Moduli Dependence of One–Loop Gauge Couplings in (0,2) Compactifications[*]

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Abstract

We derive the moduli dependence of the one–loop gauge couplings for non–vanishing gauge background fields in a four–dimensional heterotic (0,2) string compactification. Remarkably, these functions turn out to have a representation as modular functions on an auxiliary Riemann surface on appropriate truncations of the full moduli space. In particular, a certain kind of one–loop functions is given by the free energy of two–dimensional solitons on this surface.

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1 Introduction

One loop gauge couplings in string theories have been a subject of alive interest over the last four years for two obvious reasons [1, 2, 3]. Firstly, these so–called threshold functions represent the boundary conditions for the running gauge couplings of the effective field theory at the string scale and determine in this way the values of the low–energy gauge couplings [1]. Therefore their knowledge is a basic ingredient for string phenomenology. Secondly, their dependence on the vacuum expectation values of the moduli fields is of great theoretical interest: it defines the largest possible subgroup of the tree–level target space duality symmetries [4] which is realized in the quantum theory. It is important to keep in mind this relation: in general, the thresholds can restrict the quantum symmetries and cannot be determined by imposing the tree–level symmetries [5].

Recently, the moduli dependence of the one–loop couplings gained in importance in the light of the fascinating subject of S–duality in N=2 supersymmetric field theories [6]. The discussion of strong–weak coupling symmetries as true symmetries of string theories or relations between them requires the knowledge of the moduli dependence of the gauge couplings. In particular, this dependence determines the classical and perturbative monodromies in N=2 supersymmetric theories [7, 8]. In fact as we will explain below the moduli dependent part of the one–loop couplings in the N=1 theory agrees with the corresponding couplings in a closely related N=2 theory. Therefore our results can be directly applied to the above problem for this specific N=2 theory.

Roughly speaking there are two types of moduli in a four–dimensional string theory with an interpretation as a compactified ten–dimensional string theory: the moduli which describe the geometry and complex structure of the six compactified internal dimensions and the Wilson line moduli in the gauge sector which have the interpretation of flat but homotopically non–trivial gauge connections wrapping around the non–trivial cycles of the compactification manifold. While the dependence of the one–loop gauge coupling on the first type has been considered in great detail in the past, the same cannot be said about the Wilson line type of moduli. The only exactly known Wilson line dependent gauge couplings are restricted to theories, where the background gauge fields are quantized [9]. While the string theories considered there are of phenomenological relevance, no information about the global structure of the moduli space of continuously connected string vacua can be drawn. Lowest order results in an expansion in the vacuum expectation values of the Wilson moduli have been obtained in [10] by a string computation and more recently[1] in [11] using the concept of the topological free energy. It is clear that such an expansion naturally misses the global structure of moduli space. In addition, technical difficulties in the latter approach make necessary an unsystematic truncation even in the lowest order of the background values and an assumption about the duality symmetry of the one–loop couplings rather than a proof of it.

To close this gap, we will derive the duality symmetries and functional dependence of Wilson line dependent gauge couplings in an orbifold theory [12] with a twist embedding of the space group. Interestingly, our results show a strong connection to 2d physics on an auxiliary Riemann surface combining the metric and gauge moduli in a common setting.

Another interesting implication of the presence of moduli dependent threshold corrections in general is the fact that those subspaces of moduli space which enter the one–loop

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[1]see also ref. [8].

couplings have to be compatible with N=2 space–time supersymmetry [13]. The geometry of such subspaces is closely related to the special geometry of the N=2 theory which consists only of the N=2 twisted and the untwisted sector. In fact the Wilsonian gauge couplings and the Kähler potential of these special moduli fields can be obtained from a holomorphic prepotential F in the usual way. The duality transformations become a subset of the symplectic transformations acting on the symplectic vectors (X^A, F_A) of N=2 supergravity [14]. The situation is similar to the one which has been used to infer the special geometry structure of the CY moduli spaces in the case of vanishing Wilson lines [15]. While in these cases the N=2 theory whose consistency implied the special geometry of the moduli subspace was the compactification of a type II superstring theory, in the present case this rôle is taken over by the N=2 theory which is related to the N=1 theory by omitting the N=1 sectors. This theory appropriately includes the vector multiplets whose scalar components are the Wilson line moduli which could not be described by a type II compactification.

2 Symmetries and string amplitudes of the orbifold theory

Our model is a $\mathbb{Z}_8$ orbifold defined on the six–dimensional torus lattice $\Lambda_6 = SO(4) \times SO(9)$. In the complex basis the twist has the eigenvalues $\theta = \exp[\frac{2\pi i}{8}(-4, 1, 3)]$. The twist embedding in the gauge lattice $\Lambda_{16} = E_8 \times E_8'$ is chosen to be $\Theta = \mathbb{Z}_2^{(1)} \times \mathbb{Z}_2^{(1)} \times \mathbb{Z}_8^{(5)} \times \mathbb{Z}_4'^{(3)} \times \mathbb{Z}_2'^{(3)}$ [16]. The resulting gauge group for zero values of the Wilson lines is $SU(6) \times SU(2) \times U(1)^2 \times SU(4)' \times SU(3)' \times SU(2)'^2 \times U(1)'$. In addition, we introduce two complex non–vanishing Wilson lines in the first E_8:

$$a_1^I = (\lambda_1, \lambda_2; 0, 0; 0, 0; 0, 0) \quad ; \quad a_2^I = (\mu_1, \mu_2; 0, 0; 0, 0; 0, 0) \quad \text{with} \quad \lambda_i, \mu_i \in \mathbb{R} , \quad (2.1)$$

where the entries are w.r.t to the two weights $d_1 = (1/\sqrt{2}, 0), d_2 = (0, 1/\sqrt{2})$ of $SU(2)^2$. The sublattice $SO(4) \simeq SU(2) \times SU(2)$ remains unrotated in the N=2 sector which consists of all boundary conditions along the two cycles of the world–sheet torus created out of $1, \theta^2, \theta^4$ and θ^6. The windings and momenta are denoted by n^1, n^2 and m_1, m_2, respectively. The complex moduli fields which belong to this plane and contain the Wilson lines can be defined along [17]:

$$
\begin{aligned}
U &= \frac{R_2}{R_1} e^{i\phi} , \\
T &= \tilde{T} - \frac{1}{4}(\lambda_1 \mu_1 + \lambda_2 \mu_2) + \frac{1}{4}U(\lambda_1^2 + \lambda_2^2) , \\
B &= \frac{1}{2}(\mu_2 + i\mu_1) - \frac{1}{2}U(\lambda_2 + i\lambda_1) , \\
C &= \frac{1}{2}(-\mu_2 + i\mu_1) - \frac{1}{2}U(-\lambda_2 + i\lambda_1) ,
\end{aligned}
\quad (2.2)
$$

with $\tilde{T} = 2b + 2iR_1 R_2 \sin\phi$ being the Kähler–modulus without Wilson lines. R_1 and R_2 are the radii of the two underlying $SU(2)$ root lattices, respectively and ϕ is their relative orientation. The four–dimensional subspace of the E_8 which is left fixed under Θ^2 is an

$SO(8)$ root lattice described by the set of quantum numbers $k = (k_1, k_2, k_3, k_4)$ and the metric $g_{SO(8)}$ with $k^t g_{SO(8)} k = 2(k_1^2 - k_1 k_2 + k_2^2 - k_2 k_3 - k_2 k_4 + k_3^2 + k_4^2)$ [13]. There are various possibilities for the gauge groups at special values of the gauge background fields.

The starting point to calculate the non–universal part of the one–loop threshold corrections $\triangle_a$ to the inverse gauge coupling g_a^2 is the general formula of ref. [1]. Only the N=2 sector of an orbifold gives rise to moduli dependent threshold corrections [2]. For a generic gauge group factor the general formula can be simplified to [18, 13]:

$$\triangle(T, \bar{T}, U, \bar{U}, B, \bar{B}, C, \bar{C}) = \frac{1}{3} b_0^{N=2} \int_{\tilde{\mathcal{F}}} \frac{d^2\tau}{\tau_2} Z^{(1,\Theta^2)}_{SU(2)^2 \times E_8^{\text{inv.}}}(\tau, \bar{\tau}) \, \mathcal{C}_{(1,\Theta^2)}(\tau) \, , \qquad (2.3)$$

with:

$$
\begin{aligned}
Z^{(1,\Theta^2)}_{SU(2)^2 \times E_8^{\text{inv.}}}(\tau, \bar{\tau}) &= \sum_{\substack{n^1, n^2 \in \mathbb{Z} \\ m_1, m_2 \in \mathbb{Z}}} \sum_{k \in \mathbb{Z}^4} e^{\pi i \tau \left[2 m_1 n^1 + 2 m_2 n^2 + k^t g_{SO(8)} k \right]} e^{-2\pi \tau_2 |p_R|^2} \, , \\
\mathcal{C}^{-1}_{(1,\Theta^2)}(\tau) &= \sum_{l \in \mathbb{Z}^4} e^{\pi i \tau l^t g_{SO(8)} l} \, , \\
p_R &= \frac{1}{\sqrt{2 \mathrm{Im} \tilde{T} \mathrm{Im} U}} \left[(TU + BC) \, n^2 + T n^1 - U m_1 + m_2 + i(B+C)(k_1 - \tfrac{1}{2} k_2) - \tfrac{1}{2}(B - C) k_2 \right] \, .
\end{aligned}
$$
$$(2.4)$$

Here, $b_0^{N=2}$ is the β–function coefficient of the $N = 2$ sector for the case $B = C = 0$. The region of integration is extended to $\tilde{\mathcal{F}} = \{1, S, ST\} \mathcal{F}_1$ to take into account[2] the *different* contributions of the twisted sectors. Note that the plane (k_1, k_2) is not orthogonal to the plane (k_3, k_4). On the other hand, this is important to ensure the integrand to be invariant[3] under $\Gamma_0(2)_\tau$. The analogous expressions for gauge groups with additional massless charged particles for special values of the moduli fields can be found in [13].

It can be proven that (2.3) is invariant under the following transformations together with unimodular transformations on the momentum and winding numbers: Firstly, there are the generalizations of the $SL(2, \mathbb{Z})_T \times SL(2, \mathbb{Z})_U$ transformations

$$
\begin{aligned}
T &\longrightarrow -\frac{1}{T} \, , \; B \longrightarrow \frac{B}{T} \, , \; C \longrightarrow \frac{C}{T} \, , \; U \longrightarrow U + \frac{BC}{T} \, , & (2.5) \\
T &\longrightarrow T + 1 \, , & (2.6) \\
U &\longrightarrow -\frac{1}{U} \, , \; B \longrightarrow \frac{B}{U} \, , \; C \longrightarrow \frac{C}{U} \, , \; T \longrightarrow T + \frac{BC}{U} \, , & (2.7) \\
U &\longrightarrow U + 1 \, , & (2.8)
\end{aligned}
$$

Moreover there are the two kinds of shifts acting on the Wilson lines:

$$
B \longrightarrow B + ir + s \, , \; C \longrightarrow C + ir - s \, , \qquad (2.9)
$$

$$
\begin{aligned}
B &\longrightarrow B - iUx - Uy, \qquad C \longrightarrow C - iUx + Uy, & (2.10) \\
T &\longrightarrow T + (x^2 + y^2)U - y(B - C) + ix(B + C) \, ,
\end{aligned}
$$

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[2] This is explained in more detail in [5].

[3] Modular invariance is important to reproduce the full moduli dependence as well as to calculate the topological free energy.

with $r, s, x, y \in \mathbb{Z}$. In addition, there is a mirror transformation exchanging T and U, a symmetry under the exchange of B and C and a parity transformation:

$$T \longleftrightarrow U \,, \tag{2.11}$$

$$B \longleftrightarrow C \,, \tag{2.12}$$

$$B \longrightarrow -B \quad , \quad C \longrightarrow -C \,. \tag{2.13}$$

The transformations (2.5) – (2.13) represent a (non–minimal) set of generators for the duality symmetries of the moduli dependent gauge couplings given in (2.3).

3 Automorphic functions

Having determined the symmetries of the one–loop gauge couplings depending on the four–moduli subspace parametrized by T, U, B and C we will derive now their functional dependence. In addition to the symmetries we have to impose boundary conditions specifying the behavior of the functions at the locus of the moduli space $\mathcal{M}$ where additional particles become massless, causing singularities of the effective field theory. In fact, we will reduce the moduli space once more by restricting to the subspace $\mathcal{M}_i : \lambda_i, \mu_i = 0$ for $i = 1 \vee i = 2$ and consider for the moment this three moduli problem. The Kähler potential [19, 20, 17] can be written as $K = -\ln Y$ with

$$Y = (T - \bar{T})(U - \bar{U}) + (B + \bar{C})(\bar{B} + C) \;=\; \det(M - M^\dagger) \,, \tag{3.1}$$

where

$$M = \begin{pmatrix} T & B \\ -C & U \end{pmatrix} \,. \tag{3.2}$$

A $Sp(4, \mathbb{Z})$ subgroup of the duality symmetries (2.5) – (2.13) is realized on M by the action

$$M \to (A\,M + B)\,(C\,M + D)^{-1} \,, \qquad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp\,(4, \mathbb{Z}) \,. \tag{3.3}$$

Indeed on $\mathcal{M}_i$ we can define the new moduli $B_i = \frac{1}{2}\mu_i - \frac{1}{2}U\lambda_i$ and the transformations (3.3) acting on the matrices

$$M_i = \begin{pmatrix} T & B_i \\ B_i & U \end{pmatrix} \tag{3.4}$$

becomes identical to the standard action of $Sp(4, \mathbb{Z})$ on an element of the Siegel upper half plane $\mathcal{S}_2$. Note that the defining condition Im $M_i > 0$ is ensured by the positivity of both Im T and $-Y$.

In fact, the above treatment is motivated by a non–trivial property of the moduli dependent one–loop gauge couplings $\triangle_a$ for vanishing Wilson lines obtained in [2]:

$$\triangle_a \sim \ln(T - \bar{T})|\eta_T|^4 \; (U - \bar{U})|\eta_U|^4 \,, \tag{3.5}$$

where $\eta_T = \eta(T)$, η_U are Dedekind's functions. The non–holomorphic piece proportional to the Kähler potential is of pure field–theoretical origin and arises from the coupling of

the fermions to the Kähler and sigma–model connections [21, 22]. Eq. (3.5) is nothing but the sum of free energies of solitonic configurations [23] of a complex boson on a Riemann surface given by the product of two tori with Teichmüller parameters T and U, respectively (fig. 1a):

$$\triangle_a \sim \sum_\alpha \ln \left[(\text{Im } T)^{\frac{1}{2}} |\theta_\alpha(T)|^2 \right] + \sum_\alpha \ln \left[(\text{Im } U)^{\frac{1}{2}} |\theta_\alpha(U)|^2 \right] . \tag{3.6}$$

Here the θ's are the even $g = 1$ Riemann theta constants and the sum is over the CP even boundary conditions along the two plus two cycles of the product of two tori. Amazingly, this interpretation has an immediate generalization to the above defined three moduli problem with non–vanishing Wilson line. Combining T, U and B_i as in (3.4) we consider the sum over the free energies of a complex boson on a genus two Riemann surface K with period matrix M_i (fig.1b):

$$\sum_\alpha \ln \left[(\det \text{Im } M_i)^{\frac{1}{2}} |\vartheta_\alpha(M_i)|^2 \right] , \tag{3.7}$$

where now the ϑ's are the ten even $g = 2$ theta constants and the summation is over the CP even boundary conditions along the nontrivial cycles of K. As we will argue this is precisely the functional dependence of the threshold function w.r.t. a gauge group coupled to charged particles which become massless for $B_i = 0$ corresponding to the degeneration limit shown in fig.1c ! Moreover note that the direct product of tori is resolved into a common Riemann surface, a circumstance which is certainly necessary for a possible two–dimensional interpretation of the other kinds of threshold functions considered below.

We will make use now of the theory of Siegel modular forms of degree two to determine the functional dependence of the gauge coupling functions much in the same way as it can be done in the one–moduli case with the modular forms of degree one playing the central rôle. To clarify the philosophy let us sketch briefly the ingredients from which $SL(2, \mathbb{Z})$ invariant threshold functions depending on one modulus T can be derived. The relevant facts are: i) the graded ring of modular forms of $SL(2, \mathbb{Z})$ is generated by two modular forms $\mathcal{E}_4$ and $\mathcal{E}_6$ of weight 4 and 6, respectively[4] ii) there is a unique cusp form $\mathcal{C}_{12}$ of weight 12 without zeros or poles inside the Siegel fundamental region $\mathcal{F}_1$ and iii) any modular invariant function can be written as a rational function $F(j)$ in the j–invariant. In fact, one can argue that this information is sufficient to determine a $SL(2, \mathbb{Z})$ invariant function of a given divisor uniquely up to a constant. The various physical boundary conditions can be distinguished corresponding to the presence or absence of holomorphic anomalies and exceptional massless charged states at some special points in $\mathcal{M}$, respectively. Holomorphic anomalies lead to a non–holomorphic piece in the coupling function as in (3.5) and therefore the holomorphic part will have to transform covariantly rather than to be invariant under the duality transformations. Exceptional massless states require zeros (or poles) inside the fundamental domain leading to the expected logarithmic singularity in the gauge coupling. In addition, there is the singularity in the decompactification limit $T \to i\infty$ (and its mirror partner $U \to i\infty$) where the gauge couplings can diverge as $\lim_{T \to i\infty} \triangle_a \sim T - \bar{T}$ due to the infinite number of light Kaluza Klein states.

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[4]See e.g. [24].

Without giving the mathematical details [13] let us describe the relevant solutions for the three moduli case corresponding to $g = 2$. The ring of modular forms of $Sp(4, \mathbb{Z})$ is generated by two modular forms $\mathcal{E}_4$ and $\mathcal{E}_6$ and three cusp forms $\mathcal{C}_{10}$, $\mathcal{C}_{12}$ and $\mathcal{C}_{35}$, where again the subscript denotes the modular weight [25]. In the following we will omit the odd generator $\mathcal{C}_{35}$ since it factorizes into expressions of the same kind due to the algebraic dependence of its square on the generators of the modular forms of even weight. The description of these modular forms in terms of genus two theta–functions has been given in [25]. The physical boundary conditions we consider are the decompactification limits $\lim_{T \to i\infty}$, $\lim_{U \to i\infty}$, the behavior for $B_i \to 0$, the covariant or invariant transformation properties w.r.t. $Sp(4, \mathbb{Z})$ and exceptional massless states inside $\mathcal{F}_2$.

1. *Thresholds with a singularity only for $B_i \to 0$*: In this case there are particles charged under the gauge group under consideration which become massless for vanishing Wilson lines. On the other hand, there are no other singularities in the Siegel fundamental domain $\mathcal{F}_2$ apart from the decompactification singularities. The modular form with these properties is $\mathcal{C}_{10}$ and the thresholds are given by the expression

$$\triangle_a^I = \frac{b_a^I}{10} \ln Y^{10} |\mathcal{C}_{10}|^2 = \frac{b_a^I}{10} \ln Y^{10} \left| \prod_{k=1}^{10} \vartheta_k(0, M_i) \right|^4 , \tag{3.8}$$

where b_a^I is a model–dependent beta–function coefficient, Y is given in (3.1) and the ϑ_k are the ten even theta–functions at genus two. In the limit $B_i \to 0$ the expansion of $\triangle_a^I$ becomes up to order $\mathcal{O}(B_i^4)$ terms:

$$\triangle_a^I \to b_a^I \left(\ln \ Y + \ln \ |\eta_T \ \eta_U|^4 + \frac{1}{5} \ln \ |1 + 6 B_i^2 \ \partial_T \ln \eta_T^2 \ \partial_U \ln \eta_U^2|^2 \ |\eta_T \ \eta_U|^4 \ |B_i|^2 \right) . \tag{3.9}$$

Eq. (3.8) is just (3.7) times a constant.

2. *Thresholds without singularities in $\mathcal{F}_2$*: This is the generic case where there are no points inside $\mathcal{F}_2$ with additional charged particles w.r.t. the considered gauge group. The unique cusp form with this behavior is $\mathcal{C}_{12}$ and the thresholds read:

$$\triangle_a^{II} = \frac{b_a^{II}}{12} \ln Y^{12} |\mathcal{C}_{12}|^2 . \tag{3.10}$$

In the limit $B_i \to 0$ we find the following expansion:

$$\triangle_a^{II} \to \frac{b_a^{II}}{12} \ln Y^{12} |\eta_T^{24} \eta_U^{24}|^2 |1 + 12 B_i^2 \partial_T \ln \eta_T^2 \ \partial_U \ln \eta_U^2|^2 , \tag{3.11}$$

in agreement with the result of [10].

3. *Thresholds with special singularities in $\mathcal{F}_2$*: In this case there are additional massless states at finite points in $\mathcal{F}_2$. For $B_i \neq 0$ these singularities can move in the moduli space but still exist [11]. Since the exact expression depends on the spectrum let us consider the simplest example of the $\mathbb{Z}_2$ twisted plane in our $\mathbb{Z}_8$ model. For $B_i = 0$ there are enhanced gauge symmetries for the points in moduli space where $T = U$, $T = U = i$, $T = U = e^{2\pi i/3}$ with extra gauge group factors $U(1)$, $U(1)^2$ and $SU(2)$, respectively [26]. A threshold function describing the extra massless states at these points has been proposed in [11] based on symmetry assumptions:

$$\triangle_a \sim \ln(j_T - j_U) = \ln \left(\left[\frac{1}{2} \sum_\alpha \left(\frac{\theta_\alpha(T)}{\eta(T)} \right)^8 \right]^3 - \left[\frac{1}{2} \sum_\alpha \left(\frac{\theta_\alpha(U)}{\eta(U)} \right)^8 \right]^3 \right) , \tag{3.12}$$

which has indeed the correct number of zeros at the special points to describe the massless spectrum[5]. The following function combines the $B_i = 0$ limit (3.12) with $Sp(4, \mathbb{Z})$ invariance:

$$\triangle_a^{III} = b_a^{III} \ln \frac{\mathcal{C}_{12}^2 - 4\mathcal{E}_4^3\mathcal{E}_6^2 + 243 \, \mathcal{C}_{10}\mathcal{E}_4^2\mathcal{E}_6}{(\mathcal{E}_4^3 + \mathcal{E}_6^2 - \mathcal{C}_{12})^2} \, . \tag{3.13}$$

The $B_i \to 0$ expansion is

$$\triangle_a^{III} \to b_a^{III} \ln \left[(j_T - j_U)^2 + 2B_i^2(j_T - j_U)(\partial_U \ln \eta_U^2 \, j_T' - \partial_T \ln \eta_T^2 \, j_U') \right] \, , \tag{3.14}$$

where $j_T' = \partial_T \, j_T$ etc. Although the modular form (3.13) is a good candidate for the Wilson line dependent one–loop coupling for the enhanced gauge symmetries from the six–dimensional compactification, we stress that in this case we have no proof at the moment that it has the correct singularity structure at higher orders in an expansion in B_i. The reason is that due to the complicated singularity structure of $\triangle_a^{III}$ its factorization as in (3.13) could involve modular forms of higher weight introducing a small additional number of available coefficients. We hope to clarify the situation in [13].

Consider now the functional dependence of (3.12) and its generalization (3.13). From a two–dimensional point of view (3.12) hardly has a natural interpretation since it correlates T and U dependent quantities defined on the direct product manifold of fig.1a. However we have seen that the correct picture in the presence of deformations in the direction of the marginal perturbations B_i is that of the genus 2 Riemann surface in fig.1b. with the limit $B = 0$ corresponding to the degeneration of pulling the two tori away from each other (fig.1c). It is tempting to interpret the massless states at the points $T = U$, $T = U = \sqrt{-1 - BC}$, $T = U = -\frac{1}{2} + \sqrt{-\frac{3}{4} - BC}$ in terms of a 2d theory on an exceptionally symmetric surface K. In terms of K the above relations on the moduli reflect symmetries of the period matrix and imply the vanishing of the integrals of abelian differentials along certain cycles of K. We have indicated the relation of the j–function to the partition function of E_8 solitons in (3.12).

We have seen that we can describe the functional dependence of the Wilson line dependent thresholds in a codimension 1 subspace of the original moduli space (which already was a reduction of the full moduli space of the $\mathbb{Z}_2$ plane) by the modular forms of genus two. Interestingly, there is another subspace of the moduli space which can be described by elliptic functions of genus one. First observe that in the degeneration limes $T \to i\infty$ the threshold corrections of the first kind become

$$\triangle_a^I \to b_a^I \left(\ln \, Y + \ln \, |\eta_U|^4 + \frac{1}{5} \ln \left| \frac{\theta_1(B_i, U)}{\eta_U} \right|^2 + \frac{i\pi}{5}(T - \bar{T}) \right) + \mathcal{O}(e^{-2\pi T_2}) \, , \tag{3.15}$$

where $T_2 = \mathrm{Im}T$ and θ_1 is the single odd theta–function at genus one. There exists also the analogous limit for $U \to i\infty$. Interestingly, we can write for these limits an invariant expression for $B \neq \pm C$:

$$\frac{\triangle_a^{I'}}{b_a^{I'}} = \ln Y + \ln |\eta_U|^4 + \frac{1}{4} \left(\ln \left| \frac{\theta_1(\frac{B-C}{2}, U)}{\eta_U} \right|^2 + \ln \left| \frac{\theta_1(\frac{i(B+C)}{2}, U)}{\eta_U} \right|^2 + i\pi(T - \bar{T}) \right) + \mathcal{O}(e^{-2\pi T_2}) \, . \tag{3.16}$$

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[5]Rigorous proofs of (3.12) have been given recently in [7, 8].

This expression is determined uniquely by the singularities and the global symmetries. In fact, an expansion in powers of $q_T = e^{2\pi i T}$ is a very convenient and systematic method for the construction of the general threshold functions for any number of Wilson lines [13]. The coefficient functions of powers of q_T transform as n–th order theta–functions in the various limits of only one non–vanishing Wilson line modulus. The space of these functions is known to be generated by a basic set of theta functions [27]. Modular invariance in U reduces this set considerably and the precise linear combination of the few remaining generators is fixed by the additional symmetries. For example the $\mathcal{O}(e^{-2\pi T_2})$ correction to (3.16) can be easily determined by this method to be

$$-\frac{1}{2}q_T \sum_{\alpha,\beta} \frac{\theta_\alpha^2\left[\frac{B-C}{2},U\right]\ \theta_\beta^2\left[\frac{i(B+C)}{2},U\right]}{\theta_\alpha^2(0,U)\ \theta_\beta^2(0,U)}\ , \qquad (3.17)$$

where the θ_α are the even $g = 1$ theta functions and the sum is over the six pairs $\alpha,\beta \in \{0,2,3\},\ \alpha \neq \beta$.

To illustrate the formal derivation of the one–loop gauge couplings above, let us give two concrete examples for our orbifold model. To fix the coefficients we have calculated the $T \to i\infty$ and $B \to 0$ limits directly from the integral representation (2.3). The agreement of the functional dependence of these lowest order terms provides also a consistency check of our formal derivation. An example for the case I is the $SU(2)$ factor in the first E_8, broken to a $U(1)$ by the Wilson line. Any factor from the E_8' falls into case II. For the corresponding one–loop couplings we find:

$$\triangle_{U(1)}^I \quad = \quad \frac{b_{U(1),0}^{N=2}}{10}\ln Y^{10}|\mathcal{C}_{10}|^2\ , \qquad (3.18)$$

$$\triangle_{SU(4)'}^{II} \quad = \quad \frac{b_{SU(4)',0}^{N=2}}{12}\ln Y^{12}|\mathcal{C}_{12}|^2\ . \qquad (3.19)$$

4   Conclusions and outlook

The functional dependence of the one–loop gauge couplings considered in this paper has been determined in a specific orbifold compactification. However the previous experience with the dependence of one–loop couplings on Kähler and complex structure moduli indicates that these automorphic functions should have a broader application. This fact is a consequence of the general structure of duality symmetries which causes a large stability of the functional dependence on the moduli against variations of the specific compactification [5] and even the kind of interactions such as gauge, gravitational and Yukawa couplings [28].

An important question is that of the meaning of the Riemann surface which gives rise to the automorphic functions representing the one–loop gauge couplings. If there is any, one should find analogous structures in the results for one–loop couplings in general Calabi–Yau compactifications [29], generalizations to any number of Wilson lines and an explicit representation of the mass formula for the exceptional states at the enhanced symmetry points in terms of vanishing integrals of certain abelian differentials. An interesting feature is the resolution of the direct product structure[6] of the T and U dependent tori into a

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[6] A similar although different direct product structure plays a central rôle in the derivation of the perturbative monodromies in N=2 supersymmetric string theories in [7].

common Riemann surface for non–vanishing vev of the Wilson line. Although increasing the genus of the Riemann surface is certainly the wrong concept to introduce additional Wilson lines, there is amazingly another candidate ready to take the rôle of the Teichmüller parameter of an additional handle: the dilaton. In fact eqs. (3.1), (3.2) have an obvious genus three generalization including the dilaton and the associated modular forms of degree three represent automorphic functions of the target space and S duality symmetry groups. An identical situation is given in certain orbifold compactifications with the dilaton replaced by a Kähler modulus.

As we have mentioned at the beginning the result for the moduli–dependent threshold corrections of the N=1 supersymmetric orbifold compactification immediately takes over to the associated N=2 theory which consists of the N=2 sectors only. The knowledge of the perturbative gauge coupling and the rich structure of the moduli space make this N=2 theory an interesting subject to study the generalization of strong–weak–coupling duality in N=2 globally supersymmetric theories to local supersymmetry [13].

Finally, moduli dependent threshold corrections to gauge couplings in (0,2) orbifold models play an important rôle for the discussion of gauge coupling unification. In contrary to moduli independent threshold corrections [1, 9] they can give rise to significant contributions. A reasonable choice of the vevs of the moduli fields T, U, B and C should allow for gauge coupling unification at $M_X = 2 \cdot 10^{16} \text{GeV}$ without a grand unifying group [30].

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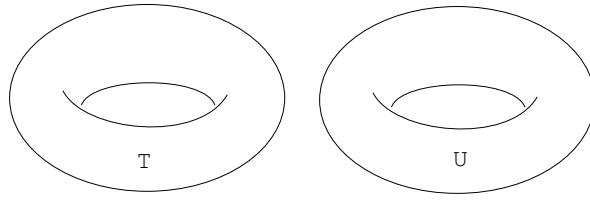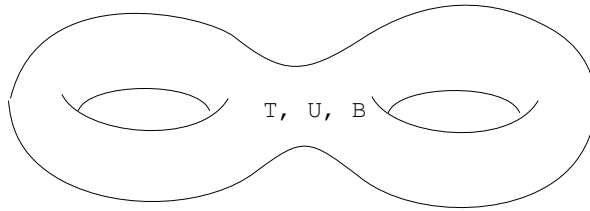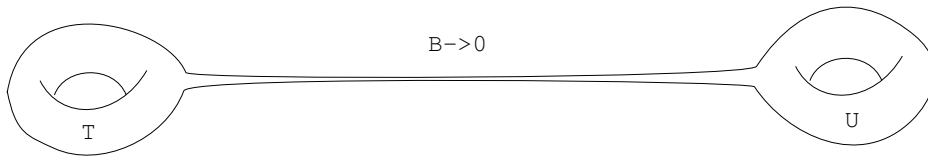[30] P. Mayr, H. P. Nilles and S. Stieberger, in preparation

Fig.1 a

Fig.1 b

Fig.1 c

Figure 1: Riemann surfaces associated to the automorphic functions representing the one–loop gauge couplings: *a*): direct product structure corresponding to eq. (3.5). *b*): g=2 surface associated to the Wilson line dependent coupling functions (3.8),(3.10) and (3.13). *c*): degeneration limit of vanishing Wilson lines replacing *a*).