

Alfio Quarteroni · Alberto Valli

# Numerical Approximation of Partial Differential Equations

With 59 Figures and 17 Tables

 Springer

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