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Bianchi-type string cosmology

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abstract

Bianchi-type string cosmology involves generalizations of the FRW backgrounds with three transitive spacelike Killing symmetries, but without any *a priori* assumption of isotropy in the 3D sections of homogeneity. With emphasis on those cases with diagonal metrics and vanishing cosmological constant which have not been previously examined in the literature, the present findings allow an overview and the classification of all Bianchi-type backgrounds. These string solutions (at least to lowest order in α') offer prototypes for the study of spatial anisotropy and its impact on the dynamics of the early universe.

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1 Introduction

Bianchi-type string cosmology involves 4D spatially homogeneous spacetimes which satisfy at least the lowest-order string beta-function eguations [1]–[7]. Disposing with the assumption of spatial isotropy, these Bianchi-type string backgrounds (BTSB) generalize (and contain as a special case) all possible Friedmann-Robertson-Walker models as well as those with asymptotic or less than $SO(3)$ isotropy. As such, they provide the best models available for the understanding of anisotropy and its impact on the dynamics of the early universe, well before the attainment of the presently observed state of isotropy [1],[5]–[7]. It is in this vaguely-understood region where most of the fundamental cosmological problems arise and also where string cosmology has its best chance of being confronted with reality.

Bianchi-type cosmologies [1],[5]–[7] can be generally defined in terms of a 3-parameter group of isometries G_3. All nine possible group *types* are classified in terms of the parameter $X = I, II, \ldots, IX$ and they are further characterized as being of G_3-class $\mathcal{A}$ or $\mathcal{B}$ according to whether their adjoint representation is traceless or not [1]. The action of G_3 is simply transitive on its orbits so that each orbit, equivalently identified with a 3D hypersurface of homogeneity Σ^3, is spanned by three independent spacelike Killing vectors ξ_i. The set of their duals $\{\sigma^i\}$, invariant under the left action of G_3, provides a natural (non-holonomic) basis for the formulation of G_3-invariant statements. For example, the characterization of a Bianchi-type metric as *diagonal* presumes the employment of a $\{\sigma^i\}$ basis because, for most types, such a metric will not remain diagonal when expressed in terms of ordinary (holonomic) coordinates.

As we will see, all BTSBs with diagonal metrics can be assembled in three classes. To facilitate the discussion (and anticipate the classification introduced later on) we denote these classes as $X(d\uparrow)$, $X(d\rightarrow)$, $X(d\nearrow)$. The arrows specify the orientation of the dual H^* of the totally antisymmetric field strength with respect to the (pictured as 'horizontal') Σ^3 sections. The $X(d\uparrow)$, recently discussed in [6], contains as a subclass all possible FRW backgrounds with vanishing cosmological constant Λ and generalizes all such previously known BTSBs [3]–[5]. Also recenty discussed was the $X(d\rightarrow)$ class [7]. It follows that, with the investigation of the remaining $X(d\nearrow)$ case, the category $X(d)$ of all $\Lambda = 0$ diagonal BTSBs can be fully uncovered. Subsequently, an overview and a classification of all possible BTSBs could be attained. These last remarks also describe the motivation and objective of this paper.

In the following section we introduce notation and certain preliminaries needed for the presentation of our main results in section 3. These are further discussed in section 4, which also contains a classification (with brief reviews) on all possible diagonal $\Lambda = 0$ BTSBs and a summarizing Table.

2 Preliminaries

We consider 4D spacetimes with Bianchi-type metrics of the form [1],[5]–[7]

$$ds^2 = -dt^2 + a_1^2(t)(\sigma^1)^2 + a_2^2(t)(\sigma^2)^2 + a_3^2(t)(\sigma^3)^2, \tag{1}$$

namely diagonal in the $\{dt, \sigma^i\}$ basis, as part of a background solution which satisfies at least the lowest-order string beta-function equations for conformal invariance. The metric coefficients a_i are functions of the cosmic time t only and, as mentioned, $\{\sigma^i\}$ is a G_3-invariant basis in Σ^3. To further fix notation we recall that these background equations can be derived from the effective action [2]

$$S_{eff} = \int d^4x\sqrt{-g}e^{\phi}(R - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} + \partial_\mu\phi\partial^\mu\phi - \Lambda), \tag{2}$$

and in the so set 'sigma-' conformal frame they are

$$\begin{aligned} R_{\mu\nu} - \frac{1}{4}H^2_{\mu\nu} - \nabla_\mu\nabla_\nu\phi &= 0, & (3)\\ \nabla^2(e^\phi H_{\mu\nu\lambda}) &= 0, & (4)\\ -R + \frac{1}{12}H^2 + 2\nabla^2\phi + (\partial_\mu\phi)^2 + \Lambda &= 0. & (5) \end{aligned}$$

The cosmological constant Λ (coming from a central charge deficit in the original theory) will be hereafter set equal to zero. In addition to the gravitational field $g_{\mu\nu}$, expressed through a_i in (1), these expressions also involve the dilaton ϕ and, in the contractions $H^2_{\mu\nu} = H_{\mu\kappa\lambda}H_\nu{}^{\kappa\lambda}$, $H^2 = H_{\mu\nu\lambda}H^{\mu\nu\lambda}$, the totally antisymmetric field strength $H_{\mu\nu\lambda}$. The latter, which may be equivalently viewed here as a closed 3-form H, is defined in terms of the potential $B_{\mu\nu}$ (equivalently the 2-form B) as

$$\begin{aligned} H_{\mu\nu\rho} &= \partial_\mu B_{\nu\rho} + \partial\rho B_{\mu\nu} + \partial_\nu B_{\rho\mu} & (6)\\ (H &= dB). \end{aligned}$$

Just like the metric (1), the dilaton and H fields must also respect the G_3-isometries, namely their Lie derivatives with respect to any Killing vector generated by the $\{\xi_i\}$ basis must vanish. This means that the dilaton field must be a constant on Σ^3, namely it can only be a function of the time t in M^4. On the other hand, the dual H^* of H must be of the form

$$H^* = H^*_0(t)dt + H^*_i(t)\sigma^i, \tag{7}$$

namely with components H^*_μ at most functions of t in the $\{\sigma^i\}$ basis. The (occasionally made) claim of necessarily vanishing H^*_i components in the present context is generally false. However, due to severe restrictions, such components can only exist in relatively few types, as we will see. In the following, t will be profitably expressed in terms of the coordinate time τ (and with a prime for $d/d\tau$) defined by

$$prime = \frac{d}{d\tau} = a^3 e^{\phi} \frac{d}{dt} \tag{8}$$

where

$$a^3 = a_1 a_2 a_3 \tag{9}$$

is the expansion factor of any co-moving volume element in Σ^3.

3 The $X(d \nearrow)$ class of Bianchi-type string backgrounds

As implied just above, and in contrast to the mentioned $X(d \uparrow)$, $X(d \rightarrow)$ classes, $X(d \nearrow)$ admits fewer Bianchi types whose spacetimes satisfy the background equations (3–5). To investigate this, one must examine all possible isometry groups (namely each type X) separately and isolate the cases in which non-vanishing H^*_i components in (7) can survive *in addition* to H^*_0. Skipping the details we state the (easily verifiable) result that the above requirements can be met only for certain G_3-class $\mathcal{B}$ types, in fact for $X = III, V, VI_h$. The projection of H^* in Σ^3 is always aligned with a particular principal direction of anisotropy. To explicitly write down these results we recall that the just mentioned types involve a 1-parameter family of isometry groups G_3 [1], which must be considered as acting on the metric (1). The commutation relations of the generators of such G_3 (given equivalently by their dual expressions) are

$$d\sigma^1 = 0, \quad d\sigma^2 = h\sigma^1 \wedge \sigma^2, \quad d\sigma^3 = \sigma^1 \wedge \sigma^3. \tag{10}$$

The values $h = 0, 1$ of the real parameter h give rise to Bianchi types III, V respectively, otherwise VI_h is realized. Choosing a specific realization of these σ^i, we can establish that the possible metrics in the $X(d \rightarrow)$ class can be written as

$$ds^2 = -dt^2 + a_1^2(t)(dx^1)^2 + a_2^2(t)(e^{hx^1}dx^2)^2 + a_3^2(t)(e^{x^1}dx^3)^2. \tag{11}$$

Comparing with (1) one can read off (11) the σ^i in terms of the x^i coordinates. In fact the metric (11) is unique up to (the only allowed but unimportant) gauge or coordinate transformations which preserve (10). We can proceed to find the 2-form B potential in (6) in terms of the above coordinates. The result of this calculation may be expressed as

$$B = \begin{cases} \frac{1}{h+1}\eta'(\tau)e^{(h+1)x^1}dx^2 \wedge dx^3 & |h \neq -1 \\ (A_0 x^1 + \zeta'(\tau))dx^2 \wedge dx^3 & |h = -1. \end{cases} \tag{12}$$

The functions $\eta(\tau)$, $\zeta(\tau)$ are to be specified and they have been introduced through their τ derivatives for later convenience. It can be subsequently verifed that (as earlier claimed) the dual H^* in (7) may be expressed as

$$H^* = a^3 H_0^*(t) e^{\phi} d\tau - A_1 e^{-\phi} \sigma^1, \tag{13}$$

with

$$a^3 H_0^* = \begin{cases} -\eta' & |h \neq -1 \\ -A_0 & |h = -1 \end{cases} \tag{14}$$

and with A_0, A_1 constants.

The background equations (3–5) can now be given explicitly. In particular the 'ii' components in the set (3) are

$$\begin{aligned} (\ln a_1^2 e^{\phi})'' - 2(h^2+1)(a_2 a_3 e^{\phi})^2 &= (A_1 a_2 a_3)^2 \\ (\ln a_2^2 e^{\phi})'' - 2h(h+1)(a_2 a_3 e^{\phi})^2 &= 0 \\ (\ln a_1^2 e^{\phi})'' - 2\,(h+1)(a_2 a_3 e^{\phi})^2 &= 0, \end{aligned} \tag{15}$$

subject to the constraint equation (the '01' in (3))

$$a_3^{-(h+1)} a_2^h a_1 = e^{-\frac{1}{2} A_1 \eta}, \tag{16}$$

plus the initial value equation (essentially the '00' in (3))

$$\begin{aligned} (\ln a_1^2 e^{\phi})'(\ln a_2^2 e^{\phi})' &+ (\ln a_2^2 e^{\phi})'(\ln a_3^2 e^{\phi})' + (\ln a_3^2 e^{\phi})'(\ln a_1^2 e^{\phi})' = \\ &= \phi'^2 + 4(h^2+h+1)(a_2 a_3 e^{\phi})^2 + (H_0^* a^3 e^{\phi})^2 + (A_1 a_2 a_3)^2. \end{aligned} \tag{17}$$

Coupled with (4),(5) these equations admit the following solutions, depending on the value of the parameter h.

For $\boldsymbol{III(d \nearrow), V(d \nearrow), VI_h(d \nearrow)}$, realized at $h = 0, h = 1, h \neq 0, \pm 1$, respectively, we find

$$\begin{aligned} a_1^2 e^{\phi} &= Q^{\frac{2(h-1)}{h+1}} \left(\frac{h+1}{P_1} \sinh P_1 \tau \right)^{\frac{-2(h^2+1)}{(h+1)^2}} \exp\left(\frac{A_1}{h+1}\eta + \frac{h-1}{h+1} P_2 \tau \right) \\ a_2^2 e^{\phi} &= Q^2 \left(\frac{h+1}{P_1} \sinh P_1 \tau \right)^{\frac{-2h}{h+1}} \exp\left(P_2 \tau\right) \\ a_3^2 e^{\phi} &= Q^{-2} \left(\frac{h+1}{P_1} \sinh P_1 \tau \right)^{\frac{-2}{h+1}} \exp\left(-P_2 \tau\right) \end{aligned} \tag{18}$$

where P_1, P_2, Q are constants (Q could be assigned any positive value). From (5),(4) for the dilaton and H field we obtain the coupled system

$$\phi'' = \left(\frac{A_1 P_1}{h+1} \right)^2 (e^{\phi} \sinh P_1 \tau)^{-2} - e^{2\phi} \eta'^2 \tag{19}$$

$$\eta'' = \frac{A_1 P_1^2}{h+1} (e^{\phi} \sinh P_1 \tau)^{-2} \tag{20}$$

and hence the functions $\phi(\tau), \eta(\tau)$, although apparently not in closed form in the general case. They are subject to the initial value equation

$$\phi'^2+e^{2\phi}\eta'^2+\frac{2A_1P_1}{h+1}(\coth P_1\tau)\eta'+\left(\frac{A_1P_1}{h+1}\right)^2(e^{\phi}\sinh P_1\tau)^{-2}=4\frac{h^2+h+1}{(h+1)^2}P_1^2-P_2^2, \tag{21}$$

as required by (17). The constraint (16) has already been satisfied in view of (18). At the $A=0$ limit, which according to (13) corresponds to a hypersurface-orthogonal H^*, the above set can be easily integrated to reproduce the already known $III(d\uparrow)$,$V(d\uparrow)$ and $VI_h(d\uparrow)$ cases [6]. A multitude of other special cases is possible. We explicitly mention the $V(d\nearrow)$, with asymptotic $SO(3)$ isotropy, realized as that spacetime expands towards an open ($k=-1$) FRW model according to (18–21) at $h=1$.

For $VI_{-1}(d\nearrow)$, realized at $h=-1$, we find

$$\begin{aligned} a_1^2e^{\phi} &= Q_1^2e^{P_1\tau}\exp(A_1\zeta+Q_1^2Q_2^2e^{2P_2\tau}) \\ a_2^2e^{\phi} &= Q_2^2|P_2|\exp{(P_2+A_0A_1/2)\tau} \\ a_3^2e^{\phi} &= Q_3^2|P_2|\exp{(P_2-A_0A_1/2)\tau} \end{aligned} \tag{22}$$

where Q_i, P_1, P_2 are constants. From (5),(4) for the dilaton and H field we obtain the coupled system

$$\phi'' = -A_0^2e^{2\phi}+(Q_0/A_0)^2e^{2P_2\tau-2\phi} \tag{23}$$
$$A_1\zeta'' = (Q_0/A_0)^2e^{2P_2\tau-2\phi} \tag{24}$$

subject to the initial value equation

$$\phi'^2-2A_1P_2\zeta'+A_0^2e^{2\phi}+(Q_0/A_0)^2e^{2P_2\tau-2\phi}=P_2^2+2P_1P_2-\frac{1}{4}(A_1A_2)^2 \tag{25}$$

as follows from (17), with $Q_0=|A_0A_1Q_2Q_3P_2|$. As in the previous case, the general solution for the functions $\phi(\tau),\zeta(\tau)$ does not seem attainable in closed form. Here, however, we observe that the dilaton field may be expressed as

$$e^{2\phi}=A_0^{-2}Q_0e^{\psi+P_2\tau}, \tag{26}$$

where $\psi(\tau)$ is a solution to

$$\psi''+4Q_0e^{P_2\tau}\sinh\psi. \tag{27}$$

For positive P_2, one can interpret the 'sinh' term as a confining potential (in relation to the $\psi=0$ convergence limit) to realize that ψ must oscillate around zero with exponentially *increasing* frequencies and *decreasing* amplitudes. Setting $\psi=0$ in (26) etc., we find

$$e^{2\phi} = A_0^{-2}Q_0e^{P_2\tau}, \tag{28}$$
$$A_1\zeta = (Q_0/P_2^2)e^{P_2\tau}. \tag{29}$$

The rest of the solution is given by (22), together with the restriction

$$3P_2^2 + 8P_1P_2 = (A_1A_2)^2 \tag{30}$$

coming from (25). The result expressed by (22) together with (28–30) does *not* give the most general $VI_{-1}(d \nearrow)$ possible, but rather the asymptotic limit of the general solution. The same result is also by itself a solution (in closed form) and, as such, it essentially reproduces as special cases each one of the $VI_{-1}(d \uparrow)$ and $VI_{-1}(d \rightarrow)$ solutions found in [6] and [7] respectively.

4 Conclusions

All possible BTSBs may be classified, so that each one is represented as $X^n(d,a)$. The parameter X specifies the isometry group G_3 and takes the values $I, II, \ldots, IX$, roughly one for each of the Bianchi types plus one for the Kantowski-Sachs class of metrics [1]. The rest of the parameters may be omitted or take values as follows. The index n specifies the isotropy group, which is $SO(3)$ if $n = 3$, $SO(2)$ if $n = 2$, or the null group (case of complete anisotropy) if n is omitted altogether. The argument d is omitted only when the metric (1) is *non*-diagonal in the $\{\sigma^i\}$ basis. The last argument, zero in the trivial case of an identically vanishing H field, specifies the orientation of H^* relative to the hypersurfaces of homogeneity Σ^3. There are no classification parameters for the dilaton field and the cosmological constant Λ. With types VI_{-1} and VII_0 counted separately, it turns out that there are in all $24^2 = 576$ cases which this classification sees as distinct BTSBs. Many of them are obviously special cases, descending from more general (less-symmetric) ones, as, for example, the '$\Rightarrow$' arrows in the Table indicate. Others (such as the '$\nexists$' cases in the Table) cannot be realized in the sense that their metrics are singular everywhere. This by no means implies that the respective geometries do not exist. It does mean, however, that such manifolds could *in principle* be realized only in the presence of appropriate sources.

Before turning to specific Bianchi types, we will briefly review certain aspects common to all cases. Let us begin by taking as an example the Bianchi-type V case, for which we copy from the Table the sequence

$$\cdots \;\; V(d \nearrow) \;\; \Rightarrow \;\; V(d \uparrow) \;\; \Rightarrow \;\; V^2(d \uparrow) \;\; \Rightarrow \;\; V^3(d \uparrow). \tag{31}$$

Any spatial component of H^* breaks $SO(3)$ isotropy, so that FRW behavior (if at all possible) can exist only for vanishing ${H^*}_i$. Clearly, one expects special interest in the cases of *asymptotic* attainment of this value. In the $V(d \nearrow)$ case, H^* is tangent to a congruence which could sustain general kinematics, namely expansion, shear (anisotropy) and, *in principle*, vorticity as well. In the $V(d \uparrow)$ case, $SO(3)$ isotropy may still be broken by other agents but vorticity must vanish identically, as it cannot be sustained for kinematical reasons. In the terminal $V^3(d\uparrow)$ case, namely the open FRW model, only isotropic expansion has survived. One can further establish that in all cases there is an initial singularity and no inflation.

Type I. The isotropy limit contains all possible flat ($k=0$) FRW cases. Until recently, only the $I^3(d0)$ with its $I^3(d\uparrow)$ and the Kasner-like $I(d0)$ generalization had been given[3],[4],[5]. They are all reproduced as special cases of $I(d\uparrow)$ [6]. The $I(d\rightarrow)$ is also known [7], but, according to our result covering all G_3-class $\mathcal{A}$ types, there exists no $I(d\nearrow)$.

Type II. The fully anisotropic $II(d\uparrow)$ given in [6] generalizes the $II(d0)$ found in [5]. For the $II(d\rightarrow)$ and $II(d\nearrow)$ cases the same hold as for type I.

Type III. $III(d\nearrow)$ exists, as we saw, and it reduces to the $III(d\uparrow)$ found in [6]. However, there also exists a general $III(d\rightarrow)$ [7], which cannot be reached as a limit of the mentioned $III(d\nearrow)$.

Type IV. All diagonal metrics are singular everywhere [6],[7].

Type V. At isotropy one obtains all open ($k=-1$) FRW cases, such as the $V^3(d0)$ and $V^3(d\uparrow)$ found in [3], [4], the first one generalized by the $V(d0)$ in [5]. They all are special cases of $V(d\uparrow)$, given in [6], with the latter further generalized by the $V(d\nearrow)$ found here. There exists no $V(d\rightarrow)$ case [7].

Type VI_h. The results of the previous case are generally valid here as well, except of course for the isotropy limit.

Type VI_{-1}. Generally, the $h=-1$ case is *not* obtained at the $h=-1$ limit from solutions of the previous type. We have seen that $VI_{-1}(d\nearrow)$ generalizes the $VI_{-1}(d\uparrow)$ in [6] as well as the $VI_{-1}(d\rightarrow)$ [7]. This is the only case in which *both* such limits can be reached from a given $X(d\nearrow)$.

Type VII_h. In this case, which also involves a 1-parameter group G_3 (here with $h^2\leq 4$), all metrics are singular everywhere, unless $h=0$. The latter case (which exceptionally involves spacetimes of G_3-class $\mathcal{A}$) is outlined next.

Type VII_0. There exists the $VII_0(d\uparrow)$ and its isotropy limits $VII_0^2(d\uparrow)$ and $VII_0^3(d\uparrow)$ are identical to those in the Type-I case [6]. The $VII_0(d\rightarrow)$ has been given recently [7] but, as we have seen, there exists no $VII_0(d\nearrow)$.

Type VIII. We have seen that there exists no $VIII(d\nearrow)$. Neither is there a $VIII(d\rightarrow)$ [7], but the $VIII^2(d\uparrow)$ has been explicitly found in [6].

Type IX. Also explicitly found is the $IX^2(d\uparrow)$ case (generalizing the well-known Taub metric to which it reduces) [6]. The complete isotropy limit therein, namely $IX^3(d\uparrow)$, reproduces the closed ($k=1$) FRW cases found in [3],[4]. In view of our present findings (cf. also [7]), the most general possible diagonal Bianchi-type IX case is $IX(d\uparrow)$ (but elusive just like its Mixmaster counterpart!).

To conclude with some generally applicable remarks, we note that the *energy of anisotropy* [1],[6] may be quite significant, compared with that of any other field present in the effective action (2), during some time near the Planck or string scale. It follows that the study of anisotropy in such strong-field regimes would require solutions which are exact to all orders in the α' organization of the string action. It is apparently not known whether some (all?) of the more general $X(d)$ backgrounds discussed here (that is, with no more than three Killing isometries) are exact solutions beyond leading order in α'. Relevant in that context, although applied to a different class of solutions, is ref. [8]. We also note that, under abelian target space duality, the *known* solutions in $X(d\uparrow)$ generate metrics in the same class [6]. Obviously, however, this cannot be the case in general. For example, inspection of (12) etc., immediately shows that the duals of $VI_h(d\nearrow)$ will involve metrics with non-vanishing '$0i$' components in the invariant basis. With the latter type of metric, even if homogeneity were not really lost, there would be no universal time to define uniquely the (presumably observable) 3D sections of homogeneity. Such spacetimes appear to be of significance very close to the initial singularity [9].

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The diagonal Bianchi-type string backgrounds

X	$d\sigma^i = \frac{1}{2}C^i_{jk}\sigma^j \wedge \sigma^k$	G_3 class	$X(d\rightarrow)$	X($d\nearrow$)	$X(d\uparrow)$	$X^2(d\uparrow)$	$X^3(d\uparrow)$ (FRW)
I	$d\sigma^i = 0$	$\mathcal{A}$	$\exists$	$\nexists$	$\exists\Rightarrow$	$\exists\Rightarrow$	$\exists$
II	$d\sigma^1 = \sigma^2 \wedge \sigma^3$ $d\sigma^2 = 0$ $d\sigma^3 = 0$	$\mathcal{A}$	$\exists$	$\nexists$	$\exists$	$\nexists$	$\nexists$
III	$d\sigma^1 = 0$ $d\sigma^2 = 0$ $d\sigma^3 = \sigma^1 \wedge \sigma^3$	$\mathcal{B}$	$\exists$	$\exists\Rightarrow$	$\exists$	$\nexists$	$\nexists$
IV	$d\sigma^1 = \sigma^1 \wedge \sigma^3 +$ $\sigma^2 \wedge \sigma^3$, $d\sigma^2 =$ $\sigma^2 \wedge \sigma^3$, $d\sigma^3 = 0$	$\mathcal{B}$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$
V	$d\sigma^1 = 0$ $\sigma^2 = \sigma^1 \wedge \sigma^2$ $d\sigma^3 = \sigma^2 \wedge \sigma^3$	$\mathcal{B}$	$\nexists$	$\exists\Rightarrow$	$\exists\Rightarrow$	$\exists\Rightarrow$	$\exists$
VI_h	$d\sigma^1 = 0$ $\sigma^2 = h\sigma^1 \wedge \sigma^2$ $d\sigma^3 = \sigma^1 \wedge \sigma^3$	$\mathcal{B}$	$\nexists$	$\exists\Rightarrow$	$\exists$	$\nexists$	$\nexists$
VI_{-1}	$(h = -1)$	$\mathcal{B}$	$\exists$	$\Leftarrow\exists\Rightarrow$	$\exists$	$\nexists$	$\nexists$
VII_h	$d\sigma^1 = -\sigma^2 \wedge \sigma^3$ $d\sigma^2 = \sigma^1 \wedge \sigma^3 +$ $h\sigma^2 \wedge \sigma^3$, $d\sigma^3 = 0$	$\mathcal{B}$	$\nexists$	$\nexists$	$\nexists$	$\nexists$	$\nexists$
VII_0	$(h = 0)$	$\mathcal{A}$	$\exists^*$	$\nexists$	$\exists^*\Rightarrow$	$\exists\Rightarrow$	$\exists$
$VIII$	$d\sigma^1 = \sigma^2 \wedge \sigma^3$ $d\sigma^2 = -\sigma^3 \wedge \sigma^1$ $d\sigma^3 = \sigma^1 \wedge \sigma^2$	$\mathcal{A}$	$\nexists$	$\nexists$	$\exists^*\Rightarrow$	$\exists$	$\nexists$
IX	$d\sigma^1 = \sigma^2 \wedge \sigma^3$ $d\sigma^2 = \sigma^3 \wedge \sigma^1$ $d\sigma^3 = \sigma^1 \wedge \sigma^2$	$\mathcal{A}$	$\nexists$	$\nexists$	$\exists^*\Rightarrow$	$\exists\Rightarrow$	$\exists$

$\exists$: solution is known (in some cases not in entirely closed form, see also [6],[7]).
$\exists^*$: solution exists but it is not known.
$\nexists$: not-realizable as a 'vacuum' background.
$\Rightarrow$: towards specialization (higher symmetry).

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