also consider monodromy induced by a rotation of the quantum scale Λ. Indeed, as will be discussed later, one may view Λ as an additional coupling constant that may be used to compactify the moduli space. A rotation $\Lambda^{2n} \to e^{2\pi i t}\Lambda^{2n}$, $t \in [0,1]$, induces a mere θ-shift, and from (3.16) and (3.7) one obtains the following matrix representation:

$$T \;=\; \begin{pmatrix} \mathbb{1} & C \\ 0 & \mathbb{1} \end{pmatrix} \;. \tag{3.19}$$

This may be viewed as "pure" quantum monodromy. Note that from the point of view of R-symmetry, only θ-shifts associated with T^2 are allowed.

To give an example, we have for $G = SU(3)$ the following semi-classical monodromies, corresponding the three singular lines in $\mathcal{M}_0$:

$$\begin{aligned} M^{(r_1)} \;&=\; \begin{pmatrix} -1 & 0 & 4 & -2 \\ 1 & 1 & -2 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ M^{(r_2)} \;&=\; \begin{pmatrix} 1 & 1 & 1 & -2 \\ 0 & -1 & -2 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \\ M^{(r_3)} \;\equiv\; M^{(r_2)} M^{(r_1)} (M^{(r_2)})^{-1} \;&=\; \begin{pmatrix} 0 & -1 & 1 & 4 \\ -1 & 0 & -2 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \end{aligned} \tag{3.20}$$

These three singular lines correspond of course to the three ways of embedding $SU(2) \hookrightarrow SU(3)$, and indeed, the matrices appropriately contain the semi-classical

monodromy in (2.4) of $SU(2)$ Yang-Mills theory as sub-matrices. They are related via conjugation by the Coxeter element

$$U \;=\; \begin{pmatrix} -1 & -1 & 1 & -2 \\ 1 & 0 & -2 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \;, \qquad U^3 = \mathbb{1} \;, \tag{3.21}$$

which represents the global symmetry of $\mathbb{Z}_3$ rotations acting on u. Since all three lines cut the $v = const.$-plane, but only two lines cut the $u = const.$-plane, we have two different kinds of monodromies at "infinity":

$$\begin{aligned} u-\text{plane}: \quad M^{(r_2)}_{\infty,u} \;&\equiv\; M^{(r_3)}M^{(r_2)}M^{(r_1)} \;=\; \begin{pmatrix} 1 & 1 & -3 & 0 \\ 0 & -1 & -6 & 6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \\ v-\text{plane}: \quad M^{(r_{cox})}_{\infty,v} \;&\equiv\; M^{(r_1)}M^{(r_2)} \;=\; \begin{pmatrix} -1 & -1 & 1 & 4 \\ 1 & 0 & -2 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \end{aligned} \tag{3.22}$$

(up to conjugation, depending on base point and offset of the chosen plane).

4. Quantum Yang-Mills Theory

4.1. Hyperelliptic curves and quantum moduli spaces

The issue is to construct auxiliary curves $\mathcal{C}$ whose moduli spaces give the supposed quantum moduli spaces of $SU(n)$ Yang-Mills theories. Such curves were found in [3,4] and are given by:

$$\mathcal{C}: \quad y^2 = p(x) \;=\; \big(W_{A_{n-1}}(x,u_i)\big)^2 - \Lambda^{2n} \;\equiv\; (x^n - \sum_{i=2}^{n} u_i x^{n-i})^2 - \Lambda^{2n} \;. \tag{4.1}$$

Since $p(x)$ factors into $W_{A_{n-1}} \pm \Lambda^n$, the situation is in some respect like two copies of the classical theory, with the top Casimir u_n shifted by $\pm\Lambda^n$. Specifically, the points e_i of the classical level surface split,

$$e_i(u_k) \;\to\; e_i^{\pm}(u_k,\Lambda) \equiv\; e_i(u_2,,...,u_{n-1},u_n \pm \Lambda^n) \;, \tag{4.2}$$

and become the $2n$ branch points of the hyperelliptic curve (4.1). The curve can thus be represented by the two-sheeted x-plane with cuts running between pairs e_i^+ and e_i^-. In addition, the "quantum" discriminant, whose zero locus Σ_Λ gives the singularities in the quantum moduli space $\mathcal{M}_\Lambda$, is easily seen to factorize as follows:

$$\begin{aligned} \Delta_\Lambda(u_k,\Lambda) &\equiv \prod_{i<j}(e_i^+ - e_j^+)^2(e_i^- - e_j^-)^2 \;=\; \text{const.}\,\Lambda^{2n^2}\,\delta_+\,\delta_-\ , \quad \text{where} \\ \delta_\pm(u_k,\Lambda) &= \Delta_0(u_2,...,u_{n-1},u_n \pm \Lambda^n)\ , \end{aligned} \tag{4.3}$$

is the shifted classical discriminant (3.10). Thus, Σ_Λ consists of two copies of Σ_0, shifted by $\pm\Lambda^n$ in the u_n direction. Obviously, for $\Lambda \to 0$, the classical moduli space is recovered: $\Sigma_\Lambda \to \Sigma_0$. That is, when the quantum corrections are switched on, a single isolated branch of Σ_0 (associated with massless gauge bosons of a particular $SU(2)$ subgroup) splits into two branches of Σ_Λ (describing massless Seiberg-Witten monopoles related to this $SU(2)$).

Specifically, for $G = SU(3)$ the curve is

$$y^2 = p(x) = (x^3 - ux - v)^2 - \Lambda^6\ , \tag{4.4}$$

and this leads to the following quantum discriminant:

$$\Delta_\Lambda = \Lambda^{18}(4u^3 - 27(v+\Lambda^3)^2)(4u^3 - 27(v-\Lambda^3)^2)\ . \tag{4.5}$$

The corresponding singular locus Σ_Λ of the quantum moduli space is depicted in Fig.4. Explicit expressions for the branch points $e_i^\pm(u,v,\Lambda)$ can easily be inferred from (3.8) and (4.2).

The genus of the hyperelliptic curve $\mathcal{C}$ (4.1) is equal to $g = n-1$, so that its $2n-2$ periods can naturally be associated with

$$\vec{\pi} \;\equiv\; \begin{pmatrix} \vec{a}_D \\ \vec{a} \end{pmatrix}\ . \tag{4.6}$$

More precisely, on such a curve there are $n-1$ holomorphic differentials (abelian differentials of the first kind) [12] $\omega_{n-i} = \frac{x^{i-1}\,dx}{y}$, $i = 1,\ldots,g$, out of which one can construct $n-1$ sets of periods $\int_{\gamma_j}\omega_i$. (Here γ_j, $j = 1,\ldots,2g$, is any basis of $H_1(\Sigma_g,\mathbf{Z})$.) All periods together can be combined in the $(g,2g)$-dimensional period matrix

$$\Pi_{ij} = \int_{\gamma_j}\omega_i\ . \tag{4.7}$$

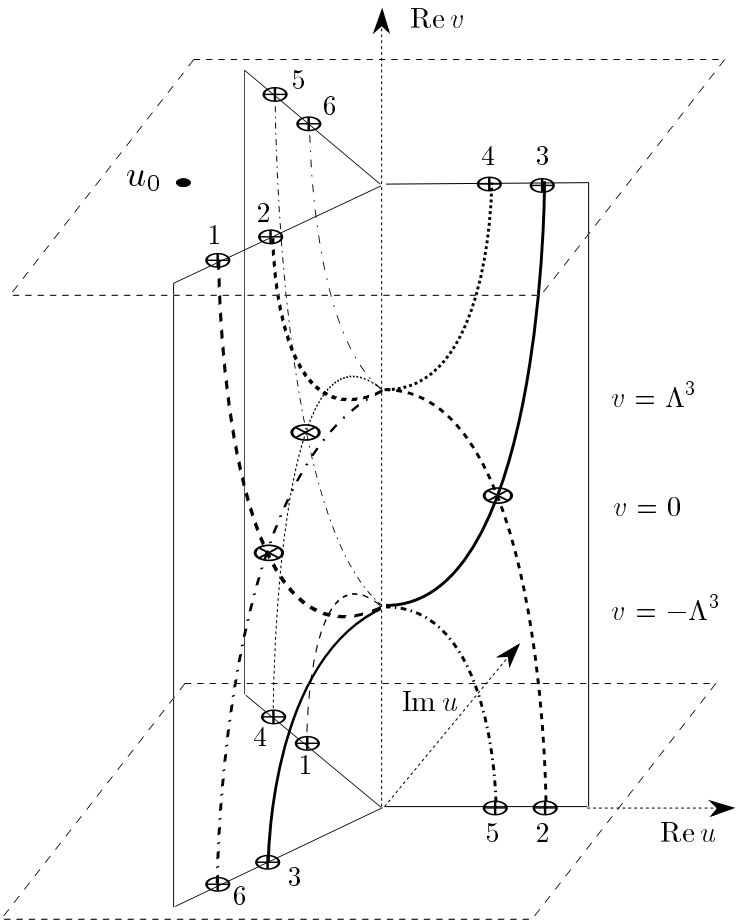


Fig.4: *Quantum moduli space for $G = SU(3)$ at real v. The six lines are the singular loci where $\Delta_\Lambda = 0$ and where certain dyons become massless. Asymptotically, for $\Lambda \to 0$ the classical moduli space is recovered. Each of the six pairs of outgoing lines represents a copy of the $SU(2)$ strong coupling singularities. The various markings of the lines indicate how the association with particular monodromy matrices changes when moving through the cusps, and u_0 is the basepoint that defines our monodromies.*

If we chose a symplectic homology basis, i.e. $\alpha_i = \gamma_i$, $\beta_i = \gamma_{g+i}$, $i = 1, \ldots, g$, with intersection pairing[†] $(\alpha_i \cap \beta_i) = \delta_{ij}$, $(\alpha_i \cap \alpha_j) = (\beta_i \cap \beta_j) = 0$, and if we write $\Pi =$

† We use the convention that a crossing between the cycles α, β counts positively to the intersection $(\alpha \cap \beta)$, if looking in the direction of the arrow of α the arrow of β points to the right.

(A, B), then $\tau \equiv A^{-1}B$ is the metric on the quantum moduli space. By Riemann's second relation $\mathrm{Im}\tau \equiv 8\pi^2/g_{eff}^2$ is positive, which is important for unitarity of the effective $N = 2$ supersymmetric gauge theory.

The precise relation between the periods and the components of the section $\vec{\pi}$ is given by:

$$\begin{aligned} A_{ij} &= \int_{\alpha_j} \omega_i = \frac{\partial}{\partial u_{i+1}} a_j \\ B_{ij} &= \int_{\beta_j} \omega_i = \frac{\partial}{\partial u_{i+1}} a_{D_j} \end{aligned} \tag{4.8}$$

(where $i, j = 1, \ldots, n-1$). From the explicit expression (4.1) for the family of hyperelliptic curves, one immediately verifies that the integrability conditions $\partial_{i+1} A_{jk} = \partial_{j+1} A_{ik}$, $\partial_{i+1} B_{jk} = \partial_{j+1} B_{ik}$ are satisfied. It also follows that $\tau_{ij} \equiv \partial_{a_i} \partial_{a_j} \mathcal{F}(a)$. This reflects the special geometry of the quantum moduli space, and implies that the components of $\vec{\pi}$ can directly be expressed as integrals over a suitably chosen abelian differential of the second kind λ:

$$a_{D_i} = \int_{\beta_i} \lambda\,, \qquad a_i = \int_{\alpha_i} \lambda \tag{4.9}$$

Indeed, one verifies that e.g.,

$$\lambda = \text{const.}\, \frac{1}{2\pi i} \Big(\frac{\partial}{\partial x} W_{A_{n-1}}(x, u_i) \Big) \frac{x\, dx}{y} \tag{4.10}$$

does the job [4] (as well as the choice of λ given in [3]). The normalization is fixed by matching a_i to the Casimirs u_k in the semi-classical limit (c.f. below).

In order to compute the exact quantum effective action $\mathcal{F}(A)$, one first needs to determine the section $\vec{\pi} = (\vec{a}_D, \vec{a})^t$ in terms of the coordinates u_k. One way to do this would be to evaluate the integrals (4.9) explicitly. This was done in ref. [5] to lowest order near the point in moduli space where there are massless dyons. However, it is rather difficult to perform the period integrals explicitly or, at least, in a series expansion to higher order. This would be needed for obtaining the instanton corrections in the effective Lagrangian.

In fact, the integrals $\int_{\gamma_i} \omega$, where ω is any abelian diffential of the second kind, satisfy as functions of the moduli u_i a system of partial differental equations, namely the Picard-Fuchs equations. It turns out that it is relatively straightforward to derive and solve these equations, and this will be discussed below in section 5. Note, though, that this will not save us completely from having to evaluate some integrals, because that will be necessary in order to identify the correct linear combinations of the solutions. However, for this one needs to compute the integrals only to low order. This is done in Appendix A, and will be used in section 6.

4.2. BPS states, vanishing cycles and Picard-Lefshetz monodromy

We will now discuss how one can obtain the quantum numbers of the various massless BPS dyons, as well as the associated strong coupling monodromies. The basic point is to relate all quantum numbers and monodromy properties to properties of the homology cycles $\vec{\alpha}, \vec{\beta} \in H_1(\Sigma_g, \mathbb{Z})$, which are involved in the definition of the periods $\vec{\pi}$ (4.9). Specifically, we expect certain dyons to be massless on the various branches of the singular locus Σ_Λ in the quantum moduli space. On any such isolated branch, the surface Σ_g becomes singular in that a particular homology cycle ν vanishes; for a schematic sketch, see Fig.5. Now, any such cycle can be expanded in terms of the basis cycles as follows:

$$\nu \;=\; \vec{q}\cdot\vec{\alpha} + \vec{g}\cdot\vec{\beta}\;, \qquad q_i, g_i \in \mathbb{Z}\;. \tag{4.11}$$

Since this cycle vanishes, it immediately follows that

$$\begin{aligned} 0 \;=\; \int_\nu \lambda \;&=\; \big(\vec{q}\int_{\vec{\alpha}} + \vec{g}\int_{\vec{\beta}}\big)\,\lambda \\ &=\; \vec{q}\cdot\vec{a} + \vec{g}\cdot\vec{a}_D \\ &\equiv Z_{(\vec{q},\vec{g})}\;. \end{aligned} \tag{4.12}$$

Here, Z is the central charge that enters in the BPS mass formula: $m^2 = 2|Z|^2$. This means that on the branch of the singular locus where some cycle ν as defined in (4.11) vanishes, a dyon with (magnetic,electric) charges equal to $\vec{\nu} = (\vec{g}, \vec{q})$ becomes massless. Clearly, under a change of homology basis, the charges change as well, but this is nothing but a duality rotation. What remains invariant is the intersection number

$$\nu_i \cap \nu_j \;=\; \vec{\nu}^t\cdot\Omega\cdot\vec{\nu} \;= \vec{g}_i\cdot\vec{q}_j - \vec{g}_j\cdot\vec{q}_i \;\in\; \mathbb{Z}\;, \tag{4.13}$$

where

$$\Omega \;=\; \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \tag{4.14}$$

is the standard symplectic metric. Note that (4.13) represents the well-known Dirac-Zwanziger quantization condition for the possible electric and magnetic charges. The vanishing of the r.h.s. is required for two dyons to be local with respect to each other and, in particular, to be able to condense simultaneously [13]. Thus, only states that are related to non-intersecting vanishing cycles are mutually local, and can be simultaneously represented by a local effective lagrangian. For example, since monopoles are associated with β-cycles and the gauge bosons with (combinations of) α-cycles, these fields can in general not be represented together in a local lagrangian.

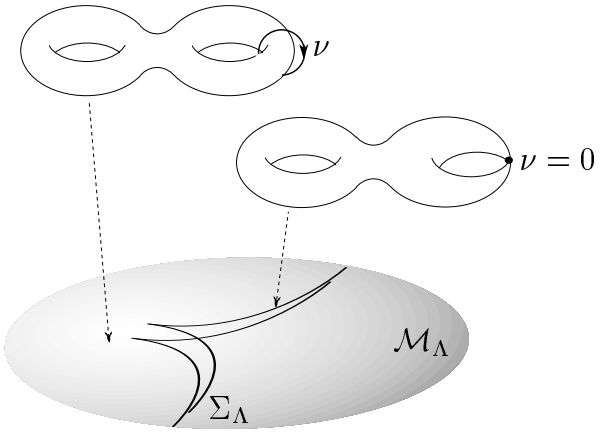


Fig.5: *On the singular locus Σ_Λ, the level surface degenerates by pinching of vanishing cycles ν. The coordinates of any such cycle with respect to some symplectic basis of $H_1(\mathcal{C},\mathbb{Z})$ gives the electric and magnetic quantum numbers of the corresponding massless dyon.*

Note that according to (3.9), the electric charges are root lattice vectors in Dynkin basis, $\vec{q}\in\Lambda_R$. In order for products $\vec{q}_i\vec{g}_j$ to make sense, it follows that $\vec{g}$ are vectors in the dual, ie., simple root basis, such that $\vec{q}\,\vec{g}\equiv q_i\langle\lambda_i,\alpha_j\rangle g_j$. The α-cycles can thus be thought of as to generate the weight lattice and the β-cycles to generate the root lattice, so that the period lattice can be interpreted as $H_1(\mathcal{C},\mathbb{Z})\cong\Lambda_W\oplus\tau\Lambda_R$. For our particular curves (4.1), the α-cycles do not vanish anywhere in the moduli space, so that the gauge bosons (or any other purely electrically charged states) are never massless. This means that the Milnor lattice generated by the vanishing cycles is a sub-lattice of the above period lattice.

If we like to represent both electric and magnetic charges in the same basis, then, of course, the metric (4.14) changes. For example, we can take the electric charges in the simple root basis as well, by conjugation with

$$W \;=\; \begin{pmatrix} \mathbb{1} & 0 \\ 0 & C \end{pmatrix}\,, \tag{4.15}$$

where C is the Cartan matrix. In this basis, the semi-classical monodromies (3.17) and intersection metric are

$$M_W^{(r_i)} \;=\; \begin{pmatrix} (r^{-1})^t & \alpha_i\otimes\lambda_i \\ 0 & r \end{pmatrix}\,, \qquad \Omega_W \;=\; \begin{pmatrix} 0 & C \\ -C & 0 \end{pmatrix}\,. \tag{4.16}$$

This basis corresponds to the one which was used in the first paper of Seiberg and Witten and is appropriate to the $\Gamma(2)$ curve (2.11).

We now turn to the monodromies that arise when we loop around the various branches of Σ_Λ. We noted that there is a particular vanishing cycle ν associated with any such branch. The monodromy action on any given cycle $\delta \in H_1(\Sigma_g, \mathbb{Z})$ is very simply determined in terms of this vanishing cycle, by means of the Picard-Lefshetz formula [11]:

$$M_\nu : \quad \delta \;\longrightarrow\; \delta - (\delta \cap \nu)\,\nu \ . \tag{4.17}$$

From this one can find for a vanishing cycle of the form (4.11) the following monodromy matrix [10],

$$M_{(g,q)} = \begin{pmatrix} \mathbb{1} + \vec{q}\otimes\vec{g} & \vec{q}\otimes\vec{q} \\ -\vec{g}\otimes\vec{g} & \mathbb{1} - \vec{g}\otimes\vec{q} \end{pmatrix} \;\in\; Sp(2n-2,\mathbb{Z}) \ , \tag{4.18}$$

which obeys $(\vec{g},\vec{q})M_{(g,q)} = (\vec{g},\vec{q})$. Under a change of basis, one has $S^{-1}M_\nu S = M_{\vec{\nu}\cdot S}$. Also observe that for $\nu_i \cap \nu_j = 0$ the corresponding monodromies commute: $[M_{\nu_i}, M_{\nu_j}] = 0$, as it should be for two mutually local states. The actual charges $(\vec{g},\vec{q})$ of the $SU(n)$ curves (4.1) will generically be given by root vectors in simple root and Dynkin bases, respectively.

As far as the semi-classical monodromies are concerned, we already noted that the α-cycles do not vanish anywhere in the quantum moduli space. However, we can formally compactify the moduli space by considering Λ as an extra modulus, and study monodromy around $\Lambda = 0$. For $\Lambda \to 0$, the α-cycles vanish since: $e_i^+ \to e_i^- = e_i$, and the classical level surface $V_u = \{e_i\}$ is recovered. Under a 2π rotation of Λ^{2n} (which leaves the curve (4.1) invariant), e_i^+ simply exchanges with e_i^-, if we take Λ sufficiently small as compared to u_n. Accordingly, since the α-cycles correspond to the weights of the fundamental representation of $SU(n)$, the monodromy corresponding to this simultaneous braid is

$$\prod_{\substack{\text{fund. rep} \\ \text{weights } \lambda_i}} M_{(0,\lambda_i)} \;=\; T$$

and thus we reproduce the quantum monodromy (3.19) directly from the curves (4.1). The θ-shifts (3.18) associated with the classical monodromies are similarly given by $T_{\alpha_i} = M_{(0,\alpha_i)}$.

Note that the Picard-Lefshetz formula (4.17) directly expresses the correct logarithmic monodromy property of the corresponding β-function, and thus automatically

guarantees a consistent physical picture. That is, near the vanishing of some $\nu = \vec{\nu}\cdot\vec{\gamma}$ (where $\vec{\gamma} \equiv (\vec{\beta}, \vec{\alpha})$), the monodromy shift of the gauge coupling, when expressed in suitable local variables, is

$$\Delta\tau_{ij} \;= -(\gamma_i^* \cap \nu)\,\frac{\partial}{\partial\pi_j}\int_\nu \lambda \;\equiv\; -\sum \nu_k\,(\gamma_i^* \cap \gamma_k)\,\frac{\partial}{\partial\pi_j} Z_\nu \;=\; -\nu_i\,\nu_j \qquad (4.19)$$

where $Z_\nu \equiv \vec{\nu}\cdot\vec{\pi}$ and γ^* is the cycle dual to γ. This is indeed the monodromy associated with the corresponding one-loop effective action near the singular line $\Sigma_\Lambda^{(\nu)}$:

$$\mathcal{F}_\nu \;=\; \frac{1}{4\pi i} {Z_\nu}^2 \log\big[\frac{Z_\nu}{\Lambda}\big]\ .$$

4.3. Strong coupling monodromies and dyon charge spectrum for $G = SU(3)$

We now consider $G = SU(3)$ in some more detail, extending our previous work discussed in [3,10]. We first need to fix some symplectic basis for the homology cycles. It turns out that a convenient basis is the one depicted in Fig.6, since it will directly reproduce the semi-classical monodromies and is adapted to the vanishing cycles.

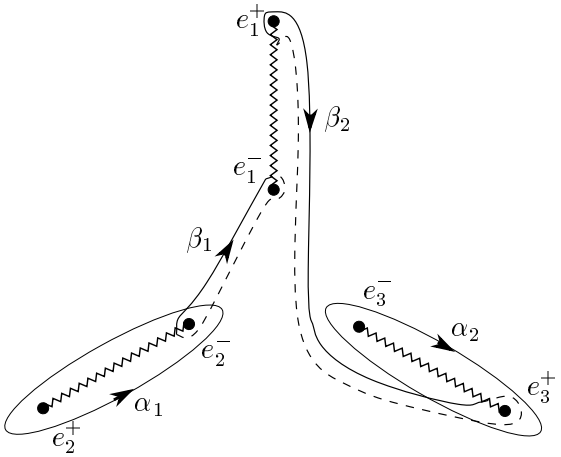


Fig.6: *The genus two curve for $G = SU(3)$ is represented as branched x-plane with cuts linking pairs of roots (4.2) of $p(x) = 0$. The locations of the cuts refer to $u = 0$, $\mathrm{Im}\,v = 0$, $\mathrm{Re}\,v > 1$. We depicted our choice of homology basis that is adapted to the vanishing β-cycles. The cycles α_i can be associated with the fundamental weights, and β_i with the simple roots.*

We now consider the monodromies generated by loops around the six singular lines in the quantum moduli space Fig.4. With reference to results by Zariski and van Kampen (c.f., [14] and references therein) it suffices to study loops in a generic complex line through the base point. Across cusps and nodes the monodromies are related through the "van Kampen relations". Specifically, we fix a plane in $\mathcal{M}_\Lambda$ at $\mathrm{Re}(v) = const > \Lambda^3,\ \mathrm{Im}(v) = 0$, as well as a base point u_0, and consider a family of loops as given in figure Fig.7. By carefully tracing the effects of loops in moduli space on the motions of the branch points in the x-plane, we find the corresponding vanishing cycles to be as depicted in Fig.8.

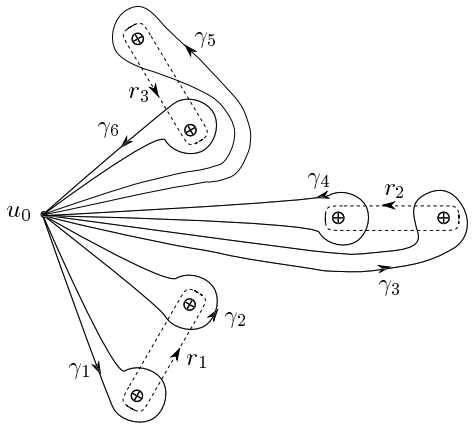


Fig.7: *Loops γ_i in the u-plane at $\mathrm{Re}(v) = const > \Lambda^3,\ \mathrm{Im}(v) = 0$, starting from the base point u_0 (cf., Fig.4). The composite loops r_i give the semi-classical monodromies (3.20).*

According to what we said above, the quantum numbers of the dyons that become massless at the various singular lines in moduli space can be directly obtained from Fig.8, by comparing the vanishing cycles with the basis cycles in Fig.6. Our basis is such that the massless excitations at $u = (\frac{27}{4})^{1/3}\Lambda^2$ are pure monopoles with charges $(\vec{g}, \vec{q})$ equal to $(1,0,0,0)$, and $(0,1,0,0)$, respectively. (For $G = SU(n)$ one can always choose a basis for the β-cycles so as to have $n-1$ massless monopoles with unit charges at a given node of $n-1$ intersecting singular lines.)

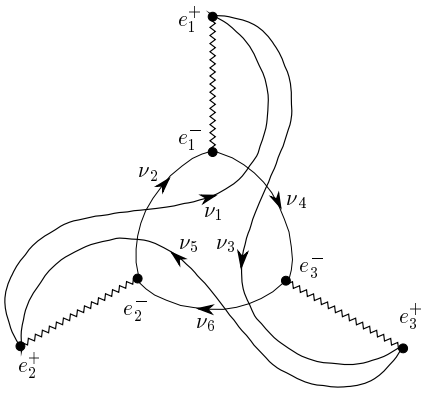


Fig.8: *Vanishing cycles ν_i in the x-plane, associated with the loops γ_i in Fig.7. We have depicted here only the paths on the upper sheet, and not the return paths on the lower sheet.*

The quantum numbers of the other massless dyons then simply follow from the global $\mathbb{Z}_3$ symmetry U (3.21). More precisely, the charges $\nu_i \equiv (\vec{g}_i, \vec{q}_i)$ and monodromies associated with all the six singular lines in $\mathcal{M}_\Lambda$ are given by matrices (4.18) with

$$M_{\nu_1} = M_{(1,0,-2,1)} \equiv \begin{pmatrix} -1 & 0 & 4 & -2 \\ 1 & 1 & -2 & 1 \\ -1 & 0 & 3 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$M_{\nu_2} = M_{(1,0,0,0)} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$M_{\nu_3} = M_{(0,1,0,0)} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

$$
\begin{aligned}
M_{\nu_4} &= M_{(0,1,-1,2)} \equiv \begin{pmatrix} 1 & -1 & 1 & -2 \\ 0 & 3 & -2 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & -1 \end{pmatrix} \\
M_{\nu_5} &= M_{(-1,-1,2,-1)} \equiv \begin{pmatrix} -1 & -2 & 4 & -2 \\ 1 & 2 & -2 & 1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & 2 & 0 \end{pmatrix} \\
M_{\nu_6} &= M_{(-1,-1,1,-2)} \equiv \begin{pmatrix} 0 & -1 & 1 & -2 \\ 2 & 3 & -2 & 4 \\ -1 & -1 & 2 & -2 \\ -1 & -1 & 1 & -1 \end{pmatrix} .
\end{aligned}
\tag{4.20}
$$

They form indeed two orbits under conjugation by U, and even form a single orbit under

$$
A = \begin{pmatrix} -1 & -1 & 2 & -1 \\ 0 & 1 & -1 & 2 \\ -1 & -1 & 1 & -1 \\ 0 & -1 & 0 & -1 \end{pmatrix} \equiv U^{-1}V^{-1} , \tag{4.21}
$$

with $A^2 = U$ and $A^6 = U^3 = V^2 = \mathbb{1}$. These global R-symmetries act simultaneously on the moduli space in Fig.4 and on the vanishing cycles in Fig.8. That is, U rotates the moduli space by $e^{2\pi i/3}$ and V represents $v \to -v$. In fact, A is the generalization of the global $\mathbb{Z}_4$ symmetry of $G = SU(2)$ [1]. Note that these matrices do not belong to the monodromy group.

From (4.20) we see that the magnetic and electric quantum numbers of the massless dyons are indeed given by root vectors (in simple root and Dynkin bases, respectively). This is in accordance with semi-classical stability [15]. Though the above quantum numbers can formally be changed via monodromy to other points in the root lattices, one expects [1,5] dyons with higher charges to become unstable when crossing lines of marginal stability. Nevertheless, from a group theoretical point of view, the possible charges are given by root lattices, and this mirrors the special properties of the lattice generated by the vanishing cycles of the curve (4.4). One may also perform a basis change via conjugation with W (4.15), under which M_{ν_i} remain integral. The matrices then contain as sub-matrices the $SU(2)$ strong coupling monodromies in the basis of [1], which are the monodromies of the isogenous $SU(2)$ curve (2.11). This suggests that there might be a similar isogenous curve for $G = SU(3)$, or even all $SU(n)$.

It has been observed [4]that the charge vectors of each pair of lines that intersect in the nodes at $v = 0$ satisfy $\vec{\nu}_i^t \, \Omega \, \vec{\nu}_j = 0$, since the cycles do not intersect (these pairs are given by $(\vec{\nu}_1, \vec{\nu}_6)$, $(\vec{\nu}_2, \vec{\nu}_3)$ and $(\vec{\nu}_4, \vec{\nu}_5)$). Thus, at any of the three nodes, two mutually local dyons become massless and both dual $U(1)$'s are weakly coupled. We will find below in section 6.2 that the exact beta function of the effective dual theory indeed reflects two massless monopole hypermultiplets. On the other hand, near the cusps at $u = 0, v = \pm\Lambda^3$, three dyons become massless simultaneously, though they are not local with respect to each other. The effective theory near these regions has recently been discussed in [8].

The strong coupling monodromies (4.20) contain, as expected, the strong coupling monodromies (2.4) of $G = SU(2)$ embedded as sub-matrices. Also, the monodromies of the three pairs of lines that go out to $v \to +\infty$ reproduce precisely the $SU(3)$ semi-classical monodromies (3.20):

$$\begin{aligned} M_{\nu_2} \cdot M_{\nu_1} &= M^{(r_1)} \\ M_{\nu_4} \cdot M_{\nu_3} &= M^{(r_2)} \\ M_{\nu_6} \cdot M_{\nu_5} &= M^{(r_3)} \ , \end{aligned} \tag{4.22}$$

(c.f., Fig.7) and thus: $M_{\nu_2} M_{\nu_1} M_{\nu_6} M_{\nu_5} M_{\nu_4} M_{\nu_3} = M_{\infty,u}^{(r_3)}$. (For the three pairs that go out $v \to -\infty$ the same is true up to a change of basis). Note that this semi-classical monodromy is defined by a loop around infinity in the u-plane located at $\mathrm{Im}\, v = 0, \mathrm{Re}\, v > 1$, as shown in Fig.4. For the same loop in a u-plane located at $-1 < \mathrm{Re}\, v < 1$, the monodromy is different because some of the singular lines interchange at the cusp at $\mathrm{Re}\, v = 1$. This monodromy is given by

$$\widetilde{M}_{\infty,u}^{(r_3)} \ = \ M_{\nu_6} M_{\nu_1} M_{\nu_4} M_{\nu_5} M_{\nu_2} M_{\nu_3} \ = \ \begin{pmatrix} 0 & 5 & -3 & 6 \\ -1 & -3 & 6 & -3 \\ 0 & -1 & 0 & -1 \\ 1 & 3 & -7 & 3 \end{pmatrix} \ . \tag{4.23}$$

5. Picard-Fuchs Equations for $G = SU(3)$

5.1. Derivation

Starting from the $SU(3)$ curve (4.4), we will first derive the system of equations for the periods $\int_{\gamma_i} w = \partial_v \pi$. Subsequently, we will obtain the system for $\partial_u \vec{\pi}$ as well as directly for $\vec{\pi} = (\vec{a}_D, \vec{a})^t$ itself.

There is a systematic, though tedious way to set up the Picard-Fuchs equations, by considering derivatives of $\omega_2 = \frac{dx}{y} \equiv \frac{dx}{\sqrt{p(x)}}$ with respect to u and v. These produce terms of the form $\frac{\phi(x)}{y^n}$ for some polynomials $\phi(x)$. The idea is to reduce the order of $\phi(x)$, to integrate by parts and re-express the result in terms of the abelian differentials. For this, we will use the fact that the discriminant (4.5) can always be written in the form [16]

$$\Delta = a(x)p(x) + b(x)p'(x) \tag{5.1}$$

where, for the $SU(3)$ curve, a and b are polynomials of order four and five, respectively. These polynomials are straightforwardly determined to be

$$a(x) = \sum_{i=0}^{4} a_i x^i\,, \qquad b(x) = \sum_{i=0}^{5} b_i x^i$$

with

$$\begin{aligned}
a_0 &= -729\Lambda^6 + 216u^3 - 16u^6 + 729v^2 + 108u^3v^2 \\
a_1 &= 9uv(-135\Lambda^6 - 4u^3 + 27v^2) \\
a_2 &= 18u^2(-27\Lambda^6 + 4u^3 - 27v^2) \\
a_3 &= 27v(27\Lambda^6 + 4u^3 - 27v^2) \\
a_4 &= 18u(27\Lambda^6 - 4u^3 + 27v^2)
\end{aligned} \tag{5.2}$$

and

$$\begin{aligned}
b_0 &= 2u^2v(-81\Lambda^6 + 4u^3 - 27v^2) \\
b_1 &= (-27\Lambda^6 + 4u^3 - 27v^2)(-9\Lambda^6 + 4u^3 + 9v^2)/2 \\
b_2 &= 3uv(243\Lambda^6 + 4u^3 - 27v^2)/2 \\
b_3 &= 5u^2(27\Lambda^6 - 4u^3 + 27v^2) \\
b_4 &= 9v(-27\Lambda^6 - 4u^3 + 27v^2)/2 \\
b_5 &= -3u(27\Lambda^6 - 4u^3 + 27v^2)
\end{aligned}$$

We can thus write under the integral sign:

$$\frac{\phi(x)}{y^n} = \frac{1}{\Delta}\frac{a\phi + \frac{2}{n-2}(b\phi)'}{y^{n-2}}\,. \tag{5.3}$$

If the order of the polynomial in the numerator is equal to or exceeding the power of $p'(x)$, we can reduce it by expressing the highest power in terms of p or p' and lower powers and integrating by parts. This procedure allows one, after some work, to find

a set of two second order differential equations that are satisfied by the periods $\partial_v\vec{\pi}$: $\widetilde{\mathcal{L}}_i(\partial_v\vec{\pi}) = 0$ with

$$\begin{aligned}\widetilde{\mathcal{L}}_1 &= (27\Lambda^6 - 4u^3 - 27v^2)\partial_u^2 - 12u^2v\partial_u\partial_v - 12u^2\partial_u - 21uv\partial_v - 4u \\ \widetilde{\mathcal{L}}_2 &= (27\Lambda^6 - 4u^3 - 27v^2)\partial_v^2 - 36uv\partial_u\partial_v - 36u\partial_u - 63v\partial_v - 12\end{aligned} \tag{5.4}$$

Note that these equations imply $(u\partial_v^2 - 3\partial_u^2)\partial_v\vec{\pi} = 0$, which can also be verified directly. In fact, $(u\partial_v^2 - 3\partial_u^2)\frac{1}{y} = \partial_x\frac{W_{A_2}}{y^3}$. If we introduce the gauge and $\mathbf{Z}_{12}$ invariant dimensionless moduli $\alpha = \frac{4u^3}{27\Lambda^6}$, $\beta = \frac{v^2}{\Lambda^6}$, the Picard-Fuchs equations take the form

$$\begin{aligned}\widetilde{\mathcal{L}}_1 &= \alpha(1-\alpha)\partial_\alpha^2 - \beta^2\partial_\beta^2 - 2\alpha\beta\partial_\alpha\partial_\beta + \frac{1}{3}(2-5\alpha)\partial_\alpha - \frac{5}{3}\beta\partial_\beta - \frac{1}{9} \\ \widetilde{\mathcal{L}}_2 &= \beta(1-\beta)\partial_\beta^2 - \alpha^2\partial_\alpha^2 - 2\alpha\beta\partial_\alpha\partial_\beta + (\frac{1}{2} - \frac{5}{3}\beta)\partial_\beta - \frac{5}{3}\alpha\partial_\alpha - \frac{1}{9}\ .\end{aligned} \tag{5.5}$$

Expressed in terms of the logarithmic derivatives $\theta_\alpha = \alpha\partial_\alpha$, $\theta_\beta = \beta\partial_\beta$, the two differential operators become

$$\begin{aligned}\widetilde{\mathcal{L}}_1 &= \theta_\alpha(\theta_\alpha + c - 1) - \alpha(\theta_\alpha + \theta_\beta + a)(\theta_\alpha + \theta_\beta + b) \\ \widetilde{\mathcal{L}}_2 &= \theta_\beta(\theta_\beta + c' - 1) - \beta(\theta_\alpha + \theta_\beta + a)(\theta_\alpha + \theta_\beta + b)\ ,\end{aligned} \tag{5.6}$$

with $a = b = \frac{1}{3}, c = \frac{2}{3}, c' = \frac{1}{2}$. The system (5.6) is in fact the generalized hypergeometric system $F_4(a, b; c, c'; \alpha, \beta)$ of Appell [17,18]. Unfortunately, not much appears to be known in the literature about analytic continuation and non-trivial transformations of its solutions, especially for the present special values of $(a, b; c, c')$, so that we have to work out the solutions ourselves.

If we write the differential equations (5.5) in the form $\widetilde{\mathcal{L}}_i\partial_v\vec{\pi} = 0$, we can pull the partial derivative operator through to get $\partial_v\mathcal{L}_i\vec{\pi} = 0$ with

$$\begin{aligned}\mathcal{L}_1 &= (27\Lambda^6 - 4u^3 - 27v^2)\partial_u^2 - 12u^2v\partial_u\partial_v - 3uv\partial_v - u \\ \mathcal{L}_2 &= (27\Lambda^6 - 4u^3 - 27v^2)\partial_v^2 - 36uv\partial_u\partial_v - 9v\partial_v - 3\end{aligned} \tag{5.7}$$

and also $(u\partial_v^2 - 3\partial_u^2)\vec{\pi} = 0$. When we express this in terms of the variables α, β, we find that this system (that is satisfied directly by the sections a_{D_i}, a_i) is equivalent to an Appell system of type $F_4(\frac{1}{6}, \frac{1}{6}; \frac{1}{3}, \frac{1}{2}; \alpha, \beta)$. Similarly, we can find the Picard-Fuchs operators also for the periods $\partial_u\vec{\pi}$; they constitute the system

In order to solve the Picard-Fuchs equations, note that they are always Fuchsian [19,18] and thus have only regular singularities. These can be described as follows [18].

Denote the linear partial differential operators of degree m, defined in a neighborhood U of a point z of the g dimensional moduli space $\mathcal{M}$ by $\mathcal{L}_i = \sum_{|p|\leq m} a_1^p(z) \left(\frac{d}{dz}\right)^p$ where we have used the notation $\left(\frac{d}{dz}\right)^p = \prod_{j=1}^g \frac{\partial}{\partial z_j}^{p_j}$. They define a left ideal I in the ring of partial differential operators on U. We now introduce the symbol of $\mathcal{L}_i$: $\sigma(\mathcal{L}_i) = \sum_{|p|=m} a_i^p(z)\xi_1^{p_1}\cdots\xi_g^{p_g}$, where $\xi_1,\ldots,\xi_g$ is a coordinate system in the fiber of the cotangent bundle T^*U at z. The ideal of symbols is defined by $\sigma(I) = \{\sigma(\mathcal{L})|\mathcal{L} \in I\}$. The singular locus is then $\Delta(I) = \pi(Ch(I) - U \times \{0\})$, where the characteristic variety $Ch(I)$ is the subvariety in T^*U specified by the ideal of symbols, and π is the projection along the fiber of T^*U. The fact that $\sigma(I)$ is generated by $\sigma(\mathcal{L}_i)$ is a special property of Picard-Fuchs systems, and this is not the case, for example, for the Appell system F_1.

Let us now find the singular locus of the general system $F_4(a,b;c,c';\alpha,\beta)$ (5.6), for which the symbols turn out to be independent of the parameters (a,b,c,c'):

$$\begin{aligned}\sigma_1 &= \alpha^2\xi_\alpha^2 - \alpha(\alpha\xi_\alpha + \beta\xi_\beta)^2 \\ \sigma_2 &= \beta^2\xi_\beta^2 - \beta(\alpha\xi_\alpha + \beta\xi_\beta)^2\end{aligned} \quad . \tag{5.8}$$

It is straightforward to find the discriminant to be $\Delta(I) : \alpha\beta(1+\alpha^2+\beta^2-2(\alpha\beta+\alpha+\beta)) = 0$. This coincides, up to the factor $\alpha\beta$, with the quantum discriminant Δ_Λ (4.5) of the $SU(3)$ curve (note that if we had computed the singular locus of the Picard-Fuchs equations in the form (5.4), we would have gotten the discriminant given in eq.(4.5) only when taking also into account the vanishing of $(u\partial_v^2 - 3\partial_u^2)\partial_v\vec{\pi}$). The additional lines $\alpha = 0$ and $\beta = 0$ are due to the change of variables $(u,v) \to (\alpha,\beta)$.

The formalism described in this section applies to every hyperelliptic curve and can be used to obtain the Picard-Fuchs equations for all $SU(n)$. Let us point out here some features of the Picard-Fuchs system, which are due to the special symmetries of our curves and hold for all $SU(n)$. We start by characterizing differential operators, which are pure second order in the derivatives. These are given by

$$\begin{aligned}\mathcal{L}^{SU(n)}_{p,q,r,s} &= \partial_{u_p}\partial_{u_q} - \partial_{u_r}\partial_{u_s}, \qquad \text{with } \; p+q = r+s \\ \mathcal{L}^{SU(n)}_0 &= \sum_{k=1}^{n-2} k u_{n-k}\partial_{u_n}\partial_{u_{n-k+1}} - n\partial_{u_2}\partial_{u_{n-1}}, \text{for } \; n > 2\end{aligned}$$

and follow directly from $\mathcal{L}^{SU(n)}_{p,q,r,s}\left(\frac{1}{y}\right) = 0$ and $\mathcal{L}^{SU(n)}_{0}\left(\frac{1}{y}\right) = \partial_x \partial_{u_n}\left(\frac{1}{y}\right)$. In general, exact forms of the type $\partial_x\left(\frac{x^r \mathcal{W}_{A_{n-1}}}{y^3}\right)$ lead to differential operators involving first and second order derivatives, namely

$$\mathcal{L}^{SU(n)}_{r} = \sum_{k=1}^{n-2} k u_{n-k} \partial_{u_{n-r+1}} \partial_{u_{n-k}} - n \partial_{u_{n-r-1}} \partial_{u_2} + r \partial_{u_{n+1-r}} \qquad \text{for} \quad r < n-2 \ .$$

For instance, for $SU(3)$ the complete system of PF-operators is given by $\mathcal{L}^{SU(3)}_{0} = u\partial_v^2 - 3\partial_u^2$ and any one of the operators from (5.4). Similarly, for $SU(4)$ the complete system consists of $\mathcal{L}^{SU(4)}_{2,4,3,3}$, $\mathcal{L}^{SU(4)}_{0}$, $\mathcal{L}^{SU(4)}_{1}$ and

$$\begin{aligned}\mathcal{L} = (64u^2w^2 - 32uv^2w + 9v^4 - 64u^2)\partial_w^2 + 4v(12u^3 - 9v^2 + 32uw)\partial_u\partial_v + \\ 8u(2u^3 + 9v^2 + 16uw)\partial_u^2 + 2u(64uw - 9v^2)\partial_w + 108u^2v\partial_v + 64u^3\partial_u 36u^2,\end{aligned}$$

where we set $\Lambda = 1$. Completeness can be checked by calculating the rank of the linear system of symbols. For generic values of the moduli the rank has to be maximal, while the rank drops precisely at the principal locus (all $\xi_i \neq 0$) of the discriminant.

5.2. Solutions in the semi-classical regime

We will be particularly interested in solving the Picard-Fuchs equations in the semi-classical regime, to which we so far, somewhat vaguely, referred to as "infinity" in the moduli space $\mathcal{M}_\Lambda$. We now like to make precise what we mean by this, by compactifying the moduli space $(\alpha, \beta) \in \mathbb{C}^2$ to $\mathbb{P}^2$, by adding a line $\gamma = 0$ at infinity (where $\gamma \equiv 27\Lambda^6$). This makes contact with our discussion of the semi-classical monodromy around $\Lambda = 0$ in section 3.2. In terms of homogeneous coordinates $(\alpha : \beta : \gamma) \in \mathbb{P}^2$, we get for the discriminant

$$\Delta(I) = \alpha\beta\gamma(\alpha^2 + \beta^2 + \gamma^2 - 2(\alpha\beta + \beta\gamma + \alpha\gamma)) \ . \tag{5.9}$$

We have thus three singular lines which intersect with each other in three points P_i, as well with the discriminant locus Σ_Λ in three points Q_i. We have sketched the singular locus of the system F_4 in Fig.9.

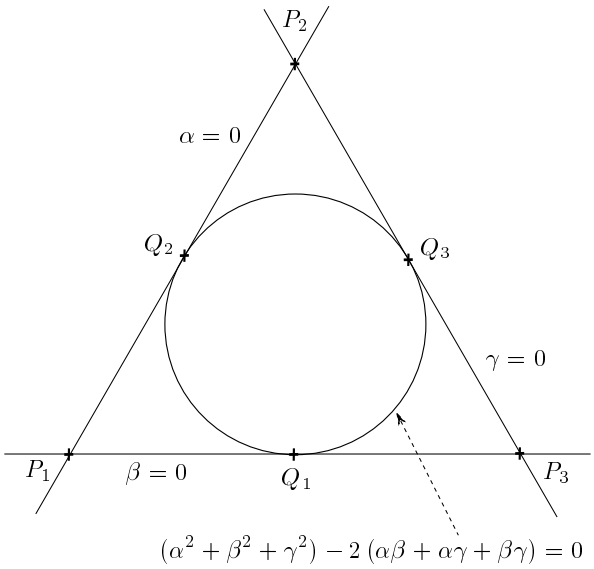


Fig.9: *The singular locus $\Sigma(I)$ of the system F_4 for the compactification of the (α, β)-plane to $\mathbb{P}^2$, with $\alpha = 4u^3$, $\beta = 27v^2$ and $\gamma = 27\Lambda^6$. The semi-classical regions correspond to the neighborhoods of P_2 and P_3, and the magnetic dual semi-classical region to Q_1. A full set of two power series and two logarithmic solutions can be found only in these regions. The cusp point is mapped to Q_2, where the theory is badly behaved, and Q_3 represents the intersection of the six singular lines with infinity; at the origin, P_1, nothing special happens.*

We can cover $\mathbb{P}^2$ with three coordinate patches (centered on the points P_i), and in each such patch there is a set of preferred inhomogeneous coordinates, ie.,

$$\begin{aligned} P_1: &\quad (\frac{\alpha}{\gamma} : \frac{\beta}{\gamma} : 1) \equiv (x_1 : y_1 : 1) \\ P_2: &\quad (\frac{\alpha}{\beta} : 1 : \frac{\gamma}{\beta}) \equiv (x_2 : 1 : y_2) \\ P_3: &\quad (1 : \frac{\beta}{\alpha} : \frac{\gamma}{\alpha}) \equiv (1 : x_3 : y_3)\ . \end{aligned} \tag{5.10}$$

We thus have two natural coordinate patches corresponding to semi-classical "infinity" in the moduli space, one roughly given by large u, the other one by large v, just as mentioned in section 3.2.

So far we have written the Picard-Fuchs equations in terms of the coordinates appropriate for P_1. In the more interesting two patches at infinity, the Picard-Fuchs

equations are ($\theta_i = x_i\partial_{x_i}$, $\theta_i' = y_i\partial_{y_i}$):

$$\begin{aligned}
P_2: \qquad & \mathcal{L}_1 = y_2\theta_2(\theta_2 + c - 1) - x_2(\theta_2' - a)(\theta_2' - b) \\
& \mathcal{L}_2 = y_2(\theta_2 + \theta_2')(\theta_2 + \theta_2' - c' + 1) - (\theta_2' - a)(\theta_2' - b) \\
P_3: \qquad & \mathcal{L}_1 = y_3(\theta_3 + \theta_3')(\theta_3 + \theta_3' - c + 1) - (\theta_3' - a)(\theta_3' - b) \\
& \mathcal{L}_2 = y_3\theta_3(\theta_3 + c' - 1) - x_3(\theta_3' - a)(\theta_3' - b)
\end{aligned} \tag{5.11}$$

with $(a, b, c, c') = (-\frac{1}{6}, -\frac{1}{6}, \frac{2}{3}, \frac{1}{2})$. Note that for these equations we have $a = b$ and $1-c$, $1-c'$, a all distinct and no pair differing by integers. Thus, we can expect them to be solved by power and logarithmic series ansätze around the origin in each patch:

$$\omega_i^{P_i}(x_i, y_i) = \sum_{n,m\geq 0} c_i(n, m) x_i^{n+s} y_i^{m+t} \tag{5.12}$$

We find the following solutions to the indicial equations:

$$\begin{aligned}
P_1: \qquad & (s, t) = (0, 0),\ (0, 1 - c'),\ (1 - c, 0),\ (1 - c, 1 - c') \\
P_2: \qquad & (s, t) = (0, a),\ (0, b),\ (1 - c, a),\ (1 - c, b) \\
P_3: \qquad & (s, t) = (0, a),\ (0, b),\ (1 - c', a),\ (1 - c', b)
\end{aligned} \tag{5.13}$$

With this it is possible to solve the recursion relations for the coefficients $c_i(n, m)$ in (5.12). The solutions in the patch P_1 are simply given by four power series, which is not very interesting. On the other hand, for each of the two semi-classical patches at infinity, we find two series and two logarithmic solutions, and this is precisely what one expects on physical grounds.

More precisely, for the power series solutions in the patches at infinity one finds (we have set $a = b$ but have left a, c, c' arbitrary and normalized $c_i^{(s,t)}(0, 0) = 1$)

$$\begin{aligned}
P_2: \qquad & c_2^{(0,a)}(n, m) = \frac{(a)_{n+m}\,(a + 1 - c')_{n+m}}{(1)_n(1)_m^2(c)_n} \\
& c_2^{(1-c,a)}(n, m) = \frac{(a + 1 - c)_{n+m}\,(a + 2 - c - c')_{n+m}}{(1)_n(1)_m^2(2 - c)_n} \\
P_3: \qquad & c_2^{(0,a)}(n, m) = \frac{(a)_{n+m}\,(a + 1 - c)_{n+m}}{(1)_n(1)_m^2(c')_n} \\
& c_2^{(1-c',a)}(n, m) = \frac{(a + 1 - c')_{n+m}\,(a + 2 - c - c')_{n+m}}{(1)_n(1)_m^2(2 - c')_n}\ .
\end{aligned} \tag{5.14}$$

In addition to the above power series solutions, we find logarithmic solutions of the form

$$(2\pi i)\Omega^{P_i}(x_i,y_i) = \sum_{n,m\geq 0} d_i(n,m)x_i^{n+s}y_i^{m+t} + \log y_i \Big\{ \sum_{n,m=\geq 0} c_i(n,m)x_i^{n+s}y_i^{m+t}\Big\} \tag{5.15}$$

One verifies that

$$d_i(n,m) = \frac{\partial}{\partial\rho}c_i(n,m,\rho)|_{\rho=0}$$

where

$$\begin{aligned} c_2(n,m,\rho) &= \frac{(s+t+\rho)_{n+m}(s+t+\rho+1-c')_{n+m}}{(1)_n(1+\rho)_m^2(c+2s)_n} \\ c_3(n,m,\rho) &= \frac{(s+t+\rho)_{n+m}(s+t+\rho+1-c)_{n+m}}{(1)_n(1+\rho)_m^2(c'+2s)_n} \ . \end{aligned}$$

With the definition $\psi(x) = \Gamma'(x)/\Gamma(x)$ we can write

$$\begin{aligned} d_2(n,m) = c_2(n,m)\Big\{&2\psi(1) - 2\psi(m+1) + \psi(n+m+s+t) - \psi(s+t) \\ &+ \psi(n+m+s+t+1-c') - \psi(s+t+1-c')\Big\} \\ d_3(n,m) = c_3(n,m)\Big\{&2\psi(1) - 2\psi(m+1) + \psi(n+m+s+t) - \psi(s+t) \\ &+ \psi(n+m+s+t+1-c) - \psi(s+t+1-c)\Big\} \ , \end{aligned}$$

where we have chosen the normalization $d(0,0) = 0$.

(5.16)

5.3. Solutions in the magnetic dual semi-classical regime Q_1

On the discriminant locus, only the patch Q_1 has an easy physics interpretation in that we can find two series and two logarithmic solutions. This reflects the fact that on Σ_Λ, only near Q_1 the theory is weakly coupled, in suitable dual local variables. The tricky point is to find good variables for which we really do have two series and two logarithmic solutions. We find that

$$\delta_\pm \ = \ (1-\alpha+\beta\pm 2\sqrt{\beta}) \ = \ \prod_{i<j}^{3}(e_i^\pm - e_j^\pm)^2 \tag{5.17}$$

($\alpha = \frac{4u^3}{27\Lambda^6}$, $\beta = \frac{v^2}{\Lambda^6}$) are suitable variables, since they vanish precisely on the discriminant (cf., (4.3)), and also incorporate all three intersection points simultaneously. In terms of these variables, the Picard-Fuchs operators for the system $F_4(-\frac{1}{6}, -\frac{1}{6}; \frac{2}{3}, \frac{1}{2}; \alpha, \beta)$ become:

$$\begin{aligned}
\mathcal{L}_1 = \Big\{ & \frac{1}{2}\delta_-(2-\delta_--\delta_+)\,\frac{\partial^2}{\partial_{\delta_-}^2} + \frac{1}{2}\,(2-\delta_--\delta_+)\,\delta_+\frac{\partial^2}{\partial_{\delta_+}^2} - \frac{1}{24}\,(9\delta_-+7\delta_+)\,\frac{\partial}{\partial_{\delta_-}} \\
& - \frac{1}{24}\,(7\delta_-+9\delta_+)\,\frac{\partial}{\partial_{\delta_+}} + \frac{1}{4}\left(4\delta_- - {\delta_-}^2 + 4\delta_+ - 6\delta_-\delta_+ - {\delta_+}^2\right)\frac{\partial^2}{\partial_{\delta_+}\partial_{\delta_-}} - \frac{1}{36}\Big\} \\
\mathcal{L}_2 = \Big\{ & \frac{1}{2}\delta_-(4-\delta_--\delta_+)\,\frac{\partial^2}{\partial_{\delta_-}^2} + \frac{1}{2}\,(4-\delta_--\delta_+)\,\delta_+\frac{\partial^2}{\partial_{\delta_+}^2} + \frac{1}{24}\,(28-9\delta_--7\delta_+)\,\frac{\partial}{\partial_{\delta_-}} \\
& + \frac{1}{24}\,(28-7\delta_--9\delta_+)\,\frac{\partial}{\partial_{\delta_+}} + \frac{1}{4}\left(8\delta_- - 16 - {\delta_-}^2 + 8\delta_+ - 6\delta_-\delta_+ - {\delta_+}^2\right)\frac{\partial^2}{\partial_{\delta_+}\partial_{\delta_-}} - \frac{1}{36}\Big\}
\end{aligned} \tag{5.18}$$

The solutions have the general form (5.12),(5.15); though we did not succeed to obtain them in a closed form, say in terms of F_4 functions, we can easily compute them up to arbitrary order. Specifically, the first terms are:

$$\begin{aligned}
\omega_1^{Q_1} &= \delta_+\Big\{1 + \frac{1}{18}\,\delta_+ + \frac{25}{3888}\,{\delta_+}^2 + \frac{7}{24}\,\delta_- + \frac{377}{3456}\,\delta_+\delta_- + \frac{25289}{746496}\,{\delta_+}^2\,\delta_- + \ldots\Big\} \\
\Omega_1^{Q_1} &= \omega_1^{Q_1}\log\delta_+ \;+\; \Big\{36 + \delta_+ \\
&\qquad + \frac{5}{36}\,{\delta_+}^2 - \frac{1}{48}\,{\delta_-}^2 + \frac{13}{24}\,\delta_+\delta_- + \frac{1609}{6912}\,{\delta_+}^2\,\delta_- + \ldots\Big\}\,.
\end{aligned} \tag{5.19}$$

The remaining solutions, $\omega_2^{Q_1}$ and $\Omega_2^{Q_1}$, are given by exchanging δ_+ with δ_- in (5.19).

Finally, as far as the solutions near the cusp points are concerned, we face the problem of finding appropriate variables near Q_2. We tried various compactifications of the moduli space, but could only find two series solutions. Actually, the non-local physics in these regions suggests that one cannot find there two series plus two logarithmic solutions, and thus a sensible prepotential $\mathcal{F}$, at all.

6. The exact quantum low energy effective action for $G = SU(3)$

6.1. Semi-classical regime

Let us write and normalize the solutions (5.14), (5.16) of the Appell system

$F_4(-\frac{1}{6},-\frac{1}{6},\frac{2}{3},\frac{1}{2})$ in the patch P_3 as follows:

$$\begin{aligned}
\big(\frac{1}{\sqrt{3}\Lambda}\big)\,\omega_1^{P_3} &= 2^{2/3}y_3^{-1/6}F_4(-\frac{1}{6},\frac{1}{6},\frac{1}{2},1;x_3,y_3)\\
&\sim\ 2^{2/3}{y_3}^{-1/6}\big(1-\frac{1}{36}y_3+...\big)\\
\big(\frac{1}{\sqrt{3}\Lambda}\big)\,\omega_2^{P_3} &= \frac{1}{3\sqrt{3}}2^{2/3}\sqrt{x_3}{y_3}^{-\frac{1}{6}}F_4(\frac{1}{3},\frac{2}{3},\frac{3}{2},1;x_3,y_3)\\
&\sim\ \frac{1}{3\sqrt{3}}2^{2/3}\sqrt{x_3}{y_3}^{-1/6}\big(1+\frac{4}{27}x_3+...\big)
\end{aligned}\tag{6.1}$$

and

$$\begin{aligned}
\big(\frac{1}{\sqrt{3}\Lambda}\big)\,\Omega_1^{P_3} &= (-1)^{-1/6}2^{2/3}12\sqrt{3}\pi\frac{\Gamma(1/3)}{\Gamma(1/6)^2}F_4(-\frac{1}{6},-\frac{1}{6},\frac{1}{2},\frac{2}{3};\frac{x_3}{y_3},\frac{1}{y_3})\\
&\quad+\big((i-\sqrt{3})\pi+4\log 2+3\log 3-5\big)\,\omega_1^{P_3}\\
&\sim\ 2^{2/3}{y_3}^{-1/6}\big(1+\frac{1}{36}y_3+...\big)+\omega_1^{P_3}\log y_3\\
\big(\frac{1}{\sqrt{3}\Lambda}\big)\,\Omega_2^{P_3} &= (-1)^{4/3}2^{-1/3}\frac{1}{3\pi}\sqrt{\frac{x_3}{y_3}}\Gamma(1/3)^2F_4(\frac{1}{3},\frac{1}{3},\frac{3}{2},\frac{2}{3};\frac{x_3}{y_3},\frac{1}{y_3})\\
&\quad+\big(1+(i+\frac{1}{\sqrt{3}})\pi+3\log 3\big)\,\omega_2^{P_3}\\
&\sim\ \frac{1}{3\sqrt{3}}2^{2/3}{y_3}^{-1/6}\sqrt{x_3}\,\big(1+\frac{22}{27}x_3+...\big)+\omega_2^{P_3}\log y_3\ ,
\end{aligned}$$

where $x_3\equiv\frac{27v^2}{4u^3}$ and $y_3\equiv\frac{27\Lambda^6}{4u^3}$. Here, we introduced the Appell function F_4, which is defined in terms of a generalized Gaussian sum [17]:

$$F_4(a,b;c,c';x,y)\ =\ \sum_{n,m\geq 0}\frac{(a)_{n+m}(b)_{n+m}}{(1)_n(1)_m(c)_n(c')_m}x^n y^m\ .\tag{6.2}$$

This sum converges only for $|\sqrt{x}|+|\sqrt{y}|<1$. For values outside this region, one can define F_4 by suitable analytic continuation. Unfortunately, formulas for analytic continuation and transformations of F_4 do not seem to be thoroughly discussed in the literature. However, for some purposes we can use the formula

$$F_4(a,b;c,c';x,y)\ =\ \sum_{k=0}^{\infty}\frac{(a)_m(b)_m}{(c)_m(1)_m}\,{}_2F_1(a+m,b+m,c';y)\,x^m\tag{6.3}$$

to analytically continue to arbitrary y, when x is sufficiently small. (There exists a similar formula given by exchanging x with y and c with c' in (6.3)).

Matching asymptotically a_1, a_2 to the Casimirs u, v (which fixes the normalization of λ), we then obtain the following identification (up to Weyl conjugation) between $\vec{\pi}$ and the solutions of F_4:

$$\begin{aligned}
a_{D1} &= -\frac{i}{4\pi}(\Omega_1^{P_3} + 3\Omega_2^{P_3}) - \frac{1}{\pi}(\alpha_1\omega_1^{P_3} - \alpha_2\omega_2^{P_3}) \\
&\sim -\frac{1}{\pi}(\frac{i}{2} + 2\alpha_1)\sqrt{u} - \frac{1}{\pi}\big(\frac{3}{4}i - \alpha_2\big)\frac{v}{u} - \frac{i}{2\pi}\big(\sqrt{u} + \frac{3}{2}\frac{v}{u}\big)\log\big[\frac{27\Lambda^6}{4u^3}\big] + ... \\
a_{D2} &= -\frac{i}{4\pi}(\Omega_1^{P_3} - 3\Omega_2^{P_3}) - \frac{1}{\pi}(\alpha_1\omega_1^{P_3} + \alpha_2\omega_2^{P_3}) \\
&\sim -\frac{1}{\pi}(\frac{i}{2} + 2\alpha_1)\sqrt{u} + \frac{1}{\pi}\big(\frac{3}{4}i - \alpha_2\big)\frac{v}{u} - \frac{i}{2\pi}\big(\sqrt{u} - \frac{3}{2}\frac{v}{u}\big)\log\big[\frac{27\Lambda^6}{4u^3}\big] + ... \\
a_1 &= \frac{1}{2}\omega_1^{P_3} + \frac{1}{2}\omega_2^{P_3} \sim \sqrt{u} + \frac{1}{2}\frac{v}{u} + ... \\
a_2 &= \frac{1}{2}\omega_1^{P_3} - \frac{1}{2}\omega_2^{P_3} \sim \sqrt{u} - \frac{1}{2}\frac{v}{u} + ...
\end{aligned} \tag{6.4}$$

Here, α_1, α_2 are parameters that cannot determined by the Picard-Fuchs equations. Their values can only be found by comparison with the asymptotic expansion of the period integrals. This is done in Appendix A with the result: $\alpha_1 = \frac{5}{4}i - i\log 2 - \frac{3}{4}i\log 3$, $\alpha_2 = \frac{3}{4}i + \frac{9}{4}i\log 3$. For a loop around $u = \infty$, (6.4) indeed gives back precisely the semi-classical monodromy $M_{\infty,u}^{(r_3)}$ in (3.22).

We can treat the semi-classical coordinate patch P_2 in a similar way, and find that for a loop around $v = \infty$ the semi-classical Coxeter monodromy $M_{\infty,v}^{(r_{cox})}$ in (3.22) is reproduced.

To obtain the prepotential $\mathcal{F}$, we need to invert the series $a_i(u, v)$, ensuring good convergence in terms of the Cartan sub-algebra variables a_1, a_2. We can start with either patch P_2 or P_3, but we will choose P_3 for convenience. Since in the patch P_3 the classical Casimir $u_0 \equiv {a_1}^2 + {a_2}^2 - a_1a_2$ is large and $v_0 \equiv a_1a_2(a_1 - a_2)$ is small, we can expand, for example, around $(a_2/a_1), (\sqrt{3}\Lambda/a_1) \sim 0$. Note that we have, essentially, a double expansion in one dimensionful and one dimensionless parameter. Though

the inversion of the double infinite series $a_i(u,v)$ in (6.4) in a closed form appears is quite hard, we can explicitly compute the quantum corrected Casimirs

$$\begin{aligned} u(a_1,a_2) &= u_0(a_1,a_2) + 27\Lambda^6\Big\{\frac{1}{72\,{a_1}^4} + \frac{a_2}{36\,{a_1}^5} + \frac{31\,{a_2}^2}{288\,{a_1}^6} + \dots\Big\} + \mathcal{O}(\Lambda^{12},\frac{a_2}{a_1}) \\ v(a_1,a_2) &= v_0(a_1,a_2) - 27\Lambda^6\Big\{\frac{a_2}{24\,{a_1}^4} + \frac{{a_2}^2}{12\,{a_1}^5} + \frac{9\,{a_2}^3}{32\,{a_1}^6} + \dots\Big\} + \mathcal{O}(\Lambda^{12},\frac{a_2}{a_1}) \end{aligned} \tag{6.5}$$

to any given order. Inserting this into $a_{D1}(u,v), a_{D2}(u,v)$ and integrating $\int a_{Di}\,da_i$, we obtain the prepotential in the form

$$\mathcal{F}(a_1,a_2) \;=\; \frac{i}{2\pi}6u_0\big(\,\log\big[\frac{a_1}{\sqrt{3}\,\Lambda}\big] - \sum_{k=0}^{\infty}\mathcal{F}_{6k}(a_1,a_2)\,\Lambda^{6k}\big)\;. \tag{6.6}$$

We find that the logarithmic term together with $\mathcal{F}_0$ gives indeed precisely the small-(a_2/a_1)-expansion of the semi-classical prepotential (3.16), ie.,

$$\frac{i}{2\pi}6u_0\big(\,\log\big[\frac{a_1}{\sqrt{3}\,\Lambda}\big] - \mathcal{F}_0(a_1,a_2)\big) \;=\; \frac{1}{6}\big(\sum_{i=1}^{3}{Z_i}^2\big)\,\tau_0 + \frac{i}{4\pi}\sum_{i=1}^{3}{Z_i}^2\log\,[{Z_i}^2/\Lambda^2]\;, \tag{6.7}$$

where $Z_i = Z_i(a_1,a_2)$ are the classical central charges (3.9), and where

$$\tau_0 \;=\; \frac{i}{2\pi}\big(\log\big[\frac{4}{27}\big] - 9\big) \;\equiv\; \frac{\theta_0}{\pi} + \frac{8\pi i}{{g_0}^2} \tag{6.8}$$

is the "bare" coupling constant. It is a priori defined up to even integers, which is a reflection of the quantum monodromy T^2 (3.19) induced by 2π rotations of Λ^3 (actually, since the curve (4.4) is unchanged even under 2π rotations of Λ^6, we have an ambiguity in τ_0 up to adding integers.) Of course, other semi-classical monodromies may induce additional integral matrix shifts for the θ-angle.

In fact, τ_0 can be tuned to an arbitrary complex number by appropriately rescaling Λ. This is why we took some effort to obtain this coupling (by evaluating the integrals in order to fix all undetermined parameters in (6.4)), since its imaginary part needs to be fixed if one eventually wants to relate the present quantum scale Λ (defined by the curve (4.4)) to the scale used in some other physical computation.

We see from (6.7) that by writing $\mathcal{F}$ in terms of the variables Z, we effectively sum up the series in the dimensionless variable (a_2/a_1). The remaining series in $(\sqrt{3}\Lambda/a_1)$

can then be interpreted in terms of non-perturbative quantum corrections. In total, we find for the asymptotic Λ-expansion of the exact quantum prepotential:

$$\mathcal{F}(a_1,a_2) \;=\; \mathcal{F}_{\rm class}(a_1,a_2)+\mathcal{F}_{\text{1-loop}}(a_1,a_2)+\mathcal{F}_{\text{non-pert}}(a_1,a_2)\;, \tag{6.9}$$

with

$$\begin{aligned}
\mathcal{F}_{\rm class} \;&=\; \frac{1}{2}\,\tau_0\,(\vec{a}^{\,t}\cdot C\cdot\vec{a})\\
\mathcal{F}_{\text{1-loop}}(a_1,a_2) \;&=\; \frac{i}{4\pi}\sum_{i=1}^{3} {Z_i}^2\log\,[{Z_i}^2/\Lambda^2]\\
\mathcal{F}_{\text{non-pert}} \;&=\; -\frac{i}{2\pi}\Big(\sum_{i=1}^{3}{Z_i}^2\Big)\sum_{k=1}^{\infty}\mathcal{F}_{6k}(Z)\Lambda^{6k}\;.
\end{aligned} \tag{6.10}$$

The leading "instanton coefficients" are given in terms of symmetric Laurent polynomials in the Z_i as follows:[†]

$$\begin{aligned}
\mathcal{F}_6(Z) \;&=\; \frac{1}{4}\frac{1}{{Z_1}^2{Z_2}^2{Z_3}^2} \;\equiv\; \frac{1}{4}\frac{1}{\Delta_0}\\
\mathcal{F}_{12}(Z) \;&=\; -\frac{1}{2^6}\Big[57\frac{1}{{Z_1}^4{Z_2}^4{Z_3}^4}-5\Big(\frac{1}{{Z_1}^6{Z_2}^6}+\text{cycl.}\Big)\Big]\\
&\equiv\; \frac{3}{2^5}\frac{1}{{\Delta_0}^3}(17{u_0}^3+189{v_0}^2)\\
\mathcal{F}_{18}(Z) \;&=\; -\frac{3}{47\,2^6}\Big[1265\Big(\frac{1}{{Z_1}^8{Z_2}^{10}}+\frac{1}{{Z_1}^{10}{Z_2}^8}+\text{cycl.}\Big)\\
&\quad -1492\Big(\frac{1}{{Z_1}^6{Z_2}^{12}}+\frac{1}{{Z_1}^{12}{Z_2}^6}+\text{cycl.}\Big)+746\Big(\frac{1}{{Z_1}^4{Z_2}^{14}}+\frac{1}{{Z_1}^{14}{Z_2}^4}+\text{cycl.}\Big)\\
&\quad -1492\Big(\frac{1}{{Z_1}^2{Z_2}^2{Z_3}^{14}}+\text{cycl.}\Big)-76701\frac{1}{{Z_1}^6{Z_2}^6{Z_3}^6}\Big]\\
&\equiv\; \frac{9}{2^5}\frac{1}{{\Delta_0}^5}\big(3080{u_0}^6+119529{u_0}^3{v_0}^2+248589{v_0}^4\big)\;,\quad\text{etc.}\;,
\end{aligned} \tag{6.11}$$

where Δ_0 is the classical discriminant (3.10), $\Delta_0 = 4{u_0}^3 - 27{v_0}^2$. As expected, the non-perturbative corrections get arbitrarily suppressed in the weak coupling limit, where $\Delta_0 \to \infty$.

Note that the prepotential (6.9) is manifestly Weyl group invariant, in contrast to the Weyl non-covariant expansions in $a_{1,2}$. This confirms that we have correctly

† The higher $\mathcal{F}_{6k}$ have no unique form when written in terms of the Z's, since some combinations vanish, but they are unique when written in terms of the a_i.

resummed the infinite series in the dimensionless variable. One can check that when starting from a different expansion that is adapted to the patch P_2, one obtains the same effective prepotential in terms of the variables Z. Thus, the Weyl invariant resummation in terms of the variables Z simultaneously covers both patches at "infinity" in $\mathcal{M}_\Lambda$.

6.2. Dual magnetic semi-classical regime

We can compute in a similar fashion the effective action $\mathcal{F}_D$ in the dual semi-classical, magnetic regime, ie., in the patch Q_1 where two monopoles become simultaneously massless. To lowest order, this has been done for all $SU(n)$ in ref. [5]. In contrast, though our techniques are not practical for computations for general n, they allow an easy determination of the corrections to $\mathcal{F}_D$ for any given $SU(n)$ group (here $G = SU(3)$) to arbitrary order.

Since the Appell functions in (6.4) cannot easily expanded or resummed in the variables $\delta_\pm$ near the nodes (and appropriate transformation formulas for F_4 do not seem to be known), we prefer to resort to the explicit series solutions (5.19) to make the following identifications:

$$\frac{1}{\sqrt{3}\,\Lambda}\begin{pmatrix} a_{D1} \\ a_{D2} \\ a_1 \\ a_2 \end{pmatrix} \;=\; \alpha_0 \begin{pmatrix} i\omega_2^{Q_1} \sim i\delta_-(1+...) \\ i\omega_1^{Q_1} \sim i\delta_+(1+...) \\ \frac{1}{2\pi}(\Omega_2^{Q_1} + \sum \alpha_{2j}\omega_j^{Q_1}) \sim \frac{1}{2\pi}\delta_- \log[\delta_-] + ... \\ \frac{1}{2\pi}(\Omega_1^{Q_1} + \sum \alpha_{1j}\omega_j^{Q_1}) \sim \frac{1}{2\pi}\delta_+ \log[\delta_+] + ... \end{pmatrix} . \qquad (6.12)$$

As before, the undetermined parameters α_{ij}, as well as the overall normalization α_0, can be fixed by asymptotically evaluating the period integrals. This is done in Appendix A, with the result: $\alpha_0 = -\frac{2^{-1/3}}{3}$, $\alpha_{11} = \alpha_{22} = -2\log 2 - 3\log 3$, $\alpha_{12} = \alpha_{21} = -2 - 2\log 2$.

Note that even though the solutions in $\delta_\pm \sim \prod_{i<j}^3 (e_i^\pm - e_j^\pm)^2$ are symmetric in $e_i^\pm - e_j^\pm$, the identification (6.12) is valid only for one given intersection, where $e_i^+ - e_j^+ = 0$ and $e_k^- - e_l^- = 0$ for some given i, j, k, l. To be specific, we have chosen the node at $u = (\frac{27}{4})^{1/3}\Lambda^2$, $v = 0$, where the lines #2 (where $e_1^- - e_2^- = 0$) and #3 (where $e_1^+ - e_3^+ = 0$) of Fig.4 intersect. Encircling the lines $\delta_\pm = 0$, (6.12) clearly reproduces the correct strong coupling monodromies, given by the matrices $M_{(1,0,0,0)}$ and $M_{(0,1,0,0)}$ in (4.20). Note that even though (6.12) represents a good solution at the remaining other two nodes, it does not represent the correct identification with the period integrals there. One rather has to conjugate the above basis with the cyclic

transformation U (3.21) to obtain the proper identifications with the period integrals at the other nodes. Of course, since all nodes are equivalent under this $\mathbb{Z}_3$ symmetry, it suffices to study the situation only at one node.

Just like in the semi-classical region, we can easily invert the series solutions $\omega_1^{Q_1}$ and $\omega_2^{Q_1}$ and integrate $\int_{a_{Di}} a_i$ to obtain for the dual effective prepotential the following result:

$$\mathcal{F}_D(a_1, a_2) \;=\; \mathcal{F}_{D,0}(a_1, a_2) + \mathcal{F}_{D,\text{1-loop}}(a_1, a_2) + \mathcal{F}_{D,\text{thresh.}}(a_1, a_2) \;, \tag{6.13}$$

where

$$\begin{aligned}
\mathcal{F}_{D,0} \;&=\; 18\frac{i}{\pi}\beta_0\Lambda(a_{D1}+a_{D2}) \\
&\quad + \frac{i}{\pi}\big(\frac{3}{8}+\frac{1}{2}\log 2+\frac{3}{4}\log 3\big)\,({a_{D1}}^2+{a_{D2}}^2) + a_{D1}a_{D2}\frac{i}{\pi}\log 2 \\
\mathcal{F}_{D,\text{1-loop}} \;&=\; \frac{1}{4\pi i}\sum_{i=1}^{2}\big({a_{Di}}^2\log\big[\frac{a_{Di}}{\beta_0\Lambda}\big]\big) \\
\mathcal{F}_{D,\text{thresh.}} \;&=\; \frac{1}{2\pi i}\sum_{k=1}^{\infty}\mathcal{F}_{D,k}(a_{Di})(432\,\beta_0\Lambda)^{-k} \;.
\end{aligned} \tag{6.14}$$

with $\beta_0 = \frac{i}{\sqrt{3}}2^{-1/3}$. Information about the massive spectrum is encoded in the threshold corrections

$$\begin{aligned}
\mathcal{F}_{D,1} \;&= -\,(a_{D1}+a_{D2})\,\big(4\,{a_{D1}}^2 - 13\,a_{D1}\,a_{D2} + 4\,{a_{D2}}^2\big) \\
\mathcal{F}_{D,2} \;&=\; \big(44\,{a_{D1}}^4 - 207\,{a_{D1}}^3\,a_{D2} - 189\,{a_{D1}}^2\,{a_{D2}}^2 - 207\,a_{D1}\,{a_{D2}}^3 + 44\,{a_{D2}}^4\big) \\
\mathcal{F}_{D,3} \;&= -\,(a_{D1}+a_{D2})\,\big(896\,{a_{D1}}^4 - 7475\,{a_{D1}}^3\,a_{D2} + 2399\,{a_{D1}}^2\,{a_{D2}}^2 \\
&\qquad - 7475\,a_{D1}\,{a_{D2}}^3 + 896\,{a_{D2}}^4\big) \;,\quad \text{etc.} \;.
\end{aligned} \tag{6.15}$$

$\mathcal{F}_D(a_{D1}, a_{D2})$ indeed represents an effective action for two $U(1)$ gauge fields, and is manifestly symmetric under exchange of a_{D1} and a_{D2}. Observe also that the corresponding β-functions are asymptotically non-free, and reflect the coupling to fundamental matter fields with charges $(1,0)$ and $(0,1)$. These correspond to pure magnetic monopoles in the original variables.

6.3. Properties of the period matrix, and various dualities

Analogous to $G = SU(2)$, there exist for $SU(3)$ certain types of dualities that relate the electric semi-classical regime near infinity in $\mathcal{M}_\Lambda$ with the dual magnetic semi-classical regime near the nodes. To see this, let us first study the physical gauge and dual gauge couplings, $\tau_{ij} \equiv \frac{\partial^2}{\partial_{a_i}\partial_{a_j}}\mathcal{F}(a)$ and $\tau_{D,ij} \equiv \frac{\partial^2}{\partial_{a_{Di}}\partial_{a_{Dj}}}\mathcal{F}_D(a_D) \equiv -(\tau_{ij})^{-1}$, when expressed in terms of $\alpha \equiv 4u^3/27\Lambda^6$, $\beta \equiv v^2/\Lambda^6$.

From our identifications (6.4), we can easily compute the period matrix τ, which is the exact quantum gauge coupling constant:

$$\Pi \;=\; \begin{pmatrix} \partial_u a_1 \, \partial_u a_2 \\ \partial_v a_1 \, \partial_v a_2 \end{pmatrix} ; \begin{pmatrix} \partial_u a_{D1} \, \partial_u a_{D2} \\ \partial_v a_{D1} \, \partial_v a_{D2} \end{pmatrix} \;\equiv\; \Big(A \,;\, B \Big) \;, \qquad \tau(u,v) \;=\; A^{-1}B \;, \quad (6.16)$$

Explicit expressions for the periods are collected in Appendix B. Of course, to make sense of $\tau(u,v)$ over the whole moduli space, we must suitably analytically continue the Appell functions. For sufficiently small v, we can resort to (6.3) in order to continue to all u. (One may also use the dual formula to continue to all v for small u. This might be useful to study the behavior at the cusps.)

In particular, we may continue τ to the origin of moduli space. This serves as a useful consistency check, since $\tau(0,0)$ is completely fixed by the $\mathbb{Z}_6$ symmetry of the curve. That is, we require that $A \equiv \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in (4.21) leaves the period matrix invariant: $\tau = (a\,\tau + b)(c\,\tau + d)^{-1}$. In addition, we know that the $\mathbb{Z}_6$ symmetry acts on the abelian differentials as follows: $A : \; \frac{dx}{y} \to -\xi\frac{dx}{y}$, $\frac{xdx}{y} \to -\xi^2\frac{xdx}{y}$, where $\xi = e^{2\pi i/6}$. This transformation has determinant equal to -1, and this must be the same as the determinant of $(c\,\tau + d)$. These conditions, as well as the positivity of $\mathrm{Im}\,\tau$, completely determine the period matrix at the origin as follows:

$$\tau(0,0) \;=\; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{i}{\sqrt{3}}\, C \;, \quad (6.17)$$

where $C \equiv \left(\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right)$ is the Cartan matrix.

Note that the identifications (6.4) were based on the comparison with the period integrals of Appendix A. For these integrals, a basis of cycles was chosen that corresponds to $\mathrm{Im}\, v = 0$, $\mathrm{Re}\, v > 1$, in order to match the basis given in Fig.6 and to reproduce the semi-classical monodromy (3.22). If we want to analytically continue to $(u,v) = 0$, we need to take into account that for $-1 < \mathrm{Re}\, v < 1$ the identifications between a_i, a_{Di} and the solutions of the PF equations change. This change of basis is

the one that relates the monodromy $M^{(r_3)}_{\infty,u}$ (3.22) (defined by a loop around infinity in the u-plane at $\mathrm{Re}\, v > 1$) with the monodromy $\widetilde{M}^{(r_3)}_{\infty,u}$ (4.23) (defined by a loop around infinity in the u-plane at $\mathrm{Re}\, v = 0$). Taking this change of basis into account, we indeed find (6.17) to hold, by evaluating the period matrix (6.16) at the origin.

Since non-trivial transformation properties of F_4 do not appear to be known, it is quite hard to find all possible duality symmetries that may act on the moduli space. What we can do is to consider transformations that act solely on $\alpha \equiv 4u^3/27\Lambda^6$ when $\beta \equiv v^2/\Lambda^6 = 0$. For this, it is helpful to rewrite the periods for $\beta = 0$ in the following form:

$$\begin{aligned}
\partial_u a_1 \;&=\; \partial_u a_2 \;=\; (\sqrt{3}\Lambda)^{-1}(-1)^{1/6}2^{-2/3}(1-\alpha)^{-1/6}\,{}_2F_1\big(\frac{1}{6},\frac{1}{6},1;\frac{1}{1-\alpha}\big)\\
\partial_v a_1 \;&=-\partial_v a_2 \;=\; (3\Lambda^2)^{-1}(-1)^{1/3}2^{-1/3}(1-\alpha)^{-1/3}\,{}_2F_1\big(\frac{1}{3},\frac{1}{3},1;\frac{1}{1-\alpha}\big)\\
\partial_u a_{D1} \;&=\; \partial_u a_{D2} \;=\; i(\sqrt{3}\Lambda)^{-1}\,2^{-2/3}\,{}_2F_1\big(\frac{1}{6},\frac{1}{6},1;1-\alpha\big)\\
\partial_v a_{D1} \;&=-\partial_v a_{D2} \;=-i(3\Lambda^2)^{-1}\,2^{-1/3}\sqrt{3}\,{}_2F_1\big(\frac{1}{3},\frac{1}{3},1;1-\alpha\big)\;.
\end{aligned} \tag{6.18}$$

From these expressions one can then infer that under

$$I:\quad \alpha \;\longrightarrow\; \widetilde{\alpha} \;=\; \frac{\alpha}{\alpha-1}\;, \tag{6.19}$$

(which just exchanges the arguments of the hypergeometric functions), the period matrix transforms as follows,

$$\tau(\alpha,\beta=0) \;=\; C\cdot\tau_D(\widetilde{\alpha},\widetilde{\beta}=0)\;, \tag{6.20}$$

provided that $(1-\alpha) \in \mathbb{R}^-$. The presence of the Cartan matrix reflects that the bases of the electric and magnetic degrees of freedom are given by the Dynkin and the simple root bases, respectively, as explained in section 4.2. This electric-magnetic duality obviously generalizes the isogeny transformation (2.15),(2.13) for $G = SU(2)$. Though we did not succeed to find an extension of (6.19) to non-zero β, we believe that such an extension does exist, and reflects a transformation property of F_4. If true, it would be very likely that analogously to $G = SU(2)$, there exist dual, isogenous forms of the hyperelliptic curve (4.1), for which electric and magnetic degrees of freedom are exchanged. This is also suggested by the form of the monodromies, as mentioned in section 4.2 and section 4.3.

In addition, analogous to $G = SU(2)$ there exists (at least for $\beta = 0$) an S-duality, which acts like

$$\tau(\alpha, \beta = 0) \;=\; \tau_D(\widehat{\alpha}, \widehat{\beta} = 0) \;, \tag{6.21}$$

with $\widehat{\alpha} = I(\widetilde{S}(I(\alpha)))$, where I is the isogeny map (6.20) and

$$\widetilde{S}: \;\alpha \;\longrightarrow\; \frac{1}{(\alpha^{1/3}-1)^3}\Big\{8+\alpha+12(-1)^{2/3}\alpha^{1/3}-6(-1)^{1/3}\alpha^{2/3}\Big\} \;. \tag{6.22}$$

Probably there exist other dualities as well, for example, a transformation that relates the cusp patches Q_2 and Q_3.

7. Conclusions and Outlook

The main result of the present paper are the explicit expressions (6.10) and (6.14) for the quantum effective prepotential, whose expansion can easily be determined to any given order in Λ. It would be very interesting to compare the instanton corrections with expressions obtained by some other kind of computation.

There are, of course, many aspects that were not touched upon in the present paper. Some of the aspects that we neglected were recently discussed, eg., in [8,9], and need not be repeated here. We just like comment on a few things.

First, it is clear that similar to $SU(2)$ (where $u(\tau)$ is given by a modular function of $\Gamma_0(4)$ or $\Gamma(2)$), the variables $u_k(\tau)$ should be given by certain higher genus modular functions (whose modular properties include the S-duality transformation (6.22)). Such functions are intrinsically defined via lattice sums of type $\sum(m+n\tau)^l$, with period lattices given here by root lattices. (For $SU(3)$, these functions could be found by inverting the expressions given in Appendix B.) One question that arises would be what the physical interpretation of such lattice sums is. Certainly one would like to think in terms of partition functions involving massive excitations, but the counting of states, of which many are unstable, would probably be subtle. Also, in contrast to $N=4$ supersymmetric theories, $|m+n\tau|^2$ does not give the mass of a state, so that these lattice functions do not give the mass spectrum but rather count electric and magnetic charges.

Among the open points is also the generalization to other groups ($G = SO(2n+1)$ was recently treated in [7]). Actually, our classical considerations in section 3.1 directly generalize to other Lie algebras. General formulae for the discriminants (3.10)

are discussed in the literature on arrangements of hyperplanes [20]. In particular, for the remaining simply laced Lie algebras of type D and E, the following simple singularities [11] are relevant:

$$\begin{aligned}
\mathcal{W}_{D_n}(x_1,x_2,u) &= {x_1}^{n-1} + \tfrac{1}{2} x_1 {x_2}^2 - \sum_{l=1}^{n-1} u_{2l}\, {x_1}^{n-l-1} - \widetilde{u}_n x_2 \\
\mathcal{W}_{E_6}(x_1,x_2,u) &= {x_1}^3 + {x_2}^4 - u_2 x_1 x_2^2 - u_5 x_1 x_2 - u_6 x_2^2 - u_8 x_1 - u_9 x_2 - u_{12} \\
\mathcal{W}_{E_7}(x_1,x_2,u) &= {x_1}^3 + {x_2}^4 - u_2 x_1^2 x_2 - u_6 x_1^2 - u_8 x_1 x_2 - u_{10} x_2^2 \\
&\quad - u_{12} x_1 - u_{14} x_2 - u_{18} \\
\mathcal{W}_{E_8}(x_1,x_2,u) &= {x_1}^3 + {x_2}^5 - u_2 x_1 x_2^3 - u_8 x_1 x_2^2 - u_{12} x_2^3 - u_{14} x_1 x_2 \\
&\quad - u_{18} x_2^2 - u_{24} x_2 - u_{30} \ ,
\end{aligned} \tag{7.1}$$

where u_k are one-to-one to the Casimirs of the corresponding algebra. The associated discriminants characterize classical Yang-Mills theories based on the simply laced Lie algebras of type D and E. As for the quantum theories, we expect the underlying curves to be of the form

$$p(x_i)\big(\mathcal{W}_{ADE}(x_i)\big)^2 \ - q(x_i)\Lambda^{2h} \ = \ 0 \ , \tag{7.2}$$

where h is the corresponding dual Coxeter number, and $p(x)$, $q(x)$ are suitable polynomials. The key point is the quadratic appearance of the simple singularity, which ensures that any singular branch of the classical singularity in $\mathcal{M}_0$ (describing an unbroken $SU(2)$) splits into two quantum branches (describing massless $SU(2)$ Seiberg-Witten monopoles). However, the genus of the curves (7.2) does not seem to easily come out correctly, although this criticism may be too naive in view of the $SO(2n+1)$ curves of [7].

We also remark that most of our considerations in section 4.2 about properties of BPS states in relation with vanishing cycles directly apply or generalize to situations involving "level" surfaces other than Riemann surfaces. In particular, they apply to classical $SU(n)$ $N=2$ Yang-Mills theory, as was already pointed out in section 3.1. They also apply to BPS states of extended supersymmetric string compactifications, where the relevant surfaces are $K3$ [21] and Calabi-Yau [22] manifolds. We think it would be interesting and important to develop a coherent and systematic picture of BPS states related to vanishing cycles, especially in view of the type II – heterotic string duality [23,24].

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Appendix A. Asymptotic evaluation of period integrals

The hyperelliptic period integrals $\int_{\gamma_i} \lambda$ provide a definite basis for the solutions of the Picard-Fuchs equations everywhere in the moduli space. In order to relate them to the various local expansion that one gets by solving the Picard-Fuchs equations, we have to compute some low order terms of the asymptotic expansions in terms of those variables by which we parametrize the vicinity of 'infinity' and the node, respectively. In this appendix we will provide some of the details of those calculations, which enable us to fix the physical relevant quadratic terms in the prepotential and constitute a check of our monodromy considerations.

The integrals to be computed are of the form

$$w_{ij} \;=\; \frac{i}{2\pi}\int_{e_i}^{e_j} \lambda \;=\; \frac{i}{2\pi}\int_{e_i}^{e_j} \frac{x(3x^2-u)dx}{y}$$

where e_i are the roots of the polynomial $y^2 = p(x) = 0$.

The first step is to find approximate expressions for the roots e_i and then to expand the integrals such as to reduce them to elementary integrals. We will do this for the semi-classical infinity at P_3 and the node at $v = 0, u = (\frac{27}{4})^{1/3}\Lambda^2$ in turn.

(i) *Period integrals at infinity*: Here we introduce variables $\alpha = \frac{v}{u^{3/2}}$, $\beta = \frac{\Lambda^3}{u^{3/2}}$ s.t. infinity is at the origin $(\alpha, \beta) = (0, 0)$ and the roots of y are approximately $e_i \equiv \sqrt{u}\widetilde{e}_i$, with:

$$\begin{aligned}
\widetilde{e}_1 &\simeq -1 + \frac{1}{2}(\alpha-\beta) + \frac{3}{8}(\alpha-\beta)^2\,, \quad \widetilde{e}_3 \simeq -\alpha-\beta\,, \\
&\qquad\qquad\qquad\qquad\qquad\qquad \widetilde{e}_5 \simeq 1 + \frac{1}{2}(\alpha-\beta) - \frac{3}{8}(\alpha-\beta)^2 \\
\widetilde{e}_2 &\simeq -1 + \frac{1}{2}(\alpha+\beta) + \frac{3}{8}(\alpha+\beta)^2\,, \quad \widetilde{e}_4 \simeq -\alpha+\beta\,, \\
&\qquad\qquad\qquad\qquad\qquad\qquad \widetilde{e}_6 \simeq 1 + \frac{1}{2}(\alpha+\beta) - \frac{3}{8}(\alpha+\beta)^2
\end{aligned}$$

with $e_{2i-1} = e_{2i}$, $i = 1, 2, 3$ at semi-classical infinity. These roots are ordered in an obvious fashion. Their relation to the branch points and cycles of Fig.6 is ambiguous

and depends on the path used in the analytic continuation. This ambiguity is physically irrelevant, and corresponds to a Weyl conjugation. By choosing a specific path, we can make the following associations: $e_1 \to e_3^-, e_2 \to e_3^+, e_3 \to e_1^+, e_4 \to e_1^-, e_5 \to e_2^-, e_6 \to e_2^+$. Thus, the integrals $w_{2i-1,2i}$ are related to the a-type periods and will be given by pure power series, whereas $w_{2i,2i+1}$, which are related to the a_D-type periods, will have logarithms.

To compute $\widetilde{w}_{12}$, we introduce the variable $w = x - \frac{1}{2}(e_2 - e_1)$ and get, after expanding the integrand in powers of small quantities:

$$2w_{12} = -a_2 \simeq \frac{\sqrt{u}}{\pi}\int_{-\beta/2}^{\beta/2} \frac{dw}{\sqrt{\frac{1}{4}\beta^2 - w^2}}(-1 + \frac{1}{2}\alpha) = -(\sqrt{u} - \frac{1}{2}\frac{v}{u})$$

Likewise we get $2w_{34} \simeq -\frac{v}{u}$ and $2w_{56} = a_1 \simeq (\sqrt{u} + \frac{1}{2}\frac{v}{u})$.

The logarithmic periods are more involved. In order to compute e.g., w_{23}, we split the range of integration into two pieces, namely $\int_{e_2}^{e_3}\lambda = \int_{e_2}^{\xi}\lambda + \int_{\xi}^{e_3}\lambda$, where $\xi - e_2 \sim e_3 - \xi$. Both integrands can then be expanded leading to elementary integrals of the form $\int \frac{p(w)}{\sqrt{w^2-a^2}\,(x+b)^n}$ with p a polynomial. We find

$$2w_{23} = a_{D2} \simeq -\frac{i}{\pi}\Big\{\sqrt{u}\Big(3 + \frac{1}{2}\log(\frac{\Lambda^6}{64u^3})\Big) - \frac{v}{u}\Big(\frac{3}{4}\log(\frac{\Lambda^6}{4u^3})\Big)\Big\}$$

and

$$2w_{45} = a_{D1} \simeq -\frac{i}{\pi}\Big\{\sqrt{u}\Big(3 + \frac{1}{2}\log(\frac{\Lambda^6}{64u^3})\Big) + \frac{v}{u}\Big(\frac{3}{4}\log(\frac{\Lambda^6}{4u^3})\Big)\Big\}$$

(ii) *Period integrals at the nodes*: parametrized by $\delta_\pm = 1 - \alpha + \beta \pm 2\sqrt{\beta}, \alpha = \frac{4u^3}{27\Lambda^6}$, $\beta = \frac{v^2}{\Lambda^6}$ the roots are approximately ($e_i \equiv 2^{-1/3}\Lambda\widetilde{e}_i$):

$$\widetilde{e}_1 = -2(1 - \frac{\delta_+}{12} - \frac{\delta_-}{36}), \quad \widetilde{e}_2 = -(1 - i\sqrt{\frac{\delta_+}{3}} - \frac{\delta_+}{36} - \frac{\delta_-}{12}),$$
$$\widetilde{e}_3 = -(1 + i\sqrt{\frac{\delta_+}{3}} - \frac{\delta_+}{36} - \frac{\delta_-}{12})$$
$$\widetilde{e}_6 = 2(1 - \frac{\delta_+}{36} - \frac{\delta_-}{12}), \quad \widetilde{e}_4 = (1 + i\sqrt{\frac{\delta_-}{3}} - \frac{\delta_+}{12} - \frac{\delta_-}{36}),$$
$$\widetilde{e}_5 = (1 - i\sqrt{\frac{\delta_-}{3}} - \frac{\delta_+}{12} - \frac{\delta_-}{36})$$

We now consider the node $v = v_0 = 0, u = u_0 = (\frac{27}{4})^{1/3}\Lambda^2$.

Close to the node we have $v = v_0 + \delta v = \frac{\Lambda^3}{4}(\delta_+ - \delta_-)$ and $u = u_0 + \delta u \simeq -\frac{1}{6}(\frac{27}{4})^{1/3}\Lambda^2(\delta_+ + \delta_-)$, or $\delta_\pm \simeq -\frac{2^{2/3}}{\Lambda^2}(u - u_0) \pm \frac{2}{\Lambda^3}v$. The computation of the a_D-type periods is straightforward. Expanding the integrand in powers of $\delta_\pm$ leads to elementary integrals, and we find

$$2w_{23} = a_{D2} \simeq -\frac{i\Lambda}{3^{1/2}2^{1/3}}\delta_+ = \frac{i\Lambda}{3^{1/2}2^{1/3}}\Big(\frac{2^{2/3}}{\Lambda^2}(u - \frac{3\Lambda^2}{2^{2/3}}) - 2\frac{v}{\Lambda^3}\Big) \ ,$$

and likewise

$$2w_{45} = a_{D1} \simeq -\frac{i\Lambda}{3^{1/2}2^{1/3}}\delta_- = \frac{i\Lambda}{3^{1/2}2^{1/3}}\Big(\frac{2^{2/3}}{\Lambda^2}(u - \frac{3\Lambda^2}{2^{2/3}}) + 2\frac{v}{\Lambda^3}\Big) \ .$$

The computation of the logarithmic solutions is more cumbersome. We again split the integral into two pieces in order to be able to deal with the singularities of the integrand separately; e.g. for w_{34} : $\int_{e_3}^{e_4}\lambda = \int_{e_3}^{\xi}\lambda + \int_{\xi}^{e_4}\lambda$ such that $e_{2,3} < \xi < e_{45}$ and $|\xi - e_3|/|e_3 - e_2|, |\xi - e_4|/|e_5 - e_4| \gg 1$. Independence of the choice of ξ serves as a check. Expanding the integrand and the limits of integration in powers of $\delta_\pm^{1/2}$ leads to elementary and elliptic integrals. For instance, taking the lowest order terms for the roots $e_i, i \neq 2, 3$ and u we are led to the integral $\int_{e_3}^{\xi}\frac{x(x+1)}{\sqrt{(x-e_2)(x-e_3)}\sqrt{4-x^2}}$. To do the integral we introduce the variable $w = -1 - x + \frac{1}{36}\delta_+ + \frac{1}{12}\delta_-$, expand $1/\sqrt{4-x^2}$ in a power series in x and then expand x^n to order $\epsilon_\pm^2$. The remaining integrals can now be performed keeping only terms to order ϵ^2 and $\epsilon^2\log(\epsilon)$. The resulting series can then be resummed. After some work, we finally get:

$$2w_{12} = -a_2 \simeq -\frac{\Lambda}{\sqrt{3}}\frac{2^{2/3}3}{\pi}\Big\{-3 - \frac{1}{12}\delta_+\Big(\log(\delta_+) - 2\log 2 - 3\log 3 - 1\Big) + \frac{1}{6}\delta_-\log 2\Big\}$$

and

$$2w_{56} = a_1 \simeq \frac{\Lambda}{\sqrt{3}}\frac{2^{2/3}3}{\pi}\Big\{-3 - \frac{1}{12}\delta_-\Big(\log(\delta_-) - 2\log 2 - 3\log 3 - 1\Big) + \frac{1}{6}\delta_+\log 2\Big\} \ .$$

Appendix B. Explicit expression for the period matrix

The period matrix, which represents the exact $SU(3)$ quantum gauge coupling constant, is given by $\tau(u,v) = A^{-1}B$, where

$$A \;=\; \begin{pmatrix} \partial_u a_1 & \partial_u a_2 \\ \partial_v a_1 & \partial_v a_2 \end{pmatrix} , \qquad B \;=\; \begin{pmatrix} \partial_u a_{D1} & \partial_u a_{D2} \\ \partial_v a_{D1} & \partial_v a_{D2} \end{pmatrix} .$$

The periods are, in terms of multi-valued Appell functions F_4 (6.2) of $\alpha \equiv 4u^3/27\Lambda^6$ and $\beta \equiv v^2\Lambda^6$, as follows:

$$\begin{aligned} \left(\sqrt{3}\Lambda\right) \partial_u a_1 =\; & 2^{-2/3}\alpha^{-1/6}\, F_4\big(\tfrac{1}{6},\tfrac{5}{6},\tfrac{1}{2},1;\tfrac{\beta}{\alpha},\tfrac{1}{\alpha}\big) \\ & - 2^{1/3}\frac{1}{3}\sqrt{\frac{\beta}{3}}\,\alpha^{-2/3} F_4\big(\tfrac{2}{3},\tfrac{4}{3},\tfrac{3}{2},1;\tfrac{\beta}{\alpha},\tfrac{1}{\alpha}\big) \end{aligned}$$

$$\begin{aligned} \left(\sqrt{3}\Lambda\right) \partial_u a_2 =\; & 2^{-2/3}\alpha^{-1/6}\, F_4\big(\tfrac{1}{6},\tfrac{5}{6},\tfrac{1}{2},1;\tfrac{\beta}{\alpha},\tfrac{1}{\alpha}\big) \\ & + 2^{1/3}\frac{1}{3}\sqrt{\frac{\beta}{3}}\,\alpha^{-2/3} F_4\big(\tfrac{2}{3},\tfrac{4}{3},\tfrac{3}{2},1;\tfrac{\beta}{\alpha},\tfrac{1}{\alpha}\big) \end{aligned}$$

$$\begin{aligned} \left(3\Lambda^2\right) \partial_v a_1 =\; & 2^{-1/3}\alpha^{-1/3}\, F_4\big(\tfrac{1}{3},\tfrac{2}{3},\tfrac{1}{2},1;\tfrac{\beta}{\alpha},\tfrac{1}{\alpha}\big) \\ & - 2^{-1/3}\sqrt{\frac{\beta}{3}}\,\alpha^{-5/6} F_4\big(\tfrac{5}{6},\tfrac{7}{6},\tfrac{3}{2},1;\tfrac{\beta}{\alpha},\tfrac{1}{\alpha}\big) \end{aligned}$$

$$\begin{aligned} \left(3\Lambda^2\right) \partial_v a_2 =\; & -2^{-1/3}\alpha^{-1/3}\, F_4\big(\tfrac{1}{3},\tfrac{2}{3},\tfrac{1}{2},1;\tfrac{\beta}{\alpha},\tfrac{1}{\alpha}\big) \\ & - 2^{-1/3}\sqrt{\frac{\beta}{3}}\,\alpha^{-5/6} F_4\big(\tfrac{5}{6},\tfrac{7}{6},\tfrac{3}{2},1;\tfrac{\beta}{\alpha},\tfrac{1}{\alpha}\big) \end{aligned}$$

$$\begin{aligned} \left(\sqrt{3}\Lambda\right) \partial_u a_{D1} =\; & (-1)^{-1/3}2^{-2/3}\alpha^{-1/6} F_4\big(\tfrac{1}{6},\tfrac{5}{6},\tfrac{1}{2},1;\tfrac{\beta}{\alpha},\tfrac{1}{\alpha}\big) \\ & - \frac{2}{3}(-1)^{1/6}2^{-2/3}\sqrt{\beta}\alpha^{-2/3} F_4\big(\tfrac{2}{3},\tfrac{4}{3},\tfrac{3}{2},1;\tfrac{\beta}{\alpha},\tfrac{1}{\alpha}\big) \\ & - (-1)^{1/6}\pi 2^{4/3}\Gamma(1/3)^{-3}\sqrt{\frac{\beta}{3}}\, F_4\big(\tfrac{2}{3},\tfrac{2}{3},\tfrac{3}{2},\tfrac{1}{3};\beta,\alpha\big) \\ & + (-1)^{2/3}2^{-5/3}\frac{\Gamma(1/6)^2}{\pi\Gamma(1/3)}\, F_4\big(\tfrac{1}{6},\tfrac{1}{6},\tfrac{1}{2},\tfrac{1}{3};\beta,\alpha\big) \end{aligned}$$

$$
\begin{aligned}
\left(\sqrt{3}\Lambda\right)\partial_u a_{D2} &= (-1)^{-1/3}2^{-2/3}\alpha^{-1/6}F_4\big(\frac{1}{6},\frac{5}{6},\frac{1}{2},1;\frac{\beta}{\alpha},\frac{1}{\alpha}\big)\\
&+\frac{2}{3}(-1)^{1/6}2^{-2/3}\sqrt{\beta}\alpha^{-2/3}F_4\big(\frac{2}{3},\frac{4}{3},\frac{3}{2},1;\frac{\beta}{\alpha},\frac{1}{\alpha}\big)\\
&+(-1)^{1/6}\pi 2^{4/3}\Gamma(1/3)^{-3}\sqrt{\frac{\beta}{3}}\,F_4\big(\frac{2}{3},\frac{2}{3},\frac{3}{2},\frac{1}{3};\beta,\alpha\big)\\
&+(-1)^{2/3}2^{-5/3}\frac{\Gamma(1/6)^2}{\pi\Gamma(1/3)}\,F_4\big(\frac{1}{6},\frac{1}{6},\frac{1}{2},\frac{1}{3};\beta,\alpha\big)
\end{aligned}
$$

$$
\begin{aligned}
\left(3\Lambda^2\right)\partial_v a_{D1} &= \sqrt{3}(-1)^{-1/6}2^{-1/3}\alpha^{-1/3}F_4\big(\frac{1}{3},\frac{2}{3},\frac{1}{2},1;\frac{\beta}{\alpha},\frac{1}{\alpha}\big)\\
&-(-1)^{1/3}2^{-1/3}\alpha^{-5/6}\sqrt{\frac{\beta}{3}}\,F_4\big(\frac{5}{6},\frac{7}{6},\frac{3}{2},1;\frac{\beta}{\alpha},\frac{1}{\alpha}\big)\\
&-3(-1)^{1/3}2^{2/3}\frac{\Gamma(1/3)}{\Gamma(1/6)^2}\sqrt{\beta}\,F_4\big(\frac{5}{6},\frac{5}{6},\frac{3}{2},\frac{2}{3};\beta,\alpha\big)\\
&+\frac{3}{4}(-1)^{5/6}2^{-1/3}\frac{\sqrt{3}}{\pi^2}\Gamma(1/3)^3\,F_4\big(\frac{1}{3},\frac{1}{3},\frac{1}{2},\frac{2}{3};\beta,\alpha\big)
\end{aligned}
$$

$$
\begin{aligned}
\left(3\Lambda^2\right)\partial_v a_{D2} &= -\sqrt{3}(-1)^{-1/6}2^{-1/3}\alpha^{-1/3}F_4\big(\frac{1}{3},\frac{2}{3},\frac{1}{2},1;\frac{\beta}{\alpha},\frac{1}{\alpha}\big)\\
&-(-1)^{1/3}2^{-1/3}\alpha^{-5/6}\sqrt{\frac{\beta}{3}}\,F_4\big(\frac{5}{6},\frac{7}{6},\frac{3}{2},1;\frac{\beta}{\alpha},\frac{1}{\alpha}\big)\\
&-3(-1)^{1/3}2^{2/3}\frac{\Gamma(1/3)}{\Gamma(1/6)^2}\sqrt{\beta}\,F_4\big(\frac{5}{6},\frac{5}{6},\frac{3}{2},\frac{2}{3};\beta,\alpha\big)\\
&-\frac{3}{4}(-1)^{5/6}2^{-1/3}\frac{\sqrt{3}}{\pi^2}\Gamma(1/3)^3\,F_4\big(\frac{1}{3},\frac{1}{3},\frac{1}{2},\frac{2}{3};\beta,\alpha\big)
\end{aligned}
$$

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